Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 32
Jeans Instability

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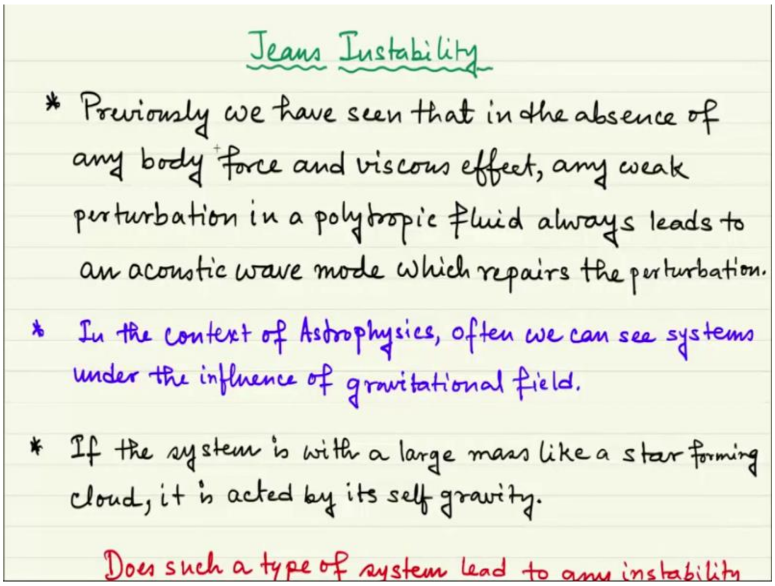

Hello, and welcome to another session of Introduction to Astrophysical Fluids. In this session, we will discuss the physics of Jeans Instability. Jeans instability is one of the most important instabilities in the framework of astrophysics, because often the stars and the galactic chunks are believed to have been formed as a result of Jeans instability. Now, what does Jeans instability mean?

So, previously if we remember that we have seen, in the absence of any body force and viscous effect, any weak perturbation of first order in a polytropic fluid for which the pressure is just only a function of density always and actually has a relation like p is equal to some constant times ρ^γ. This type of fluid always leads to an acoustic wave mode which repairs the perturbation, given the initial state is a static, I mean the state was in hydrostatic equilibrium.

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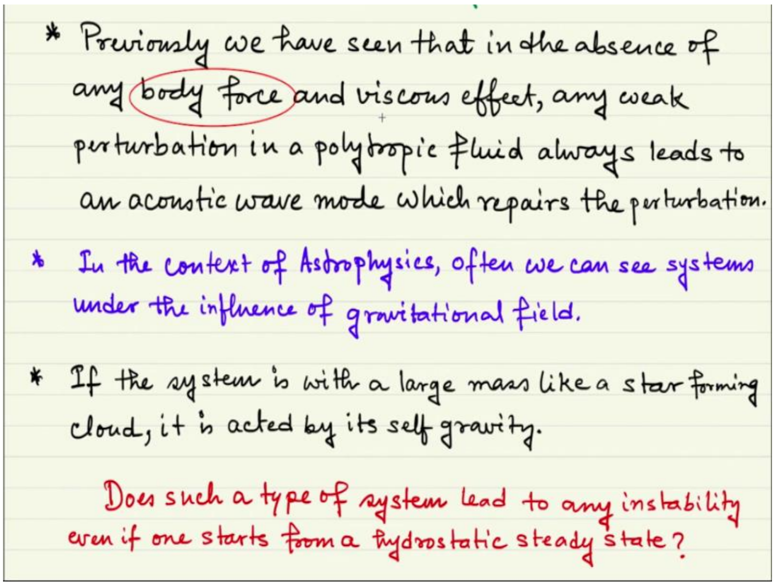

Now, in the context of astrophysics, often we can see systems under the influence of gravitational field. So, we can no longer neglect the body force effect. So, the body force and viscous effects both were neglected in the previous case, where we just saw the emission of sound waves or I mean the formation of the sound waves in order to respond to any weak linear perturbation made to the system. But, in astrophysics we have to consider the body force which is the gravitational field.

