Introduction to Astrophysical Fluids
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Lecture - 30
Rayleigh Benard convection I

Hello, and welcome to another lecture of Introduction to Astrophysical Fluids. In previous discussion, we showed that how using a very simplistic approach, we can derive or we can obtain the criterion for convective instability of a compressible fluid. Now, in this lecture, we will derive the conditions or obtain the criterion for the instability against convection, for an incompressible fluid and this is known as the famous problem of Rayleigh Benard convection.

Now, Rayleigh Benard convection is important for both the domestic life, where we can see the water is boiling and I mean when the water is heated by some burner type of thing or a stove, you can see the water is boiling and convective type of motion takes place and also, this type of convection and actually proper signature of this type of convection can be seen in the solar surface, which we call the Granules.

Now, how this type of things can be obtained; I mean a bit more analytically, how this type of the condition for stability rigor like depending on the various parameters of the system, how we can quantify such criterion that we will learn in this lecture.

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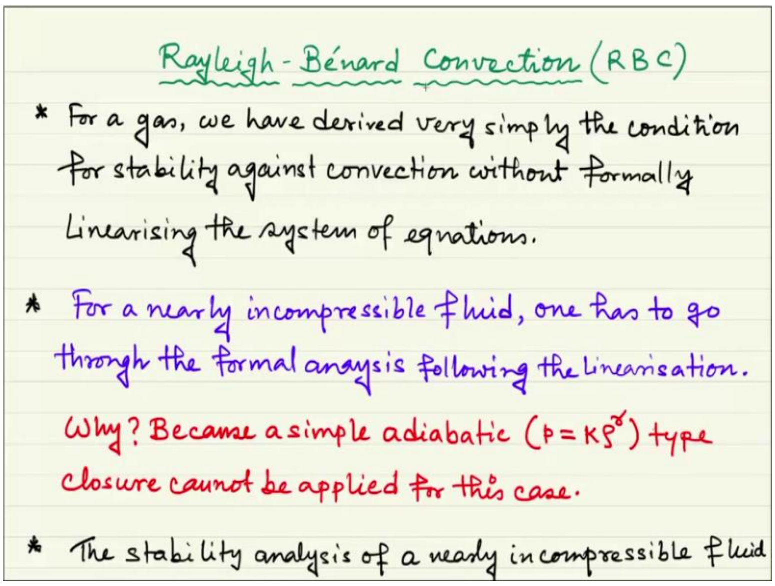

So, Rayleigh Benard convection or which we often called as RBC is important both for our household life and also, for astrophysical contexts. So, a gas, which is in general compressible, we already have derived very simply the condition for stability against convection without formally linearizing the system of the equations in the previous discussion.

We said that in general, we can do same type of analysis by proper calculations but that is a bit cumbersome and actually, in order to derive the Schwarzschild criterion of stability, we basically do not need to go through all this mathematical rigor. But this is a very analysis could lead us to the same conclusion.

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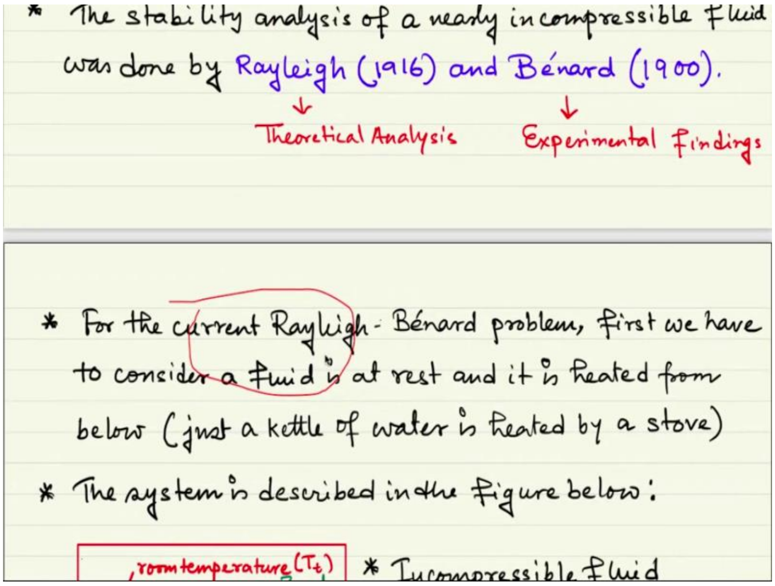

So, when we saw that if a system is in hydrostatic equilibrium and the system has a very important temperature gradient then, the system has a much more tendency to get convicted, I mean convictive I mean get unstable against convective motion.

So, now the thing is that was very good for a fairly compressible fluid. So, we can just part up the density and take the mass of the blob of the fluid to another position very rapidly, thereby undergoing an adiabatic change. But now the question is what happens if the fluid is a nearly incompressible fluid ok? Can we do something like that?

