Introduction to Astrophysical Fluids
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Lecture - 31
Rayleigh Benard convection II

Hello, and welcome to another discussion of Introduction to Astrophysical Fluids. In this part we continue our discussion of relevant convection and previously we saw that how to derive the basic equations or the governing equations for such a system, I mean for relevant convection, in this part we will see how to solve or how to like proceed for an analytical solution of such type of problem.

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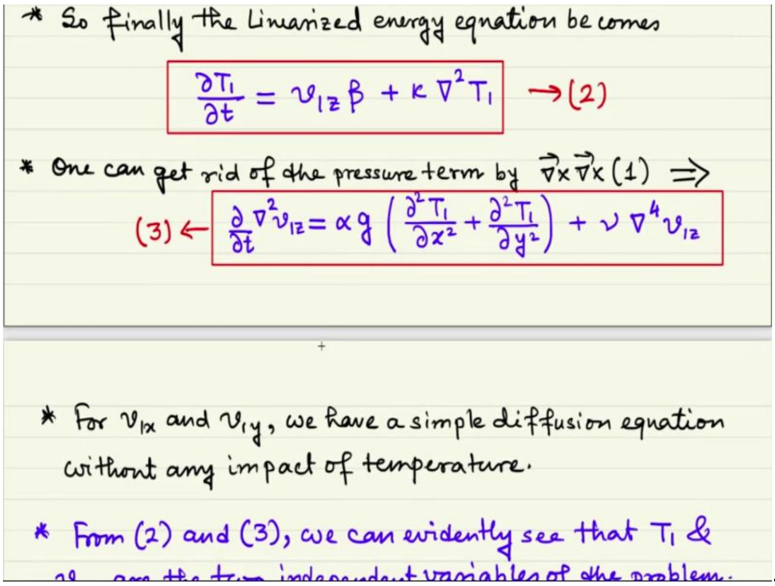

So, these are those two equations which constitute the family of the governing equations for relevant convection. The first one is of course, the evolution of the temperature I mean, of course, the perturbative part of the temperature

$$\frac{\partial T_1}{\partial t} = v_{1z}\beta + k\nabla^2 T_1 \qquad (2)$$

The second one is

$$\frac{\partial \nabla^2 v_{1z}}{\partial t} = \alpha g\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2}\right) + \vartheta\nabla^4 v_{1z} \qquad (3)$$

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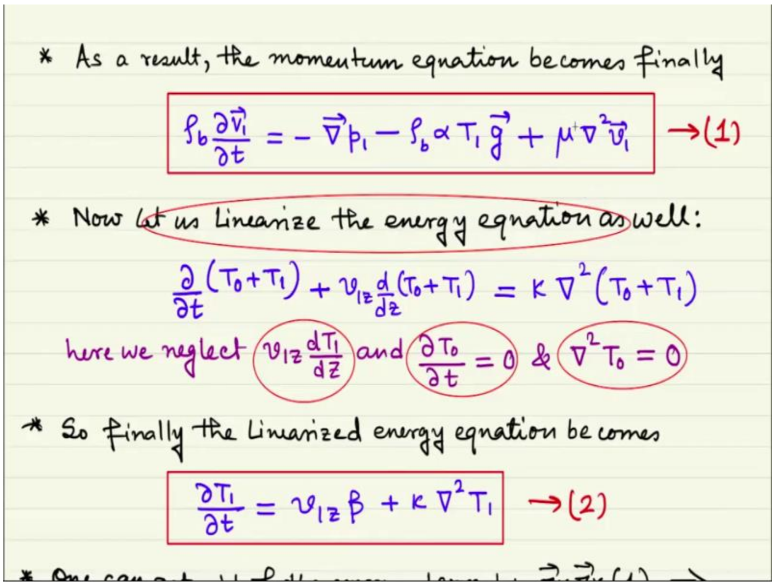

Of course, ϑ if you remember well this is nothing, but $\frac{\mu}{\rho}$. So, here for example, you can simply take this ϑ is nothing, but $\frac{\mu}{\rho_0}$ or which is also very much close to $\frac{\mu}{\rho_b}$. So, this is nothing, but the coefficient of kinematic viscosity.

