Introduction to Astrophysical Fluids
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Lecture – 29
Convective instability and Swarzschild stability criterion

Hello, and welcome to another lecture of Introduction to Astrophysical Fluids. As promised earlier, in this lecture, we will discuss more in detail the subject of instability and linear wave modes in fluid media. So, it is true that the subject has already been introduced in this course, in the context of the sound waves. Now, we know that sound wave is the way of response of a compressible fluid medium to a weak external perturbation or disturbances in order to repair the disturbance.

Now, if the disturbance is not weak enough then we actually showed that there can be non-linear effects and development of discontinuity, shocks, this type of thing. But that was very much case specific. Now, in this part and also in next week's lectures you will see that we will try to address different type of problems related to such waves and instabilities.

But for this week we will start with a very interesting, but very simple case it is called the case of convective instability. But before even starting discussing about convective instability, I would like to make a formal introduction to this waves and instability in case of a fluid.

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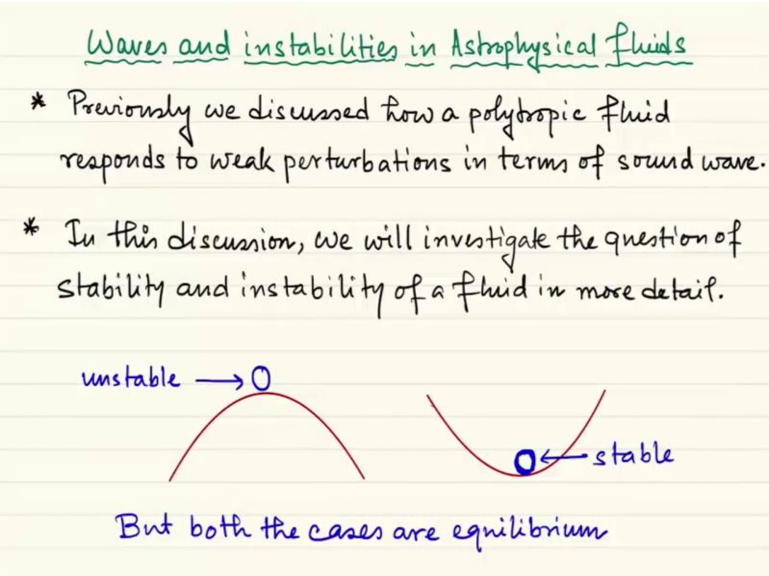

So, if you recall our previous discussion, you can easily see that a polytropic fluid generally responds to a very weak perturbation we call a first order perturbation in terms of sound wave or acoustic mode. In this discussion, we will investigate the question of stability and instability of a fluid in more detail, it is a general perspective.

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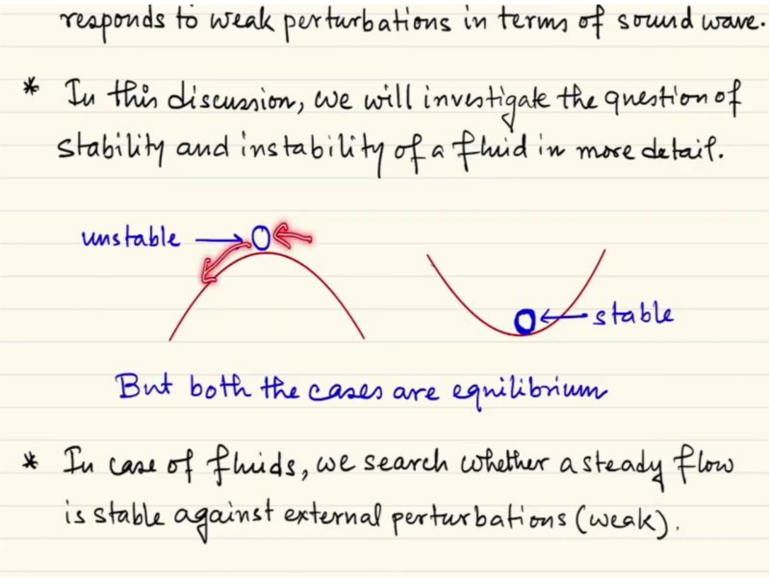

So, for that let us go back to the particle picture in Newtonian mechanics. Let us say a particle which is resting I mean staying at rest on the top of a mountain or in a valley. In both the cases, the total force on the particle is 0 and the particle is at rest. But really do the two cases correspond to the same type of mechanical state well both of them are equilibrium conditions that is true because the total force is vanishing.

Now, the mechanical equilibrium is there. Now, the question is whether same type of equilibrium are they corresponding to? Well, the answer is no. Here this is corresponding to unstable equilibrium, here this is corresponding to stable equilibrium you all know. Because a weak perturbation with respect to this configuration would just lead to a linear oscillation, I mean simple harmonic oscillation, and finally, with time it will try and finally, it will recover its original configuration.

However, for this type of case this if you perturb the particular bit very weakly this particle will go farther or if you perturb like this particle will go farther, from its original configuration or original position. So, this is the concept of stable and unstable equilibrium, whether the particle after a weak perturbation would try to fight against the perturbation or would try to support the perturbation. That distinguishes the stable equilibrium versus the instable equilibrium case.