If the system is with a large mass, like a star forming cloud, for example, then the system is acted by its self-gravity. So, the system is also acted, I mean the body force is nothing but its own gravity, own field of gravity.

So, now, the question is that does such type of system also lead to the stability and linear mode of like sound wave or something like that or this leads to any type of instability, in this case even if one starts from a hydrostatic steady state? That is the question.

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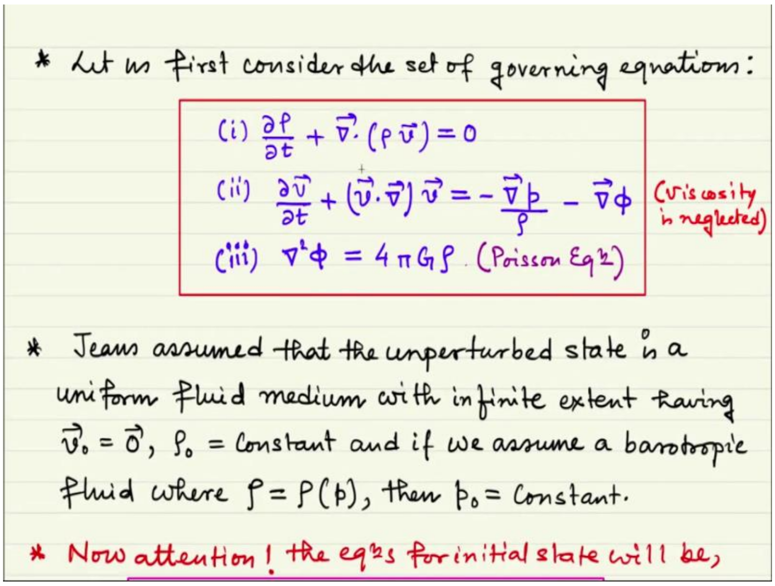

Now, in order to answer to this question, let us first consider the set of governing equations. So, the set of governing equation the 1st one you all can now recognize this is the famous continuity equation $\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0$. This is the famous Euler equation, of course, because we have neglected the viscosity. So, $\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\varphi$, here as you can easily see that we have neglected the viscosity.

The 3rd equation is the Poisson's equation for gravitational field. So, $\nabla^2\varphi = 4\pi G\rho$, here you can easily see that φ is the gravitational potential. So, g which is the gravitational intensity can be written as $-\vec{\nabla}\varphi$.

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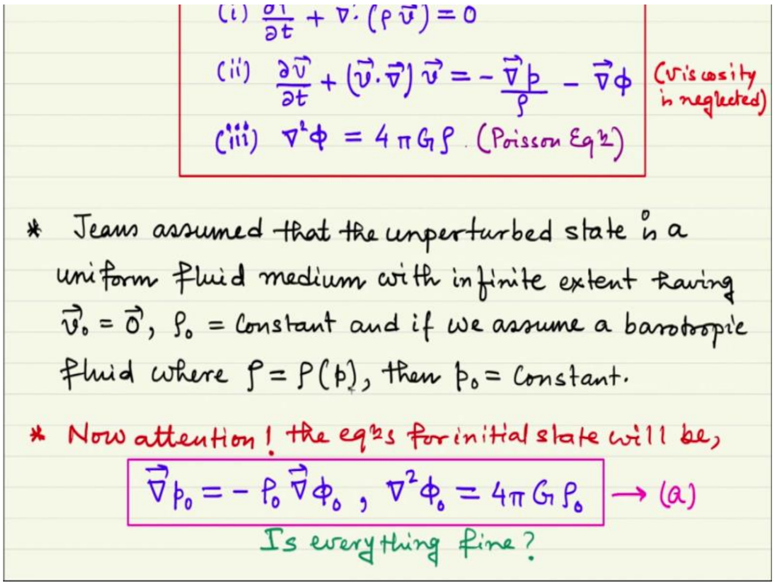

Now, Jeans assume that the unperturbed state is a uniform fluid medium with infinite extent. This state is uniform means the density and pressure they are constant everywhere, because we have assumed a polytropic which is a special case of a barotropic fluid, so where ρ is a function of p only. So, if ρ is constant as ρ_0 then p is also having constant value which is equal to p_0.

