Introduction to Astrophysical Fluids
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Lecture – 03
Collisionless Boltzmann equation

Hello and welcome to the course of Introduction to Astrophysical Fluids. So, in the previous lectures, we discussed basically the basic concepts of the dynamical theory and how we can promote from one dynamical theory to the other; starting from the very basic quantum mechanics to the classical physics of small number of particles and then, we also said that if the number of the particles are high enough, then we will use the statistical description thereby, leading to the kinetic theory.

Also, there is another level which is basically our principal objective to derive the dynamical theory for that level, this is the level of fluids or continuum, that we will do of course in the next part of this course. Gradually, we will develop the equations of a fluid. I ended the last lecture by deriving a very basic theorem of statistical mechanics which is known as Liouville's theorem or Liouville's equation and I also derived a corollary from that.

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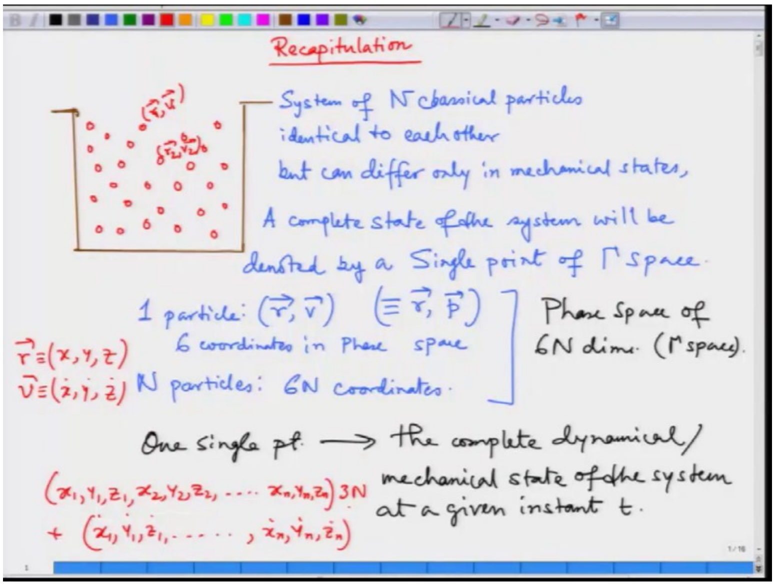

So, let us recapitulate some of the points of the last lecture in a global way. If you remember that, you have a system of N classical particles which are identical to each other, but can differ only in mechanical states then the complete state of the system will simply be denoted by a single point of gamma space.

Now, as I said that if you are just considering 3-dimensional real space and the motion of the particles in this, then for every particle, you will have 6 coordinates over which 3 coordinates are from position and 3 coordinates are from velocity. So, for one particle, you will have $(\boldsymbol{r}, \boldsymbol{v})$, this is also equivalent to $(\boldsymbol{r}, \boldsymbol{p})$ if you are using Hamiltonian mechanics and the Hamiltonian formulation, as I used in Liouville's theorem in the last lecture but $\boldsymbol{p}$ is the momentum so for non-relativistic case they are actually similar.

So, you can simply write $(\boldsymbol{r}, \boldsymbol{v})$ for one particle, in phase space, this is their designating coordinates. So, 3 coordinates for $\boldsymbol{r}$ because 3D space and corresponding 3 velocity coordinates for the particle. So, one particle gives 6 coordinates in phase space. Then, for N particles, you will have 6N coordinates.

So, if you assume your phase space to be of 6N dimensions that is basically your $\Gamma -$space as I said, then if you just freeze your time and take a snapshot; then basically only one single point will designate the complete state or rather complete dynamical state or mechanical state of the system at a given instant, let's say t.

Then, of course by varying t, you can also have other pictures. So, there can be two type of alternative description; one is that you say my phase space will be of 6N dimension and then, I vary the time as a parameter or you just say that my complete phase space will be 6N plus one dimension (time), that is possible.