Well, the answer is not evident and then actually, one has to go through the formal analysis following the linearization process. Then, we have to go through linearization. Why? Because the simple thing is that if you really remember that when the fluid blog was, I mean a small mass of the fluid was displaced from one place to the other, then we said that the blob acts like an adiabatic fluid.

Now, adiabatic means p is equal to $k\rho^{\gamma}$ and γ cannot be any other thing than the $\frac{C_p}{C_v}$ which has a specific value. Now, if our fluid is incompressible, then the mass of the blob of the fluid which is displaced should also be incompressible right. So, this type of adiabatic closure can never be applicable to that blob and we all know that for an incompressible fluid, γ actually should be equal to infinity right.

So, this type of treatment, which we did in the previous discussion cannot be applicable no longer, any longer here. Now, the stability analysis of a nearly incompressible fluid was formally done for the first time by Rayleigh in the year 1916 and he did the theoretical analysis.

But 16 year before that, Benard actually found the experimental evidences of what Rayleigh recovered or Rayleigh found by analysis. So, Benard was the person who experimentally developed the field and Rayleigh was the person who theoretically developed the field.

So, combining both, we now talk in terms of Rayleigh Benard convection. Now, for the current Rayleigh Benard problem, first we have to consider a fluid is at rest and the fluid should be heated from below. Just think of a kettle which is heated continuously by a burner or a stove from below.

Now, if you just continuously heating the system, actually it is expected that you do not change the regulator of the burner. Then, after some time, actually you can assume that the heating rate is constant or this is a steady heating rate. So, that is our point of departure. We will come to that in a moment.

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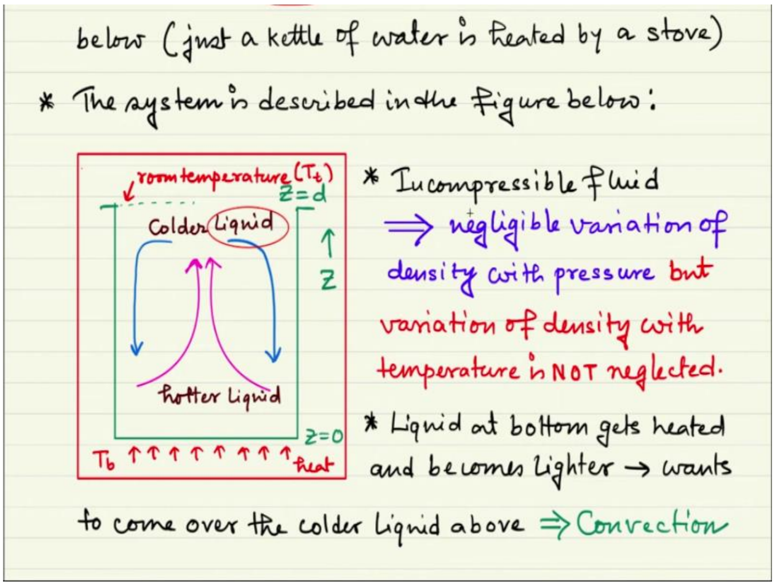

Now, we can actually show the whole system in a schematic way in the figure below. So, here you can see that we have a fluid at rest and actually, we consider an incompressible or nearly incompressible fluid. So, we sometimes call this as a liquid. So, we have a liquid at rest in a container and this liquid is heated from below.

So, there is a temperature gradient you all know that and the containers height is d. So, if we assume that the fluid is filling up the whole container or the liquid is filling up the whole container, then the liquid is also filled up till the height Z is equal to d.

What is Z? Z is nothing but the vertical upward direction. So, this point where this is heated is Z equal to 0 and this is Z is equal to d. So, the burner temperature which is now steady is called the T_b and the upper surface of the liquid which is now kept at room temperature, we just call this as T_t.

So, of course, you can easily understand in case of heating T_b is much larger than T_t. Now, incompressible fluid simply means that it has a negligible variation of density with pressure. So, that is something to be understood clearly. Incompressible fluid does not blindly say that its density is an absolute constant.

It is saying that its density is constant only against a change in pressure. But if we heat the system, if we change the temperature of the system, then the variation of density is non-negligible or this is nonzero and that is exactly what we will be considering here, that is the density is a function of temperature only.

Now, as you can see from this setup that liquid at bottom gets heated and become lighter right and when it becomes lighter, then it wants to come over the colder liquid above. Because the colder liquid is heavier and having greater density.

So, this colder liquid wants to come down. So, there is an up and down motion and this is called the very popular convection motion. You all know that convection motion is the macroscopic way of transport of heat right in a medium.