The fluid is at rest that means, v_0 is equal to 0. So, that is the unpartnered state. Now, you can see that if you just insert this initial condition in this set of equations, you can easily see that the initial state should always be steady that I said at the very beginning and in addition it is now uniform and static. So, $\frac{\partial}{\partial t}$ are all 0, v is 0. So, this is the continuity equation trivially vanishes.

This term is nonzero $\vec{\nabla} p_0 + \rho_0 \vec{\nabla}\varphi_0$ is equal to 0, so that is the unperturbed state. So, if our system is steady and static then what happens?

So, finally, from that you can write

$$\vec{\nabla} p_0 = -\rho_0 \vec{\nabla}\varphi_0, \text{ and } \nabla^2\varphi_0 = 4\pi G\rho_0. \qquad \text{(a)}$$

So, that is the Poisson equation. Now, my question, is it looks fine, right. Now, the question is that if you see in depth is it still, ok? Well, there is a catch.

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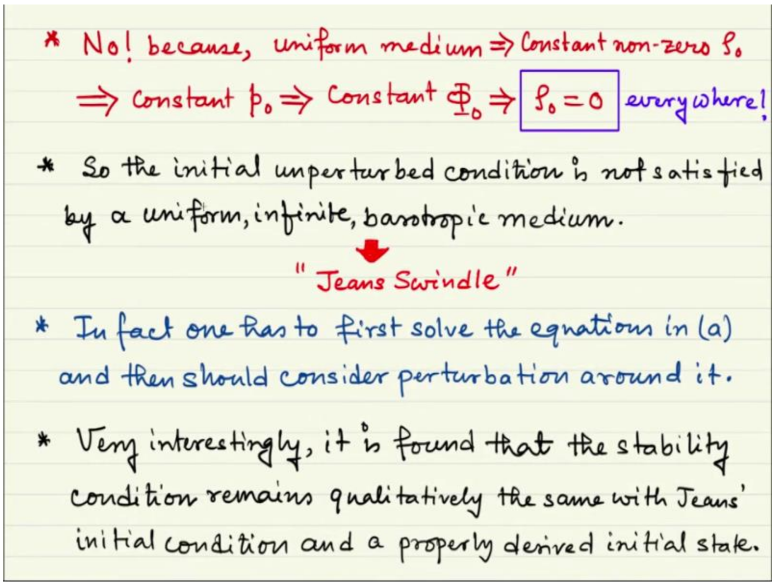

The thing is that this is not fine, that means, the equations are not auto-consistent, self-consistent. Why? Because, if you now consider for that the medium is uniform, then the medium will have a constant nonzero value for ρ_0. So, ρ_0 will be nonzero, but constant.

If we have barotropic or polytropic fluid then p is also constant, so p_0 is constant. So, ρ_0 constant means p_0 constant and p_0 constant means $\vec{\nabla} p_0$ is 0. If ρ_0 is nonzero and $\vec{\nabla} p_0$ is 0, then $\vec{\nabla}\varphi_0$ is also 0, that means, φ_0 is a constant. So, if $\vec{\nabla}\varphi_0$ is 0, now you remember divergence of something which is 0 is also 0 and that means, that $4\pi G\rho_0$ is 0, that means, ρ_0 itself is 0.

So, we started by saying that we have a uniform medium, so ρ_0 is constant, but strictly nonzero, because the matter should be there. But now, we end up by saying that ρ_0 is 0 everywhere. So, what is the problem? The actually the initial unperturbed condition is not satisfied by a uniform infinite barotropic medium. That means, that up to this unperturbed equation there is no problem in this equation, but after that this initial unperturbed steady static condition is not satisfied by a uniform medium.

That means, whenever you make the assumption that ρ_0 is constant and nonzero, then the problem is started, and this known as Jeans swindle. Jeans did not realize the fact, but he was the first one to address this problem analytically and we will come to this later and he actually somehow found qualitatively the same result.