So, finally, in our equations we can see there are only two unknowns, one is T_1 another is actually v_{1z}.

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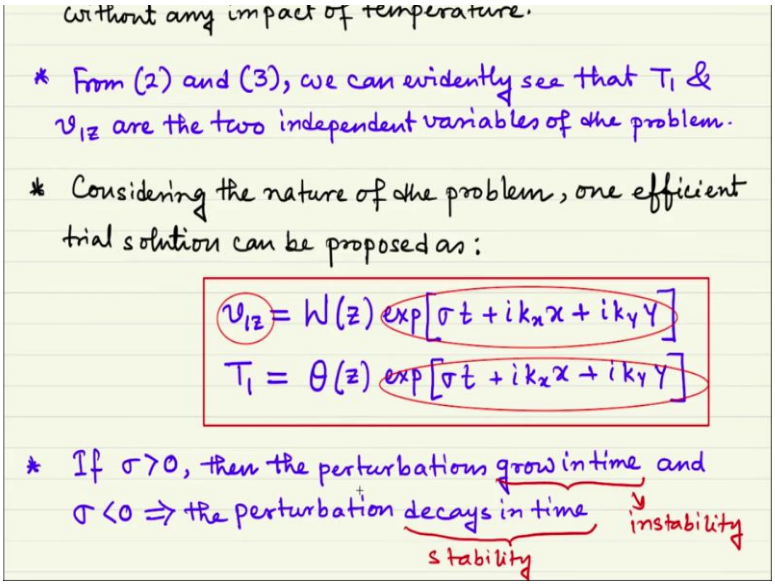

So, you can see that these two equations 2 and 3 are almost like the diffusion equations, but a bit modified. So, the first one is the diffusion equation for T_1 if this term $v_{1z}\beta$ is absent for example, the second one is a diffusion equation for $\nabla^2 v_{1z}$ if this term $\alpha g\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial y^2}\right)$ is absent.

So, of course, we can easily see that with all these terms presents, they are not quite exactly diffusion equations. But we can finally, think of solutions which can be something which look like much as diffusion solution. So, that is what our next step. We now have to propose trial reasonable solutions for T_1 and v_{1z} and there can be many possibilities and people have done a lot of trials and errors. You can go through the works of Subramaniam Chandrasekhar.

So, people had done a lot of work on this type on this problematic of relevant convection the analytical part and there you can see that there is a vast and detailed discussion to justify this type of equations, but here we are just saying that there can be many possibilities. One possibility is just to propose normal plane wave solutions as we did for sound waves.

Now, here what we will do? We will just see that the solutions for v_{1z} and T_1 have this type of nature that is some function W. So, I am just talking first for the case of v_{1z}. So, the solution will be equal to some function W which is a function of z only because if something special occurs that means, the mostly the convective motion will take place along z direction mostly, and then with t there is a change and this change can actually be easily understood by this type of dependence (equation 2).

So, if you know simply the diffusion type of solution, you can see that this is a slightly more general form of diffusion like of solution for the z and t part. So, here we have simply written exponential σ times t that is the temporal dependence of the solutions for both v_{1z} and T_1 and if σ is real then there can be a possibility of instability either growing or decaying and if σ is imaginary then of course, we can understand that in time that will be oscillatory.

Now, for x dependence and y dependence, just for simplicity, we can propose plane wave type of solutions for both v_{1z} and T_1. For T_1 the z dependent function is the θ. So, you can easily see that other than the z dependent part this time x and y dependent part they are the same for the both v_{1z} and T_1. I rather instead of telling you the intuitive reason for that I rather try to encourage you to see the form of the equations much more carefully and try to understand why intuitively these parts.

So, there is not a hard and fast reason for that because these are all the trial solutions and finally, if you take these solutions, you can see that the analysis becomes much easier and the analysis is much more physically talking, but at least why this part should be equal and that can be a reasonable approximation that you have to guess from these two equations. That is a very smart choice actually and I want you to just think over this point.

Now, for our case what we have just said that if σ is greater than 0 then the perturbation grows in time and if σ less than 0 then the perturbation decays in time.