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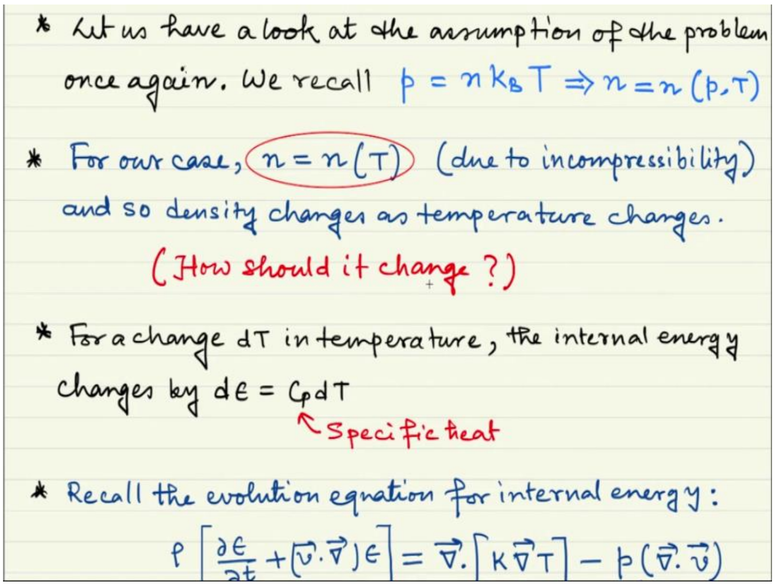

Now, what I just said that for an incompressible fluid, the density can only change when the temperature is changed that can quantitatively be understood from a very basic kinetic theory knowledge, which we already discussed in some of the previous lectures, where we recall that p or the pressure is equal to the number density times the Boltzmann constant times the temperature and this density n is equal to a function of p and T.

So, n is a function of p and T. If you write now n is equal to $\frac{p}{k_B T}$. For incompressible case, n is only a function of T, and n totally uncoupled from the variations of p. Whenever there is a variation in p, n does not care. That is the meaning of incompressibility that n is a function of T only.

So, density changes as temperature changes. How should it change? We all know if you heat the system for most of the usual materials, the density should reduce. Because of the intermolecular distances increases, so the density should reduce when the temperature increases and vice versa.

So, for a change dT in the temperature, the internal energy also changes by $d\epsilon$ But that is $d\epsilon$ is equal to C_p times dT. C_p is nothing but the specific heat of the material.

Now, if we recall the evolution equation for internal energy, you see always we make reference to those basic equations fluid equations which we derived and that is the evolution equation for internal energy and this is nothing but $\rho\left[\frac{\partial \epsilon}{\partial t}+\left(\vec{v}.\vec{\nabla}\right)\epsilon\right]=\vec{\nabla}.[k\vec{\nabla}T]$. So, this is the conductive term minus $p(\vec{\nabla}.\vec{v})$. Now, for our compressible case, this one is neglected because of the continuity equation.

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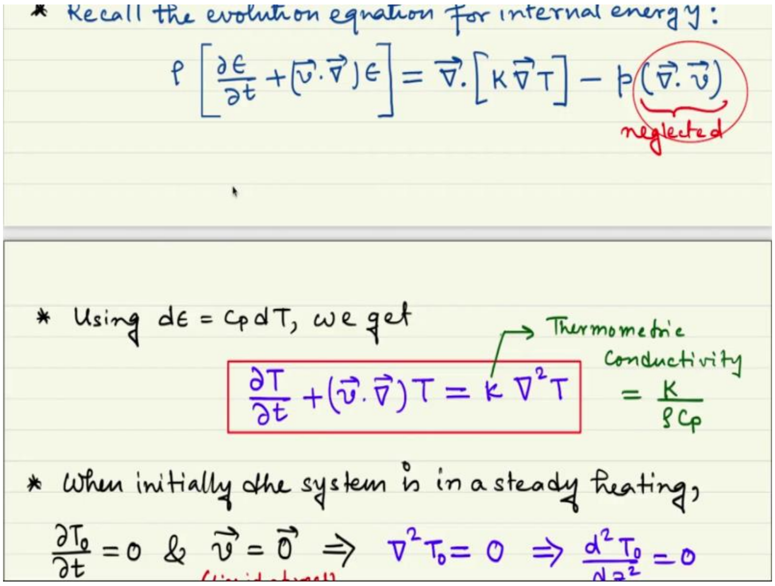

So, continuity equation actually says another thing that not only is ρ is indifferent of the variations of density, variation of pressure, ρ is also, if temperature is constant, ρ is also constant in time and space that is also very important.

If you go another level higher, you actually can see that for an incompressible fluid the material derivative or the Lagrangian derivative of the density also vanishes. So, that means, the density of a fluid particle is a constant in time, of course, when the system is not heated.