So, what is the way out? In fact, one has to first solve the equation and then should consider the perturbation around it. So, first we have to solve these set of equation and to check that which type of functions actually satisfies this set of equation, and we will see that ρ_0 is constant and nonzero can never be a solution of this equation. So, that is the proper way.

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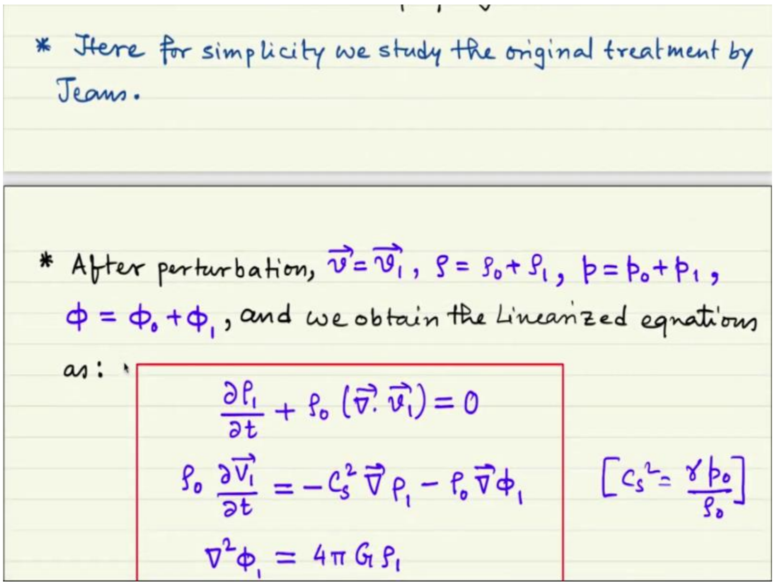

But very interestingly, historically, when Jeans finally analyzed his thing, it is found that the stability condition what Jeans obtained, actually remains qualitatively the same if we do it properly from a well-defined initial state. So, that is the proper way of treating that.

Then, in case you can just remember Lane–Emden's equation, actually, in this case one has to solve Lane–Emden's equation type of thing, I mean which is for a spherical system this type of equations can be cast into this Lane–Emden type of equation, and one can actually see solution of this problem in a proper manner.

But here for this discussion just for the sake of simplicity, we study the original treatment by jeans. That means, we will use the Jeans swindle method. Now, after perturbation what happens? So, ok why we are following this Jeans swindle method, because of two reasons, first one is that this is historically important to know what Jeans did and finally, it is simple and gives qualitatively the same result as that of a mathematically properly done method.

Now, after perturbation what happens, v is now becomes v_1, ρ is now becoming $\rho_0 + \rho_1$ and p is $p_0 + p_1$ and finally, φ is also equal to $\varphi_0 + \varphi_1$. Again, we obtained the linearized equations as

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla}.(\overrightarrow{v_1}) = 0$$

$$\rho_0 \frac{\partial \overrightarrow{v_1}}{\partial t} = -C_s^2 \vec{\nabla} \rho_1 - \rho_0 \vec{\nabla} \varphi_1$$

$$\nabla^2 \varphi_1 = 4\pi G \rho_1$$

Where C_s^2 is nothing but the polytropic sound speed. Now, you know this is nothing, but $\frac{\gamma p_0}{\rho_0}$.

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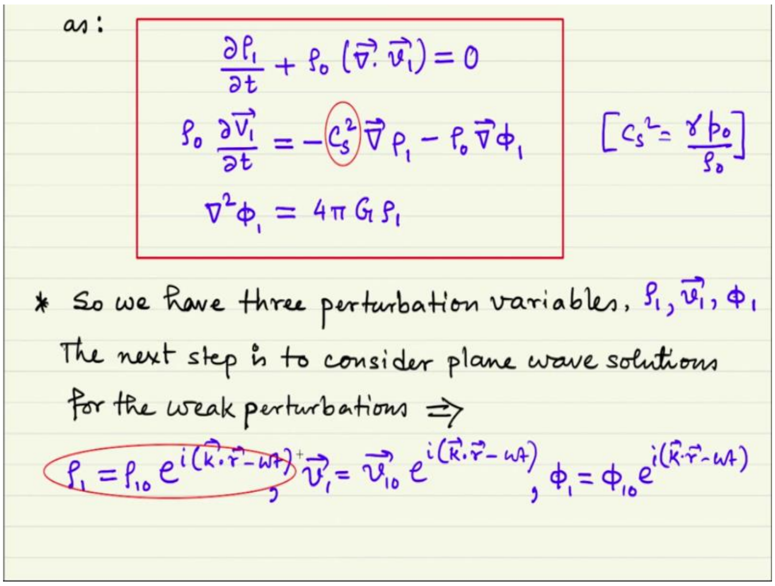