But here, we are just talking about 6N dimension phase space and we are using time just as a parameter. Then, if you now assume all the particles are there which are classical particles and at some time t you take a snapshot, then this particle (Refer Slide Time: 6.22) will have some $(\boldsymbol{r}, \boldsymbol{v})$. Let us say $(\boldsymbol{r_i}, \boldsymbol{v_i})$; this particle will have, let us say $(\boldsymbol{r_1}, \boldsymbol{v_1})$; this can be $(\boldsymbol{r_2}, \boldsymbol{v_2})$ for example. Write $\boldsymbol{r} \equiv (x, y, z)$ and $\boldsymbol{v} \equiv (\dot{x}, \dot{y}, \dot{z})$, if you now take one combination of all 6N type of coordinates i.e., $(x_1, y_1, z_1, x_2, y_2, z_2, \dots x_n, y_n, z_n, \dot{x_1}, \dot{y_1}, \dot{z_1}, \dot{x_2}, \dot{y_2}, \dot{z_2}, \dots\dots \dot{x_n}, \dot{y_n}, \dot{z_n})$ and this is one single point in $\Gamma -$ space.

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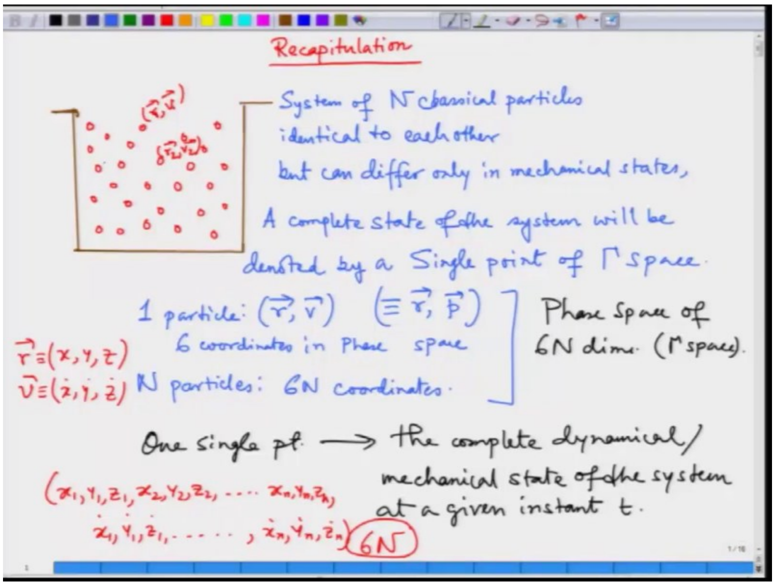

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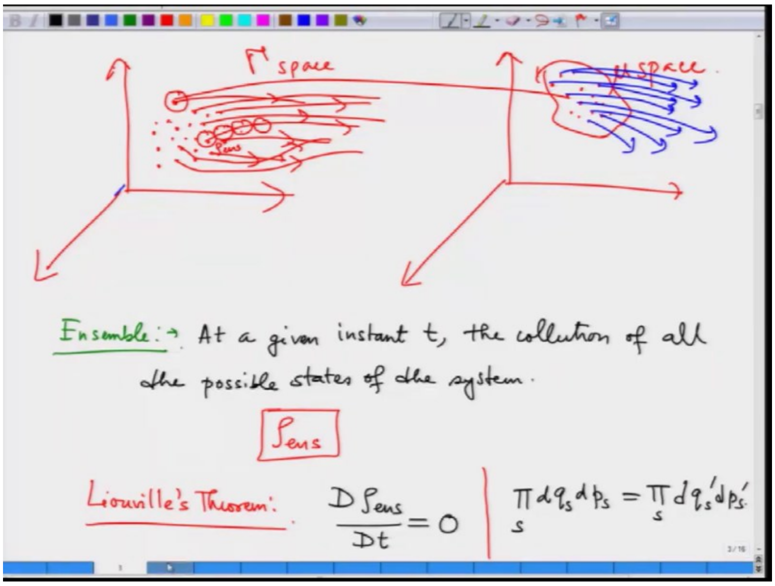

As I said that, we can also define another phase space which is μ −space, then for every single particle, the complete mechanical state at one time is designated by one single point and we can only switch from Γ −space to μ −space, if we have particles which are identical to each other and they can just differ from each other by mechanical conditions that means, in terms of their position and velocity.