So, that is a strict instability growing in time perturbation and that is a strict stability decays in time. Now we are trying to search for the conditions for marginal instability that means, exactly the condition where the solution is not automatically growing or decaying just due to its explicit time dependence.

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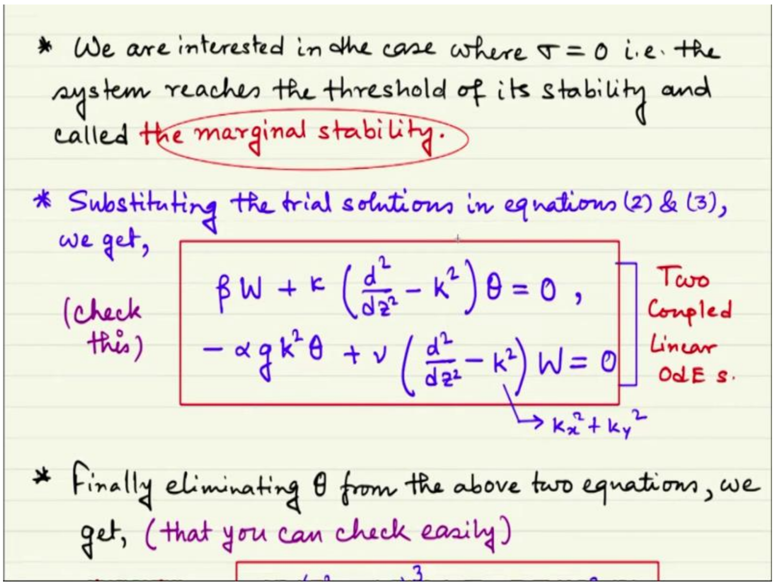

So, that is why what we will do? We say that σ positive means instability, σ negative means decaying solution. Actually, σ it is not actually stability to be very honest, but if σ is imaginary then actually we are talking about oscillations, but in both cases whether this is growing or decaying. So, we can make the marginal part where the solution does not neither grow nor decays in time and that is the part of marginal stability.

Some people say this is the part of marginal instability and that is because in this case σ is chosen to be 0 and time independent.

So, philosophically you can see that in the case of proper language both are actually non stable conditions, if the solution the perturbation grows in time. If the perturbation grows in time, then what happens that even if you have oscillatory type of solutions in space this oscillatory type of solutions will actually grow and that means, their amplitude will also grow in time makes the system unstable.

And on the other hand, if your system is oscillatory, we have something like decaying time dependence then what happens? This oscillatory type of thing will also decay in time giving you the original state which was the static hydrostatic equilibrium steady initial state. So, in that sense we say that σ is equal to 0 is the part of the marginal stability or some people actually for the same reason called marginal instability beyond which you have the strict instability.

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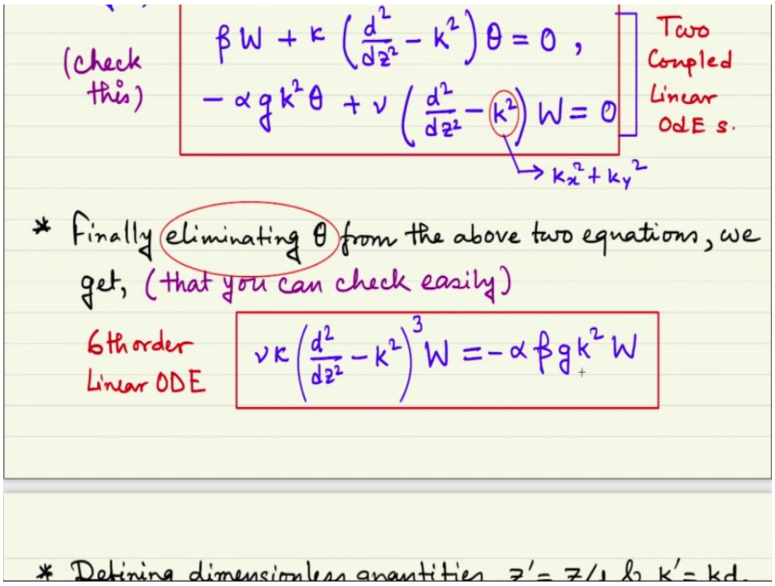

So, now and our study will be confined within this regime of marginal stability. So, if we have marginal stability then this t dependent part will go away and we simply have $v_{1z} = W(z)\exp(ik_x x + ik_y y)$ and T_1 will be equal to $\theta(z)\exp(ik_x x + ik_y y)$.