Now, we have 3 equations and 3 perturbation variables, one is ρ_1, one is v_1, one is φ_1. The p_1 is actually replaced by the polytropic condition by ρ_1. In the next step, we consider very simply plane wave solution for all weak perturbations. So, ρ_1 will be equal to some ρ_{10} which is the amplitude part $e^{i(\vec{k}.\vec{r} - \omega t)}$.

This is the plane wave type of solution for the fluctuations. Another way of saying is that as they are linear then we can think them to be composed of Fourier components and then we take one of the Fourier components. Another way just to say that, we are assuming that plane wave solutions. Where, v_1 will be equal to some $v_{10}\ e^{i(\vec{k}.\vec{r}-\omega t)}$, and then φ_1 will be equal to $\varphi_{10}\ e^{i(\vec{k}.\vec{r}-\omega t)}$.

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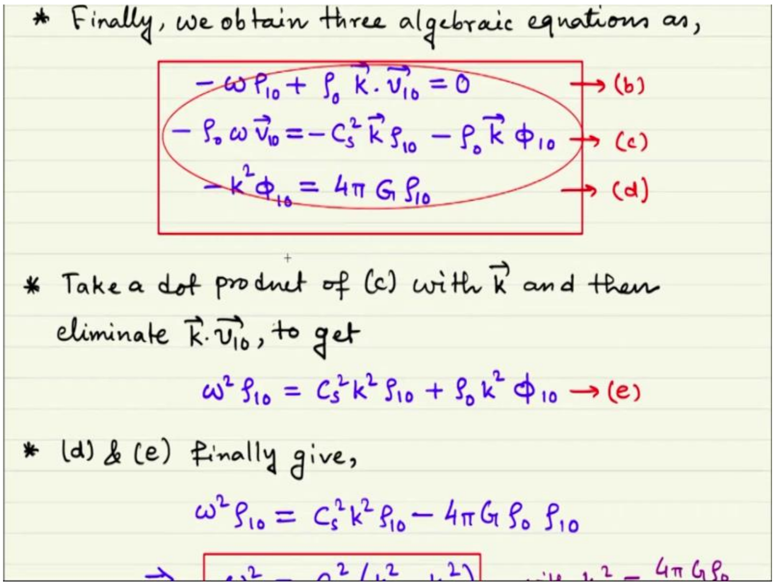

Then finally, we replace all these 3 values in all these 3 expressions. This plane waves type of solutions and we will obtain that

$$-\omega\rho_{10} + \rho_0\vec{k}.\vec{v}_{10} = 0 \qquad \text{(b)}$$

$$-\rho_0\omega\vec{v}_{10} = -C_s^2\vec{k}\rho_{10} - \rho_0\vec{k}\varphi_{10} \qquad \text{(c)}$$

$$-k^2\varphi_{10} = 4\pi G\rho_{10} \qquad \text{(d)}$$

So, we finally, have a set of 3 algebraic equations with 3 unknowns ρ_{10}, v_{10}, and φ_{10}. Of course, we have 5 unknowns because, v_{10} is a vector quantity, so you know that actually we have 5 scalar equations and 5 unknowns.