Now, using $d\epsilon$ is equal to $C_p\, dT$, we actually can replace this expression $\rho\left[\frac{\partial \epsilon}{\partial t}+(\vec{v}.\vec{\nabla})\epsilon\right]=\vec{\nabla}.[k\vec{\nabla}T]$ over here and you can easily find another equation in terms of T or the temperature in absolute scale of course which is $\frac{\partial T}{\partial t}+(\vec{v}.\vec{\nabla})T=k\nabla^2 T$. This k is nothing but the thermal conductivity by ρ times C_p.

This is called thermometric conductivity. Some people also call it thermal diffusivity. Why thermal diffusivity? Because, if you just think of the system is static then v is 0, then T is simply obeying an equation of diffusion type $\frac{\partial T}{\partial t}=K\nabla^2 T$. So, K is another type of diffusivity.

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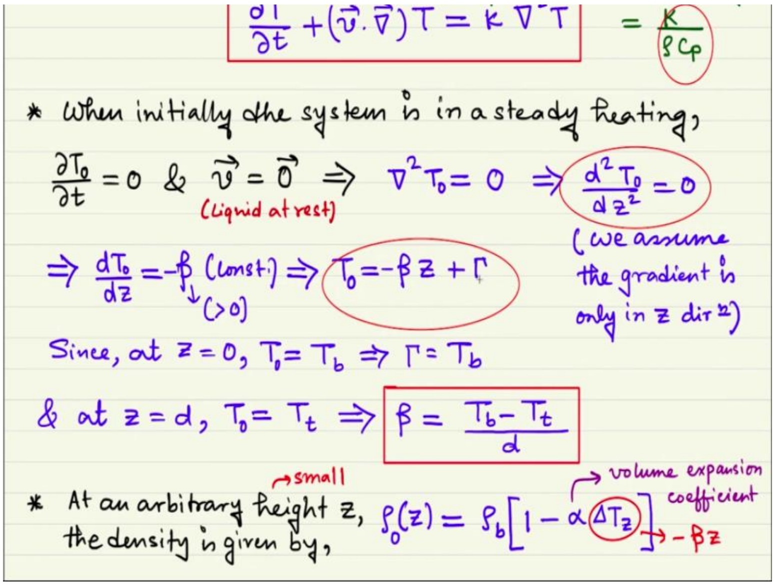

Now, we are ready to define an initial state around which we will make a perturbation and when initially the system is in steady heating. So, as I said that most of the cases, we will start from a steady condition. So, steady heating means the $\frac{\partial T_0}{\partial t}$, T_0 is that initial temperature, $\frac{\partial T_0}{\partial t}$ is equal to 0 and v is equal to 0.

So, $\nabla^2 T_0$ is equal to 0 that means that when the system is in steady heating, there is a special gradient of temperature that means, the temperature is a function of space and this is called a stratification of the liquid in terms of the temperature.

That means, you can actually see static layers of fluid liquids of different temperatures and if this $\nabla^2 T_0$ is 0, then you can actually see that as we have just assumed that our systems heating is only in the vertical direction. So, the temperature gradient can be effectively taking place in the Z direction.

So, you can actually see that $\frac{d^2T_0}{dz^2}$ is equal to 0. So, that is the initial condition and it simply says that T_0 or rather $\frac{\partial T_0}{\partial z}$ is equal to some constant and that constant should in general be negative because we know that when Z increases, T_0 must decrease because the heating source is at Z is equal to 0.

So, it should be some negative quantity, which is minus β, where we assume β is actually an algebraically positive quantity. So, T_0 will then be equal to minus βz plus Γ. This Γ is the constant of integration. Now, at Z is equal to 0, we know T_0 is nothing but T_b that is the burner temperature or the bottom temperature and so, Γ is nothing but equal to T_b because at Z_0, T_0 is T_b.

So, Γ is equal to T_b. Now, at Z equal to d, T_0 is equal to T_t right; T_t temperature at the top that is why then you evaluate β which will be simply equal to T_b minus T_t by d. Please check all these steps. These are very much easy and can be doable.

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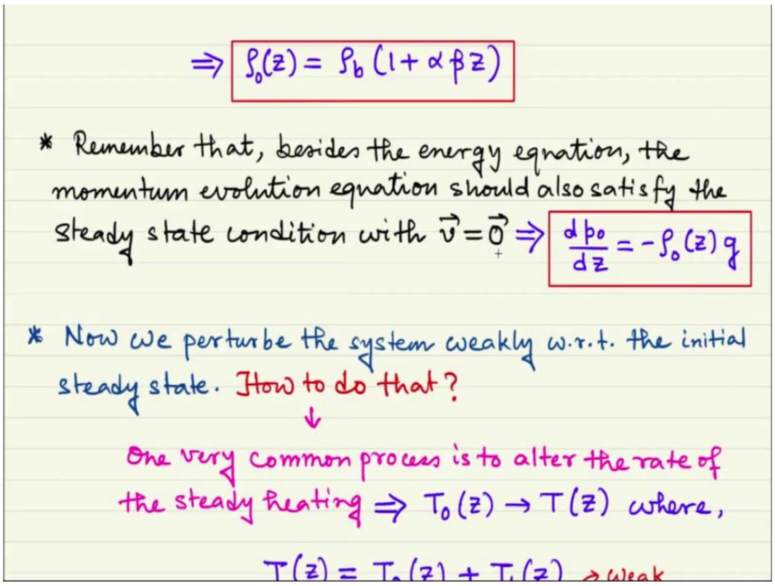