If you substitute properly these two values over equations 2 and 3, then you can see that the final equations will become $\beta W + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\theta = 0$. Here I have introduced something new small k. I am coming to another equation which will be $-\alpha g k^2 \theta + \vartheta\left(\frac{d^2}{dz^2} - k^2\right) W = 0$.

Now, this k square is nothing, but equal to $k_x^2 + k_y^2$ that is the modulus square of the horizontal wave number. So, this is the wave number which is in the plane perpendicular to the convection, I mean perpendicular to the heating direction to be very honest. So, I can say that the heating direction is in the particle direction.

So, what they are? This equation with W and θ, they constitute a system of two coupled linear ODEs. Please check these equations and I mean how to come from these solutions just by replacing those solutions in these equations. This is a very simple and elegant home task.

Now, finally, what we have to do? We can go a bit further. We have two equations to unknown, they are of course, linear coupled ODEs second order differentiation. So, we also need two boundary conditions for each, but one can actually eliminate θ, one of the two variables from the system of equations and if you eliminate θ for example, from this above two equations finally, you get a sixth order linear differential equation.

Which is nothing, but $\vartheta\kappa\left(\frac{d^2}{dz^2} - k^2\right)^3 W = \alpha\beta g k^2 W$.

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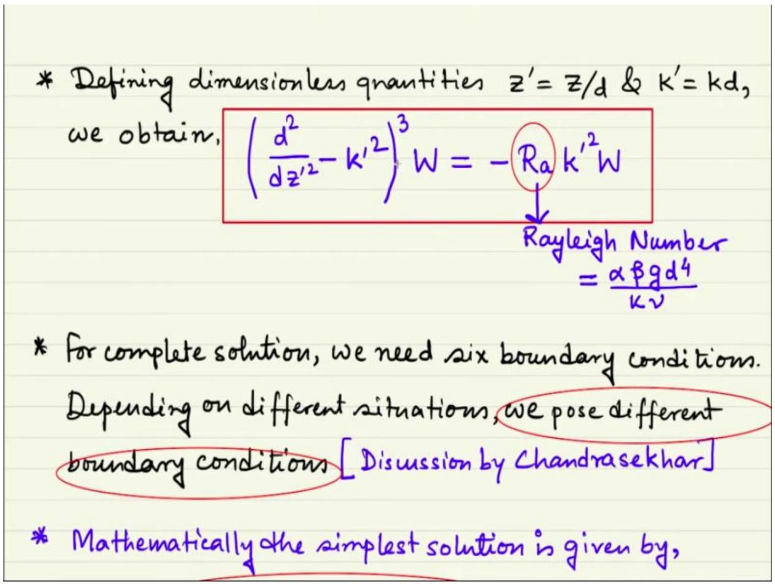

Now, if we define dimensionless quantity z' which is nothing, but $\frac{z}{d}$ and k' is equal to kd. Two dimensions, one is dimensionless height, one is dimensional length less horizontal wave number. If we define d as the characteristic length scale for our system because this is the vertical size of the container.

We finally, write the whole equation as simple as in the form $\left(\frac{d^2}{dz'^2} - k'^2\right)^3 W = -R_a k'^2 W$. This dimensionless number R_a is known as Rayleigh number and this is given by $\frac{\alpha\beta g d^4}{\vartheta\kappa}$.

So, there are one maybe two things to remember that Rayleigh number is first of all increases with β, with α, with d, of course, with g, but g is considered to be constant here, but it decreases with κ and ϑ. So, if the system is more and more viscous its Rayleigh number will go down and if the systems thermometric conductivity is more and more important than the system will also have low Rayleigh number.

On the other hand, if you make the initial temperature gradient very very higher and more and more important, then your Rayleigh number will be important. So, Rayleigh number, in some way basically gives you the importance, the effort of the system to make an important temperature gradient by the effort of the system to destroy that temperature gradient.