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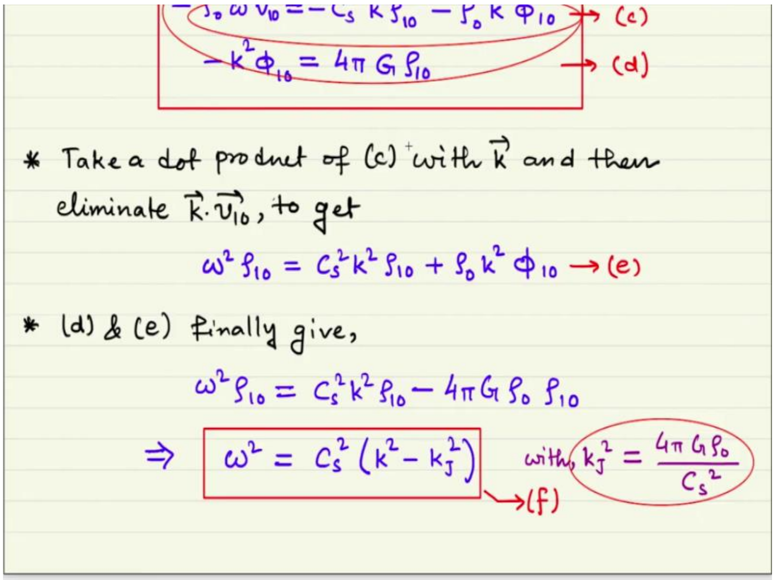

Now, take a dot product of equation (c) with $\vec{k}$, see if we just take $\vec{k}$ dot in the both sides of (c), then finally, we eliminate $\vec{k}.\vec{v}_{10}$ from (b).

Finally, we get another expression which is the expression number (e) which is

$$\omega^2 \rho_{10} = C_s^2 k^2 \rho_{10} + \rho_0 k^2 \varphi_{10} \quad \text{(e)}$$

Finally, I have from (d) and (e). I have got rid of v_{10} variable. So, combining, we get $\omega^2 \rho_{10} = C_s^2 k^2 \rho_{10} - 4\pi G \rho_0 \rho_{10}$ and then just rewriting we can get

$$\omega^2 = C_s^2 (k^2 - k_J^2) \quad \text{(f)}$$

Here, I define this k_J^2 such that k_J^2 is equal to $\frac{4\pi G \rho_0}{C_s^2}$, and this is known as the Jeans wave vector, and here you can actually see this is nothing but the dispersion relation, for the plane wave solution corresponding to the perturbation.

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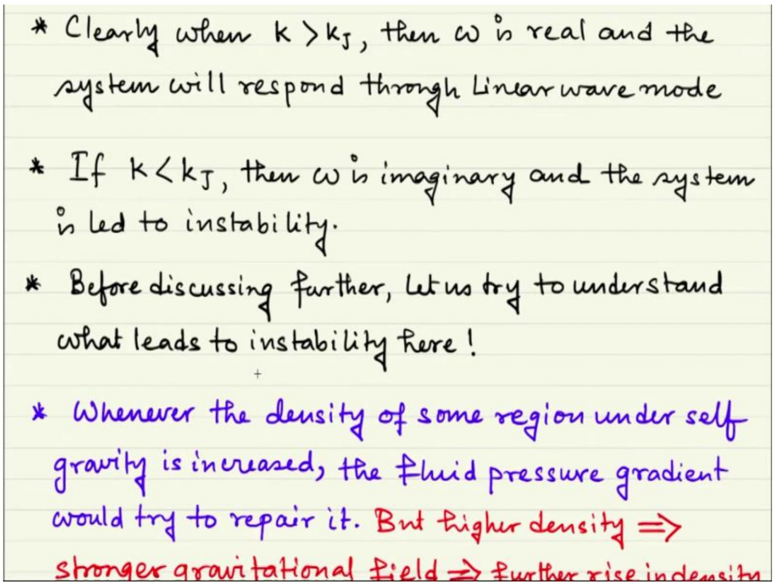

Now, clearly when k is greater than k_J, then ω is real, because ω^2 is positive and the system will respond through linear wave mode. If k is less than k_J, then ω is imaginary and the system then will be led to instability and which is called the Jeans instability.

Now, before discussing further, let us try to understand what leads to instability here actually.