In general, we are thinking that even the height of the container d is actually not very very much big or large. So, we can simply say that the z, which is a displacement I mean some arbitrary displacement from z is equal to 0 has a moderately small value.

I mean although, it can be an arbitrary height, but if it is within d and d itself is not very very large. So, this is a small height. That is why I add the point small later. So, it is an arbitrary height z from the heating ground but reasonably in our study, we are actually considering all the moderately small lengths or small heights.

So, for that arbitrary height, the initial matter density or the liquid density will be ρ_0 which will be equal to $\rho_b[1 - \alpha\Delta T_z]$, where ΔT_z is the temperature difference between the heated bottom layer and the height z and α is nothing but the volume expansion coefficient.

So, if it is for the volume, then this could have been $V_0[1 + \alpha\Delta T]$ but here for density, it should be $[1 - \alpha\Delta T_z]$. Now, what is ΔT_z? β is the temperature gradient and β times z and that is what β times z with a minus sign gives us the temperature difference between the bottom layer and an arbitrary layer at height Z.

Now, here, we have just used this linear gradient and just multiplied this with z. That is the reason why I insisted on just adding this thing that they are very small values. All these things are actually mathematically not rigorous, but they are done very frequently to get the analysis done, first of all in a simplistic manner and then, we do add gradually the sophistications.

So, finally, ρ_0 at a height Z, this is the initial density at a high height Z is equal to $\rho_b[1 + \alpha\beta z]$. Now, remember that besides the energy equation, momentum evolution equation should also satisfy the steady state condition with v is equal to 0 and then, like any other hydrostatic system under hydrostatic equilibrium in gravity, we can simply say $\frac{dp_0}{dz}$ is equal to $-\rho_0 Zg$.

Because if you remember the total setup, the setup is actually vertically standing let us say on earth on some places or in a system, where the gravity is actually acting vertically. So, that I can draw over here. The gravity is actually vertically acting downward. So, that is the direction of the gravity.

So, g is nothing but equal to $g\,(-\hat{e_z})$ and if that is true, then you can simply write $\frac{dp_0}{dz}$ is equal to minus $-\rho_0 Zg$. Because g does not have any other component in x and y. So, $\frac{dp_0}{dx}$ is equal to 0 and $\frac{dp_0}{dy}$ is also equal to 0. So, in the x and y directions, p_0 is a constant throughout.

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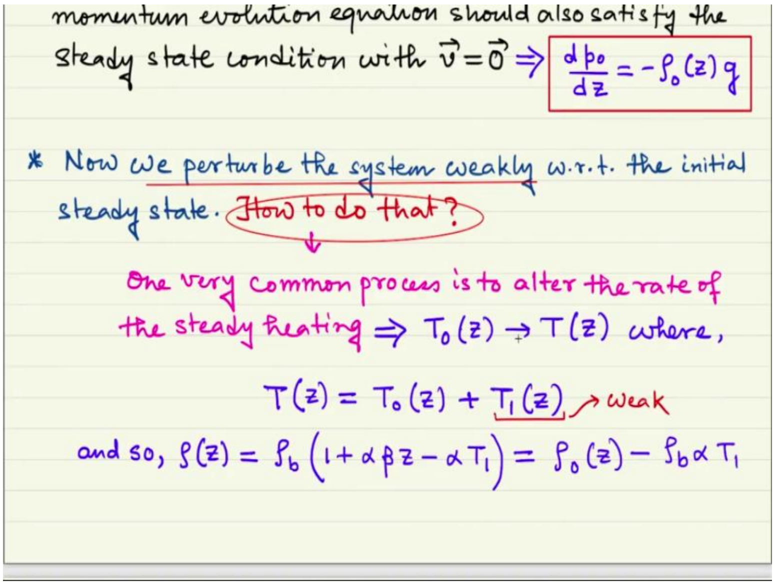

Now, this is all about the initial configuration which is a steady flow with a steady heating and moreover, we are talking about not a flow but at steady state condition at rest. So, this is a steady state hydrostatic condition. Now, we want to perturb the whole system, we want to perturb it very weakly with respect to the initial steady state.

The question is how to do that. Can we do that? One very common process is to alter the rate of the steady heating. So, you now, play with the regulator of the burner and let us say I am just increasing the regulator of the burner so that the heating rate is increased.