So, you can easily understand that if a Rayleigh number is larger and larger, then the system can be subject to more and more convective instability.

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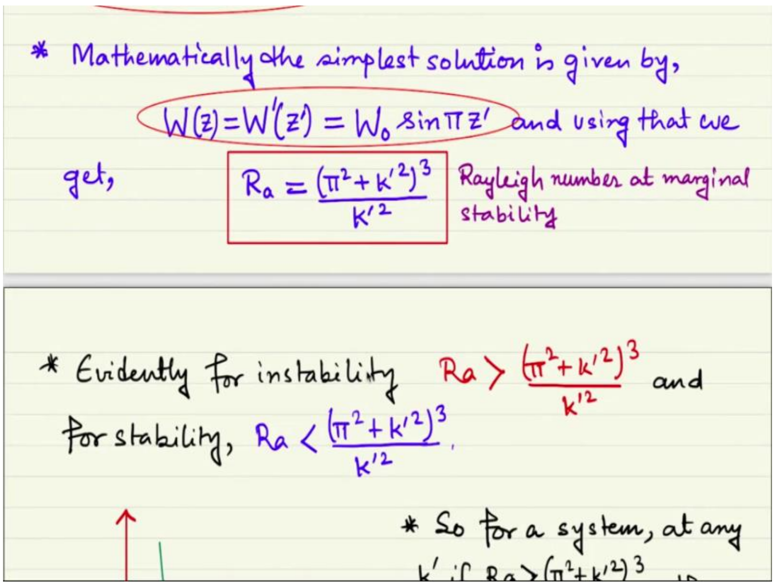

And if Rayleigh number is lower and lower than the system can efficiently repair the effect of strong temperature gradients thereby making it more and more stable. This is the qualitative idea and which is quite correct actually, one can see in the laboratory experiments.

Now, for complex solution of the system, we need six boundary conditions because this is a sixth order ODE. Now linear ODE, of course, now depending on the different situations, we need different boundary conditions. So, that is something very important. In physics we can have different type of conditions where relevant convection can be important and for which we really need different type of boundary conditions.

Another point where you can actually see the discussion by Chandrasekhar, who had a vast amount of work on this topic and also you can see the recent paper. So, always in the internet you can see the recent researches in this domain.

So, mathematically now the simplest solution is given by this. So, where we can write that $W(z)$ is equal to some $W'(z')$ another function, of course, that is why it is a formal way of writing another function of z' and which is given by $W_0 sin\pi z'$ and if we use that we can get this Rayleigh number is nothing, but $\frac{(\pi^2+k'^2)^3}{k'^2}$. So, this is a very simple solution which satisfies this equation and actually you can check that.

Just for your information, this solution is not corresponding physically, most of the cases either in household cases or in astrophysical context. So, in both cases actually this equation is nothing, but an idealization and can be used for academic purpose, but the philosophy which we will now extract will be roughly the same.

So, here for this type of equation I have a Rayleigh number which is given by $\frac{(\pi^2+k'^2)^3}{k'^2}$. So, this is a function of z' only.

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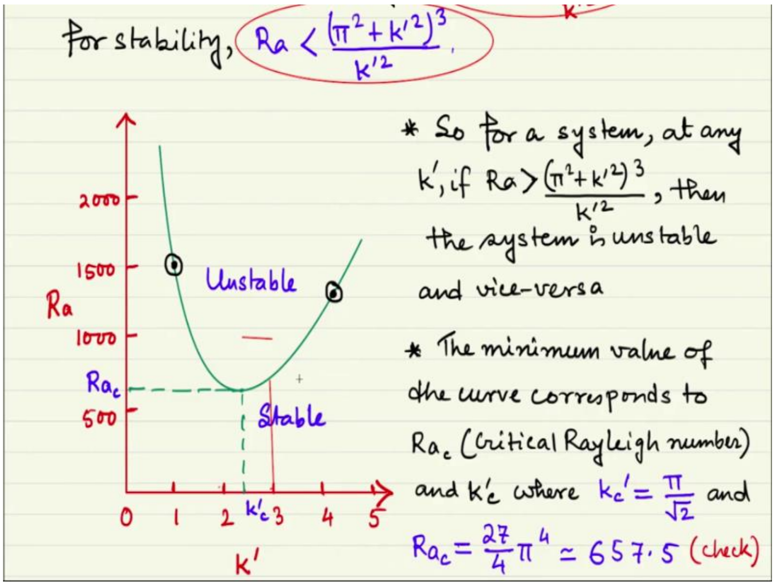