Now, you can see that one single point in Γ −space is mapped to N points in μ −space, every point having 6 coordinates and then, I said that when this particle (in Γ −space) is led to evolve in time, maybe it evolves along this trajectory; then in μ −space, they will have corresponding N number of trajectories.

Let us define a concept which is the concept of ensemble. So, the concept of ensemble; so ensemble is nothing but at a given instant t is the collection of all the possible states of the system. So, this is something which is ideally an imaginary creation. What practically we do that if the system is ergodic; the system is observed for long time then we can see that if you take one snapshot every second; then, what happens that at every second, you just capture the information of the state; for every second, you have one single point of gamma space right. Then, after a sufficiently long time, you have a collection of such states right or collection of such gamma space points.

Then, you say if the system is ergodic, then instead of observing the system throughout this long-time interval, now let me do an equivalent approach; that means, I say that the system when observed throughout a long-time interval gives all these states possible states then, if I freeze my time then I can actually imagine that it is also possible that I can make at one given instant all these possible conditions or the possible states as the replica of the system. That means, by some trick you have made or you have prepared, let us say you have taken 1000 pictures in 1000 seconds now, you say I take all the information of this 1000 pictures, every picture contains information of the 6N coordinates; I mean there will be 1000 combinations of 6N variables. Then, you say that I will now imagine that this is equivalent that I am taking the system or I am observing the system at one single time instance and I imagine that my system is prepared in site; I mean 1000 multiple replicas and this collection of replicas is known as ensembles.

So, one real point in $\Gamma -$ space is nothing but a member of the ensemble or we say a point of ensemble. We also defined that if the members of the ensemble point are sufficiently densely situated in the phase space, then we can define an effective density for the states and we say that this is the density of the ensemble and if you remember, we call that ρ_{ens}.

Then finally, we talked about Liouville's theorem, which simply said that if we follow the evolution of this ρ_{ens} along any of the trajectory of the members of the ensemble, then we will have

$$\frac{D\rho_{ens}}{Dt} = 0$$

I am now talking about a large number of points in gamma space. So, these points are nothing but the replicas of the system and every point if is left to evolve, they will evolve separately according to some law, I do not know which law, but that depends on the system ok and then, you see that basically if you follow any of those trajectories, any one, but you have to follow one trajectory and then you just continue calculating ρ_{ens} along that trajectory, with time it will be constant. So, this was the statement of Liouville's theorem.

Then, we also said that there is a corollary of Liouville's theorem which simply said that the elementary volume containing a given mass of density points or a given mass of ensemble points will be behaving like an incompressible medium. That means

$$\prod_s dq_s dp_s = \prod_s dq_s' dp_s'$$

Today, we will try to understand what is the story for μ −space. Can we also define such type of density of states in the mu space? Well, the answer is yes, we can define.

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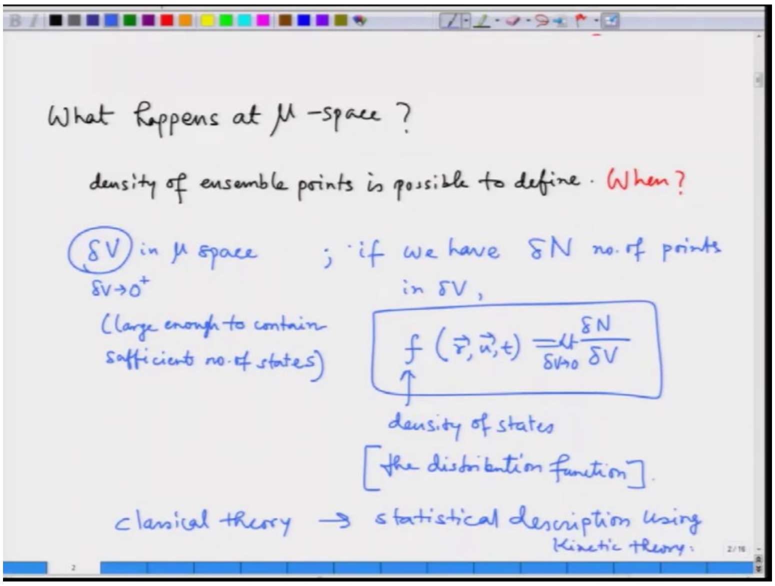