Then, $T_0(z)$ will now be coming some $T(z)$. So, $T_0(z)$ was the temperature at T is equal to 0, I mean time is equal to 0 and now, at any arbitrary time, the temperature at a layer, at a distance, at a height z from the bottom is given by simply T as a function of z.

And this $T(z)$ can be written as $T_0 + T_1$. So, this T_1 is nothing but the first order or weak perturbation over T_0 and what about ρ_z? ρ_z is nothing but $\rho_b[1 + \alpha\beta z]$ that was the initial value $-\alpha T_1$ because of this perturbed temperature and finally, you can write ρ_z is nothing but equal to $\rho_0(z) - \rho_b \alpha T_1$.

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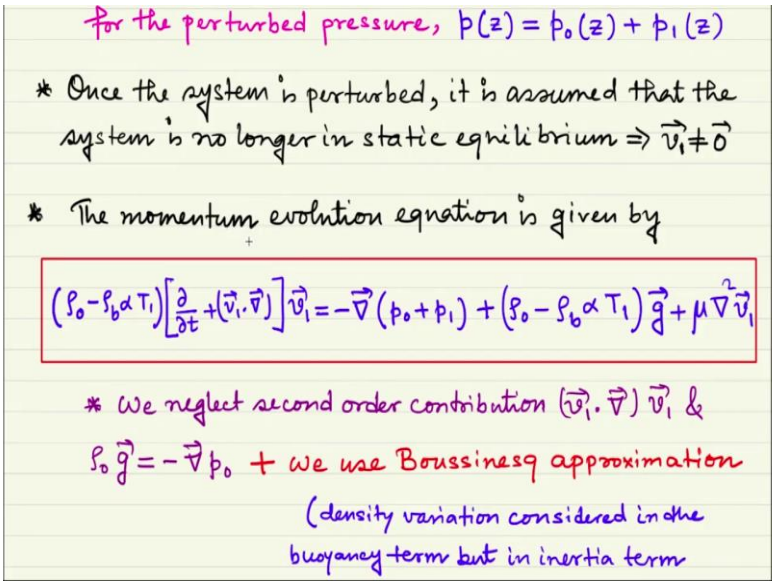

And finally, for the perturbed pressure $p(z)$ is equal to $p_0(z) - p_1(z)$. Again, this is very weak. This is weak, we can easily understand because T_1 itself is weak. So, this is also weak with respect to p_0. Now, our system is perturbed. So, it is assumed that the system is no longer in static equilibrium.

So, if the velocity is now perturbed from 0 to some v_1 that v_1 should not be equal to 0; v_1 should be and actually must be nonzero. Then, with incorporating all these quantities, in terms of its initial value plus its perturbed value finally, the momentum evolution equation can be written as $(\rho_0 - \rho_b \alpha T_1)\left[\frac{\partial}{\partial t} + (\overrightarrow{v_1} . \vec{\nabla})\right]\overrightarrow{v_1} = -\vec{\nabla}(p_0 + p_1) + (\rho_0 - \rho_b \alpha T_1)\vec{g} + \mu \nabla^2 \overrightarrow{v_1}$. Here, somehow, we have taken into account the viscous effect.

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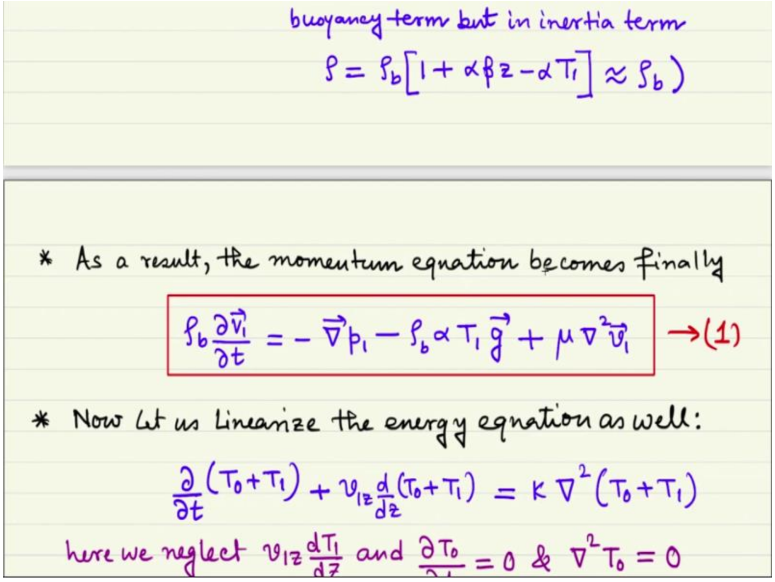

Because, most of the cases for the liquids the viscous effect is non negligible and g, we just say that we are in such a system that g does not change much and so, there we neglect. We neglect the perturbation in the g field. Now, we neglect in this equation, we neglect routine I mean this is a routine process.