Now, evidently, we know that for instability R_a must be greater than $\frac{(\pi^2+k'^2)^3}{k'^2}$. So, this is the Rayleigh number for the marginal instability part. So, for instability, as I said when Rayleigh number goes larger and larger, the system is can be subject to more and more instability. So, Rayleigh number, when Rayleigh number is greater than this marginal value $\frac{(\pi^2+k'^2)^3}{k'^2}$, then the system is in unstable and for stability Rayleigh number should be less than this value.

Now, one can actually plot this marginal Rayleigh number which you can see in the green solid lines and you can see this is the line which is giving by this R_a. So, here, in this axis R_a is plotted, along this axis k' is plotted and this curve is nothing, but this curve this one.

So, for a system now you can easily see that there are two things one is that, this solid curve is the curve for marginal instability. If your system has a Rayleigh number let us say, you have chosen a wave number let us say wave number is k' I mean normalized wave number k' is equal to 3. Corresponding to that if now your system has a Rayleigh number which is superior to the marginal, so, for this one let us say if we just make some drawing for corresponding to 3. So, corresponding to 3 the Rayleigh number corresponding to the marginal instability is here. Now if you have a Rayleigh number which is greater than this then here it should get somewhere here and it will have an instability, this is the unstable region and if your Rayleigh number is less than this value then you will have stability.

So, here, two arbitrary points, one is in the left of the minimum value and one is to the right of the minimum value, we can actually check this the same type of behaviors are there that means, for a given normalized horizontal wave vector if your Rayleigh number is greater than the marginal stability value of Rayleigh number, then the system will be unstable against convection, if not, the system will be stable against convection.

So, you can see that this curve will also have something. So, called a minimum. So, this minimum if you just calculate, this is called the critical value of the Rayleigh number. So, critical Rayleigh number or R_{ac} and for such type of solution where R_a is equal to this $\frac{(\pi^2+k'^2)^3}{k'^2}$, R_a marginal Rayleigh numbers.

Then the corresponding critical Rayleigh number will be given actually corresponding to critical horizontal wave number which will be given by $\frac{\pi}{\sqrt{2}}$ that means, that k' is equal to $\frac{\pi}{\sqrt{2}}$, you can have the marginal Rayleigh number to be minimum and if the corresponding critical Rayleigh number that is the minimum value of the marginal Rayleigh number is equal to $\frac{27}{4}\pi^4$. So, which is almost equal to 657.5.

Now, there can be a confusion between marginal Rayleigh number and critical Rayleigh number. So, marginal Rayleigh number are the Rayleigh numbers which lie on this curve and the minimum value of all the marginal Rayleigh numbers for a given system is called the critical Rayleigh number for the system.

Now if the systems Rayleigh number lies here that means, that let us say 300, for example, that means, the systems Rayleigh number is inferior even to the critical Rayleigh number then irrespective of the horizontal wave number the system is always stable.

If the systems Rayleigh number is greater than the critical Rayleigh number let us say over here. When I am saying now Rayleigh number that means, the normal Rayleigh number, I am not talking about marginal Rayleigh number. I have just calculated the Rayleigh number of the system and I am just saying I am just saying that my systems Rayleigh number is something and there you have to take into account this fact that σ is not equal to 0 in general.

Now, let us say you calculate your Rayleigh number which is let us say 1200 over here then what happens? Then you want to check that whether your system will be stable against RBC or relevant convection, or not then the thing is that if you start from higher length scale that means, lower k value, if you start increasing length decreasing the length scale or increasing the horizontal k' value.

So, here once again the length scale is in the horizontal direction because this is the horizontal wave vector that is something not to be confused with that has nothing to do with the vertical dimension. So, this is all about the horizontal size. If the horizontal length scale is decreased gradually that means, the horizontal wave number is increased gradually.