So, if so, my question is what happens at μ −space? In μ −space, we can actually define in a similar fashion, a density of ensemble points ensemble points. But whenever I talk about density, now you have to remember that every single point in μ −space is a single particle state at a given instance. So, density of ensemble point is possible to define. Now, the question is when? I take a very small volume element δV in μ −space i.e., $\delta V \rightarrow 0$.

Now, I write $\delta V \rightarrow 0^{+}$ that is because δV is a very small volume element with respect to the total volume accessible in μ −space but δV is sufficiently large is large enough to contain sufficient number of states.

So, this δV should be large enough to contain sufficient number of states so that we can define some density, that is the exactly the same thing; the same philosophy which we used for Γ − space.

Then, if we have δN number of points in δV, we can define the corresponding density of states f, we use the symbol f and which will now be a function of $(\boldsymbol{r}, \boldsymbol{u}, t)$. So, $\boldsymbol{r}$ is the position; $\boldsymbol{u}$ is the velocity; t is the time then

$$f(\boldsymbol{r}, \boldsymbol{u}, t) = \lim_{\delta V \rightarrow 0} \frac{\delta N}{\delta V}$$

If this $\lim_{\delta V \rightarrow 0} \frac{\delta N}{\delta V}$ limit exists, then only we can say that the system has or the corresponding μ −space has a legitimate density of states $f(\boldsymbol{r}, \boldsymbol{u}, t)$ and this is known as the distribution function. So, only when this distribution function is well defined, then one can go from normal classical theory level to statistical description using kinetic theory.

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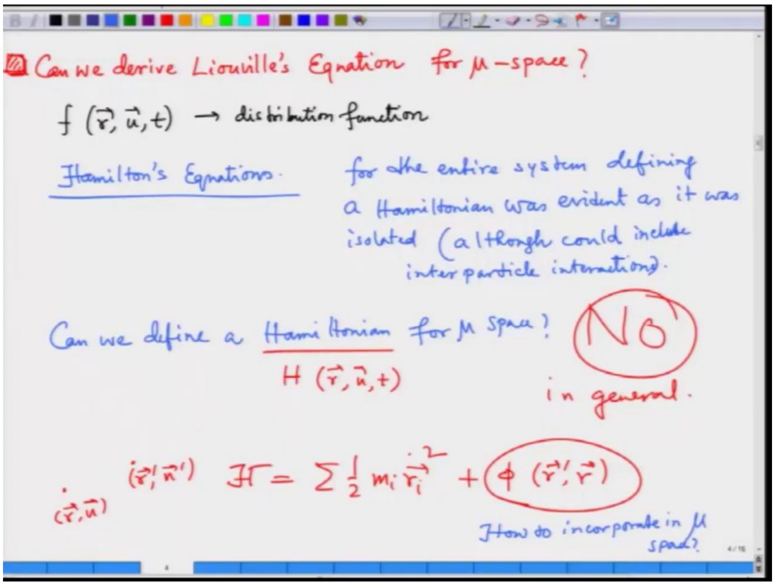

Can we derive Liouville's equation in μ −space?

We saw that under some given condition, we can define a density of states $f(\boldsymbol{r}, \boldsymbol{u}, t)$ which we call formally a distribution function. Now, the question is that can we also derive Liouville's equation for μ −space as we derived it for Γ −space? Well, that is not evident okay.

Now, of course, if you really followed attentively the derivation of Liouville's equation, you have to understand that the key factor in Liouville's theorem was the Hamilton's equations and that Hamiltonian was defined because the system was without dissipation when you took the total system under consideration; that means, the system was an isolated system okay.

If you just consider the total system as a whole, basically there can be interaction between two particles but there was no source of energy from outside or there is no sink of energy outside. So, the system as a whole does not gain any energy and similarly it is not losing any energy that is why this is a conservative system and we could define a Hamiltonian.