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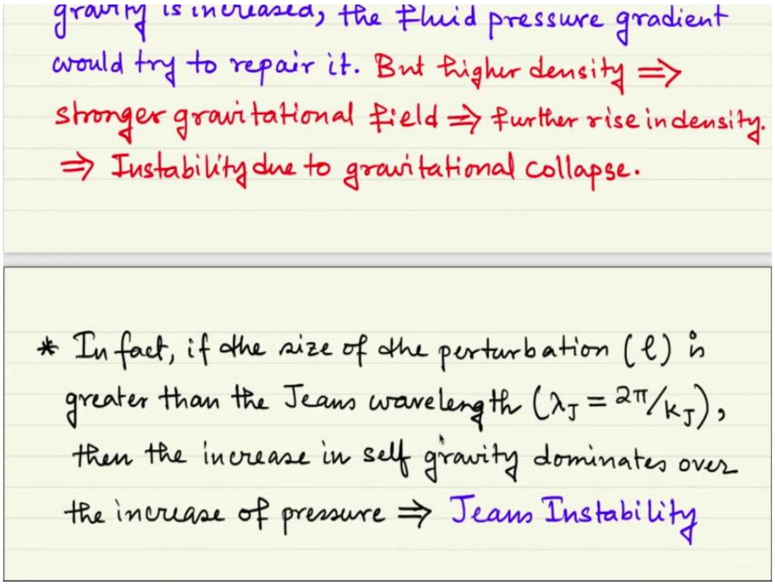

So, whenever the density of some region under cell gravity is increased the fluid pressure gradient would try to repair it, just like in a normal compressible fluid. But, when the density gets higher, it creates a stronger gravitational field, self-gravity field will be increased, that is true by Poisson's equation.

That leads to the further rise in density, and that actually leads to an instability due to gravitational collapse. In fact, if the size of the perturbation is greater than the Jeans wavelength which is defined as λ_J which is nothing but $\frac{2\pi}{k_J}$, then the increase in self gravity dominates over the increase of pressure and which is relating to the Jeans instability.

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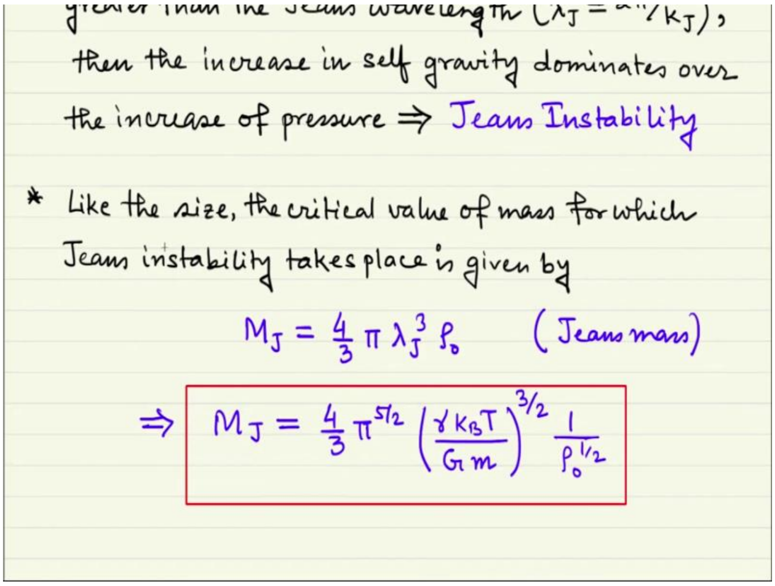

Now, analogically for the size, we have there is a critical value of mass, for which Jeans instability takes place only if the mass overcomes or mass dominates over the value of M_J, which is nothing but $\frac{4}{3}\pi\lambda_J^3\rho_0$. This is called the Jeans mass.

Now, that M_J is equal to

$$M_J = \frac{4}{3}\pi^{5/2}(\frac{\gamma k_B T}{Gm})^{3/2}\frac{1}{{\rho_0}^{1/2}}$$

So, M_J is proportional to $T^{3/2}$ and it is also proportional to $\frac{1}{{\rho_0}^{1/2}}$. Why is that, just try to understand physically? That is my question to you.