The second order contribution $[(\vec{v_1}.\vec{\nabla})]\vec{v_1}$ and we also say that for the zeroth order contribution g is equal to minus gradient of $\vec{\nabla}p$. So, that is the initial condition plus there is another thing which is super important here. This is called the Boussinesq approximation and this is the approximation with this analysis becomes much simpler otherwise, it would have been much lengthier and more complex.

What is that Boussinesq approximation? This simply says that all I mean since we are in nearly incompressible domain, the density variation only considered for the buoyancy term Because if we do not consider the density variation this one, when we just say that this total term is almost equal to ρ_0, then this $\rho_0 g$ will exactly be nullifying minus $\vec{\nabla}p_0$ and so, there will be no buoyancy force.

Actually, the convection cannot be explained then as you can easily understand. Because as I said the convection is nothing, but the interplay of a vertical displacement of different type of fluid. So, one is heated and gets lighter, but still in the lower level and one is colder, less heated, heavier and still in the upper level. So, this type of convective instability is there.

So, the lighter fluid moves up only because of buoyancy right and if the buoyancy force is absent, then the whole story is destroyed. If the buoyancy force is absent, then we cannot do anything to explain the convective motion, that is why we cannot neglect the density variation here.

On the other hand, the density variation here does not play a much important role because even with this one, the total thing because the total thing is almost equal to ρ_0. So, in this part we just neglect this increment.

So, the method is very simple. Here the zeroth order term and the first order term both are intact, that is why I can neglect the first order contribution in front of the zeroth order contribution. Here the first order contribution the zeroth order contribution is nullified by the zeroth order contribution over here, that is why I have to let the first order contribution survive.

Simply this and sometimes you can actually say ρ_0 and ρ_0 is also reasonably very near to ρ_b. And what is the advantage of using ρ_b? Because ρ_b is a constant, ρ_b is the burner temperature at in the initial state.

As a result, the momentum equation becomes finally. So, here you can see when I say about the perturbation of something like this, I said that the burner is now I have increased the regulator of the burner, this type of thing. That means, also I can actually change this ρ_b, but where I am talking about this ρ_b, this means the ρ_b in the initial state.

Of course, in even without touching the burner, we can actually make other heating source or other some source of energy which can heat up the system as well. Now, as a result, finally, the momentum evolution equation becomes

$$\rho_b \frac{\partial \overrightarrow{v_1}}{\partial t} = -\vec{\nabla} p_1 - \rho_b \alpha T_1 + \mu \nabla^2 \overrightarrow{v_1} \qquad (1)$$

So, finally, we only have this term which is in the first order, this term $\vec{\nabla} p_1$ the first order pressure gradient, the first order body force term, which is nothing but the buoyancy term and the viscous term, which is also of the first order. Now, let us linearize in the same way the energy equation.

So, the variation is only considered in the direction of z. So, we can just simply write $\vec{v}.\vec{\nabla}$ as $v_{1z}dz$ and here, within the bracket you have $T_1 + T_0$ is equal to $k\nabla^2$, then the total temperature is again $T_1 + T_0$. Now, we neglect $v_{1z}\frac{dT_1}{dz}$ because of the second order problem and $\frac{dT_0}{dt}$ is equal to 0 due to the initial steady state condition.

Also remember due to this reason in the steady state, we also had $\nabla^2 T_0$ is equal to 0. So, these three things. So, this is neglected due to being second order term.

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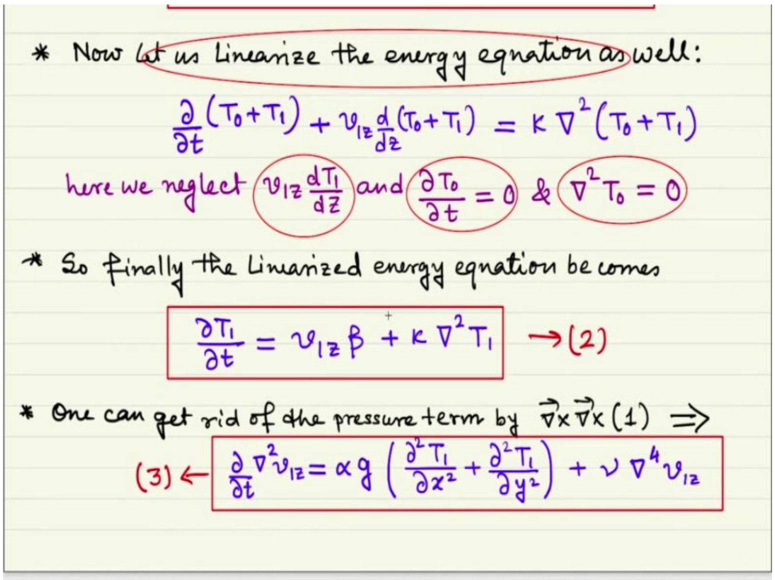