So, for the entire system defining a Hamiltonian was evident as it was isolated (although could include inter particle interactions). Now, the question is, can we define a Hamiltonian for μ −space? The first level answer is no, in general. Does that mean that we do not have any hope? Of course, here when we say Hamiltonian that would be a Hamiltonian (H) which is a function of $\boldsymbol{r}$, $\boldsymbol{u}$ and maximum a function of t i.e., $H(\boldsymbol{r}, \boldsymbol{u}, t)$.

Can we have some good news using some assumption or something? Well, now try to think, consider a particle at $(\boldsymbol{r}, \boldsymbol{u})$ interacting with another particle at $(\boldsymbol{r}', \boldsymbol{u}')$, these two particles can interact each other. Then, we know that they are just moving under their mutual interaction, we can define a Hamiltonian which simply is nothing but

$$H = \sum \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \phi(\boldsymbol{r}, \boldsymbol{r}')$$

That is possible. But how to incorporate this type of thing in μ −space? This is not possible in μ −space because, we can just permit $(\boldsymbol{r}, \boldsymbol{u})$ and t and no $(\boldsymbol{r}', \boldsymbol{u}')$, this is the single particle state.

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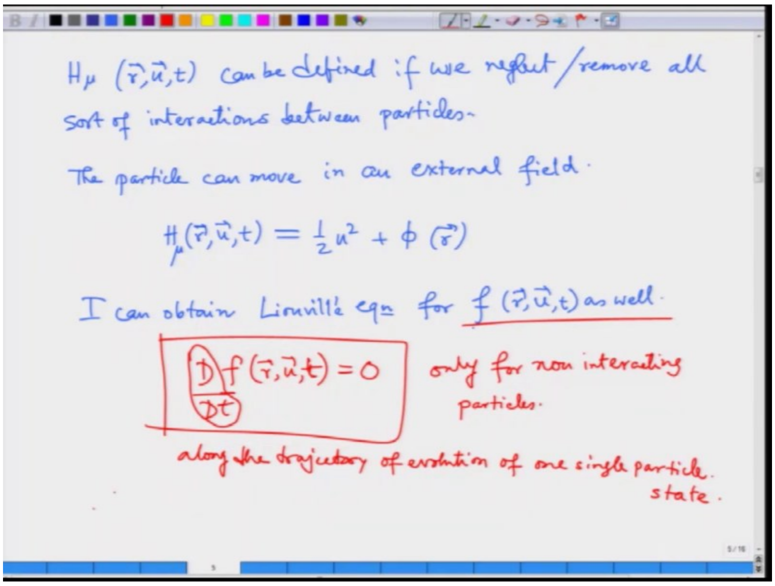

So, the answer is very clear that in general although it is not possible, $H_\mu(\boldsymbol{r}, \boldsymbol{u}, t)$ can be defined if we neglect or we have to remove all sort of interactions between particles. That does not say that on each particle, there is no force or nothing but there can be some external force or something, so the particles can be move in an external field.

For example, if you have a container of gas molecules but they are not colliding with each other, they are not attracting or they are not repelling each other, but for example, the total container is put on earth so, all of them are experiencing earths force of gravity.

So, this is the instance where you can see that there is no inter particle interaction but all the particles are moving under the action of gravity and here, you can see that in the potential form, only the instantaneous position of one single particle will be there to write the Hamiltonian of the system and external force fields usually are conservative in nature so, we can write the Hamiltonian for every single particle like this

$$H_\mu(\boldsymbol{r}, \boldsymbol{u}, t) = \frac{1}{2} u^2 + \phi(\boldsymbol{r})$$