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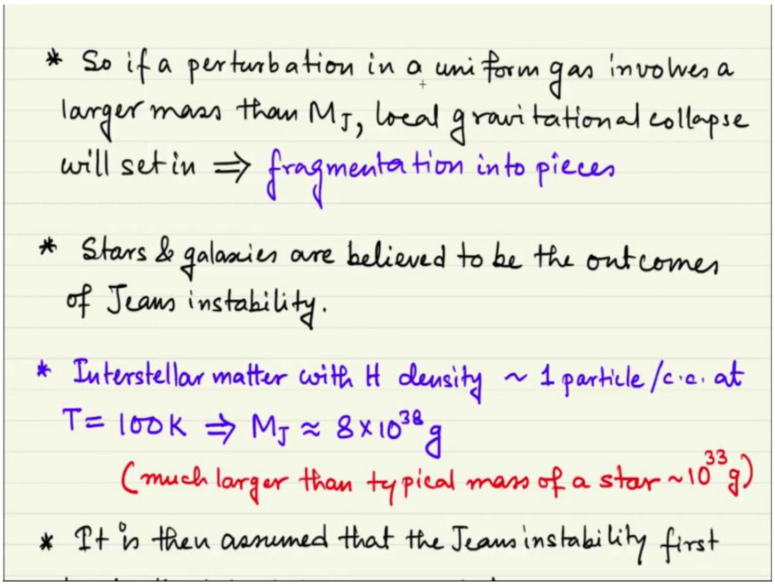

So, in a nutshell, if a perturbation in a uniform gas involves a larger mass than M_J, then the local gravitational collapse will set in and that actually, leads to Jeans instability, thereby fragmentation of the whole system into peace. So, local gravitational collapse basically makes the total continuous mass system to be fragmented, if you can understand the whole picture.

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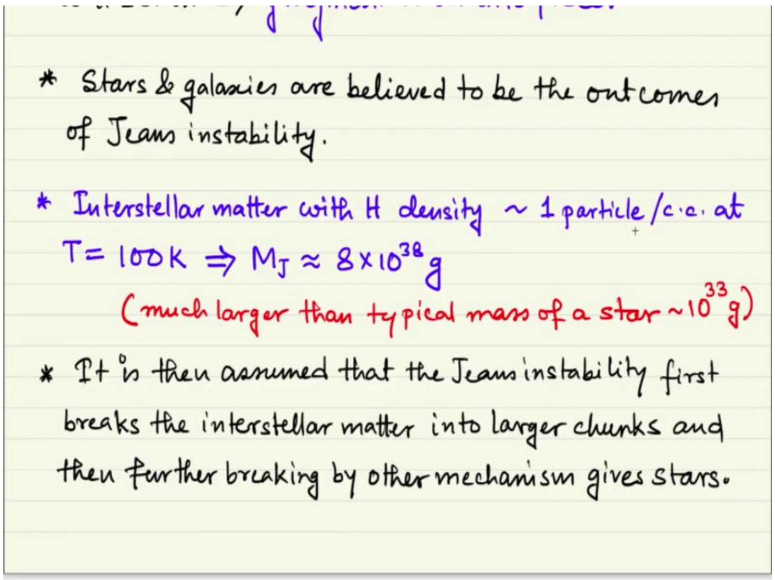

The stars and galaxies are believed to be the outcomes of such instability, but instead interstellar mass with hydrogen atomic density, nearly one particle per cubic centimeter at temperature 100 Kelvin gives a Jeans mass which is of the order of 8×10^{38} gram, which is much larger than the typical mass of a star, which is of the order of 10^{38} gram. It is then assumed that the Jeans instability first breaks the interstellar matters into larger galactic chunks, and then further breaking by other mechanisms gives starts.

So, this is not just one step, Jeans instability which can gives us the usual stars. But this is a two-step process. One step is the bigger interstellar matter, then this breaks into chunks, larger chunks and from that larger chunk that is, of course, by Jeans instability. But then from larger chunks how they get further fragmentation, that is another interesting issue in astrophysics. So, that is all about Jeans Instability.

Thank you very much.