For a system whose Rayleigh number is greater than the critical Rayleigh number, first it becomes stable for very small value of k and then it just crosses the wave vector corresponding to the marginal Rayleigh number, and then it becomes unstable for a range of wave numbers. Again, when it leaves this region from another horizontal wave number, of course, we are talking in terms of normalized wave numbers k', then again, the system becomes stable.

So, stable-unstable-stable, on the other way if you want to increase the length scale that means, decrease the wave vector the same thing, first you have a stable region then you have an unstable region then you have a stable region.

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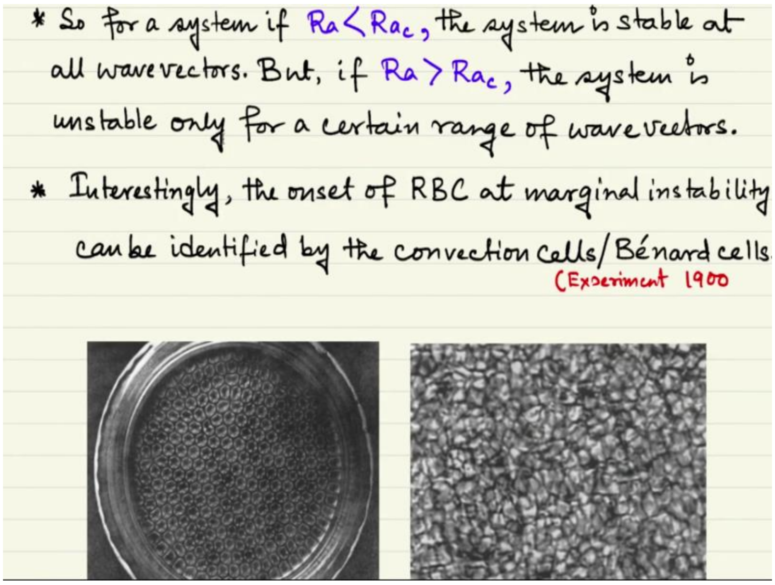

The system is unstable only for a certain range of wave vectors, this is very crucial to understand. Interestingly, the onset of RBC at marginal instability can be identified in practical cases by the appearance of convection cells, so called Benard cells. Benard for the first time in his experiment at 1900 showed the formation of this type of structures.

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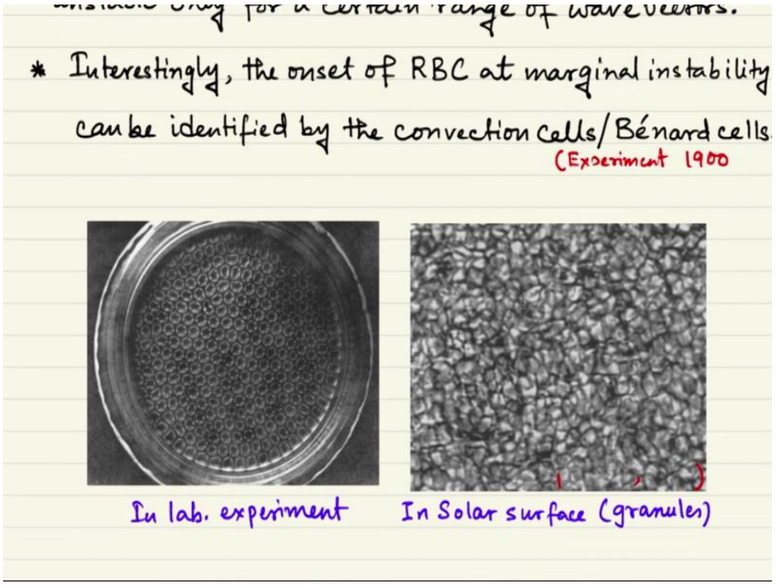

Now, this type of structures can be seen in lab experiments actually even in a vessel you can see that. You can have very beautiful pictures on internet, but here I am just showing you the appearance of this almost hexagonal structured Benard cells in the lab experiment of a Rayleigh Benard convection and this type of convection cells which are called the granules can also be seen in the solar surface near the photosphere. That exactly confirms the existence of convective instability and the relevant convection actually near the solar surface. So, this makes the whole phenomena astrophysically very much important and interesting.

Thank you very much.