Now, if we define that, then we are actually done; then you can say that I can derive or I can obtain Liouville's theorem rather equation for $f(\boldsymbol{r}, \boldsymbol{u}, t)$ as well and that is very important message. So, you can

$$\frac{Df(\boldsymbol{r},\boldsymbol{u},t)}{Dt}=0$$

only for non-interacting systems or interacting non interacting system of particles . So, if your $\frac{Df(\boldsymbol{r},\boldsymbol{u},t)}{Dt}=0$, then what is the meaning of corresponding $\frac{D}{Dt}$? That is something very interesting. In Γ-space, if you remember that $\frac{D}{Dt}$ was along the trajectory of evolution of one single point of Γ-space that means, the evolution from one global state having 6N coordinates to any other subsequent global states. So, that actually was the evolution of the whole system; the trajectory of evolution of the whole system. But here, $\frac{D}{Dt}$ has a separate meaning. Here, $\frac{D}{Dt}$ simply means that we are following the trajectory of evolution of one single particle because every single particle or one single particle state.

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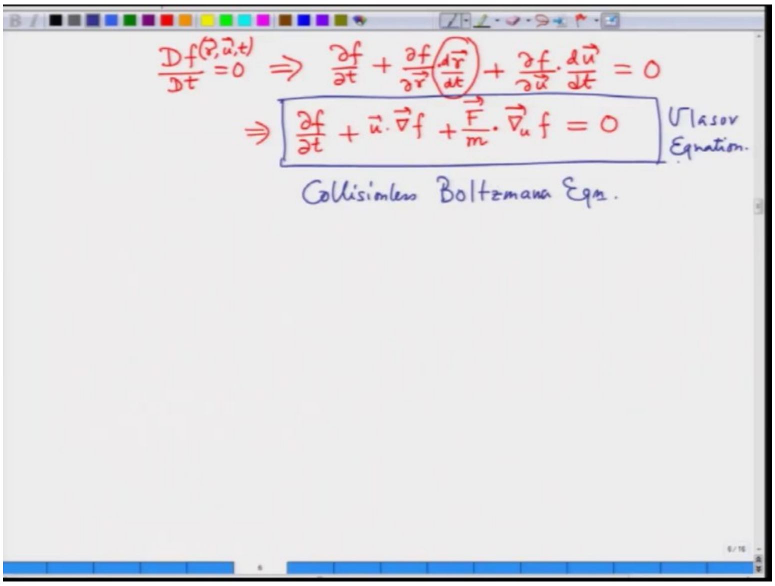

If you have $\frac{Df(\boldsymbol{r},\boldsymbol{u},t)}{Dt}=0$, you can see that

$$\frac{Df(\boldsymbol{r},\boldsymbol{u},t)}{Dt}=0\Longrightarrow\frac{\partial f(\boldsymbol{r},\boldsymbol{u},t)}{\partial t}+\frac{\partial f}{\partial\boldsymbol{r}}.\frac{d\boldsymbol{r}}{dt}+\frac{\partial f}{\partial\boldsymbol{u}}.\frac{\partial\boldsymbol{u}}{\partial t}=0$$

So, $\frac{\partial f}{\partial r}$ can be written as $\boldsymbol{\nabla} f$ and $\frac{dr}{dt}$ is the particle velocity $\boldsymbol{u}$. Now, we said that the particle can be subject to external fields, force fields. So, let us say that external force field which is assumed to be conservative of course is $\boldsymbol{F}$. Then, $\frac{\partial \boldsymbol{u}}{\partial t}$ is nothing but the acceleration. So, I can write this as $\frac{\boldsymbol{F}}{m}$; where, m is the mass of the particle; one single particle and actually for every particle because they are identical to each other in general. So, $\frac{\boldsymbol{F}}{m}.\frac{\partial f}{\partial \boldsymbol{u}}$ will be $\frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla_u} f$, $\boldsymbol{\nabla_u}$ is nothing but a gradient over velocity space and . So,

$$\frac{\partial f(\boldsymbol{r}, \boldsymbol{u}, t)}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} f + \frac{\boldsymbol{F}}{m}.\boldsymbol{\nabla_u} f = 0$$

This equation is known as Collisionless Boltzmann equation. This equation also has another name it is called Vlasov equation.

Now, why Collisionless? Because we could use the Liouville's theorem and hence, derive this only for Collisionless gas ok. As you will see that I will discuss next, that if the system is not without collision; that means, with collision, then the right-hand side is no longer zero ok.

Thank you.