So, finally, the linearized equations energy equation becomes

$$\frac{\partial T_1}{\partial t} = v_{1z}\beta + k\nabla^2 T_1 \quad (2)$$

So, finally, we can see that we have this equation 1 and 2. So, we have v_1, we have T_1 and actually we have p_1 as well and here, we also have v_{1z}. So, we have actually roughly three unknowns' v_1, p_1 and T_1. One simple trick can simplify the life just by taking double curl of this equation 1 and if you take that and if you note that double curl of any vector is equal to gradient of divergence of that vector minus Laplacian of that vector and here, since the velocity vector is divergence less double curl of the velocity vector will simply be equal to minus of Laplacian of the velocity vector.

If you correctly follow that identity and do the algebra carefully, they get combined to give

$$\frac{\partial \nabla^2 v_{1z}}{\partial t} = \alpha g \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2}\right) + \vartheta \nabla^4 v_{1z} \quad (3)$$

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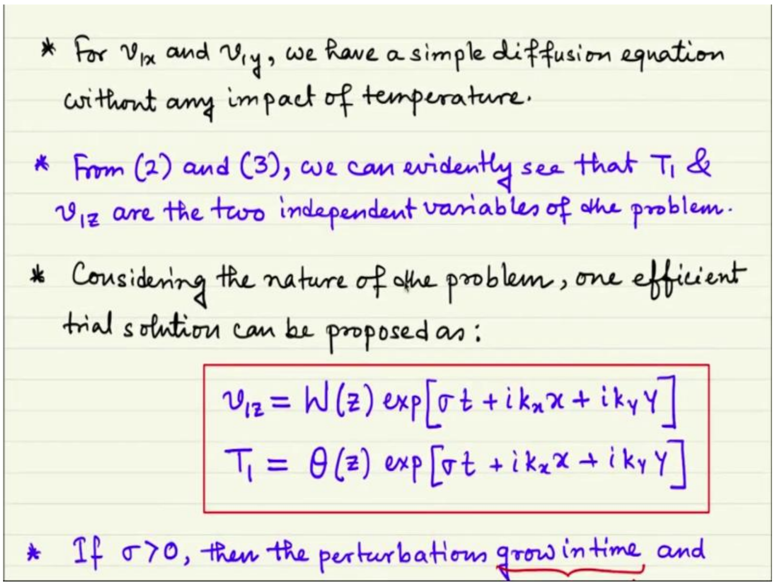

For v_{1x} and v_{1y}, we simply have a diffusion equation without any impact of temperature that you can also check. So, from equation 2 and 3 finally, we can evidently see that T_1 and v_{1z} are the two independent variables of the problem. So, finally, if you have equation 2 and 3, you can easily see that you have 2 equations and 2 unknowns; one is T_1 and one is v_{1z} that is even in more interesting. I mean handleable.

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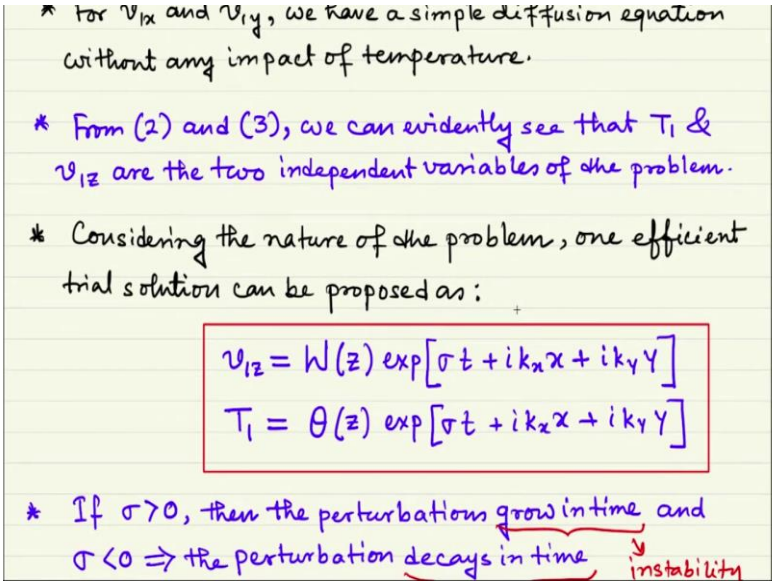

So, considering the nature of the problem, so one can actually blindly propose for various type of solutions. We will here try to do some trial type of solutions of different of several types. So, I mean for this part of the discussion, we have finally seen that I mean for the total Rayleigh Benard convection problem now gets reduced to 2 equations with 2 unknown variables. The more detailed discussion about the solution of the system will come in the next discussion.

Thank you.