Now, in case of fluids, we cannot really distinguish this type of position type of thing, but we can distinguish several types of steady state. So, in case of fluids, we do not search mechanical equilibrium position in general, we search whether steady flow is stable against external perturbations or not. Of course, the external perturbations must be weak otherwise we cannot do linear analytical theory out of it.

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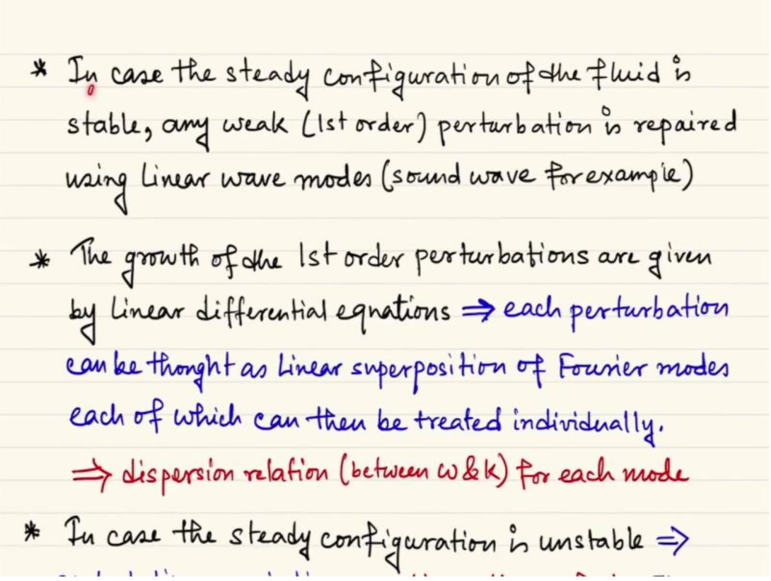

Now, in case of steady configuration, for a fluid we will say it is a steady configuration or unsteady configuration. So, if the configuration is non-steady then there is no question of searching whether we can perturb it and we search for unstable equilibrium or stable equilibrium.

But if the fluid is steady that means, the fluid variables are no longer explicitly depending on time in that case, the fluid is stable only when any weak or first order perturbation is repaired by the fluid using linear wave modes as exactly, we saw in the case of acoustic wave or sound waves.

But the growth rate of the first order perturbations, in this case, are only given by linear differential equations and what is the advantage for that. So, if we have a weak perturbation with respect to a stable steady configuration then we first call that configuration as steady, I mean stable and steady.

Then, the second thing is that we can express the growth equation or the evolution of the perturbation as linear differential equations, if you remember, that is the method of linearization which we already discussed or introduced in the context of sound waves. That means, for example, the non-linear adjective term vanishes.

What happens under this case? That each perturbation can be thought as linear superposition of Fourier modes each of which can then be treated individually. So, that is the property of a linear differential equation, so, in general you can say that every perturbation can be expressed as a superposition of various plane waves type of thing, sines and cosines.

Then once this is done finally, if you remember finally, we find out the dispersion relation relating the frequency and the wave vector for each mode.

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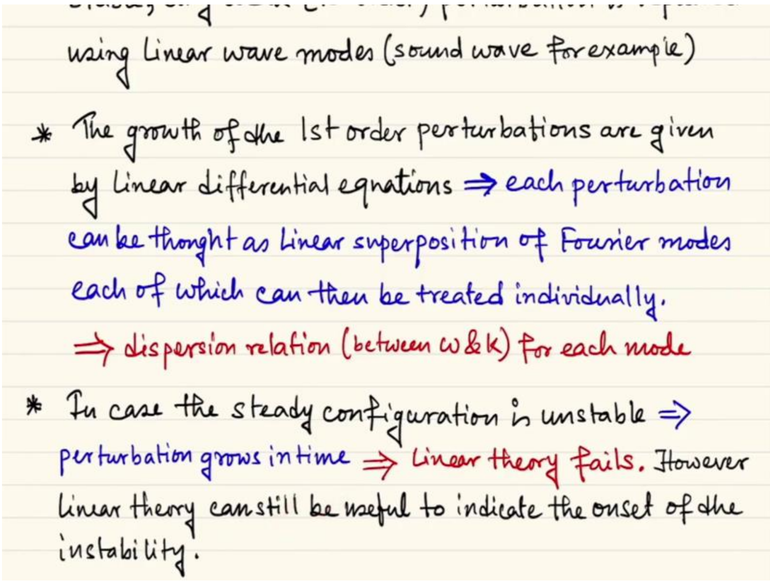

Then also we all discuss whether the fluid, I mean the wave is dispersive or not, this type of thing. Now, it is true that in the case of sound wave, we just directly showed that there is a wave mode. Now, in case the steady configuration is unstable, then what happens?

Then, this perturbation is no longer repaired by the fluid medium, then it grows in time, the system supports to aggravate the perturbation, degree of perturbation then linear theory fails. However, linear theory can still be useful to indicate the onset of the instability and also the behavior of the instability at its initial phase and that is the theory of linear instability. That we will do also in next lectures.

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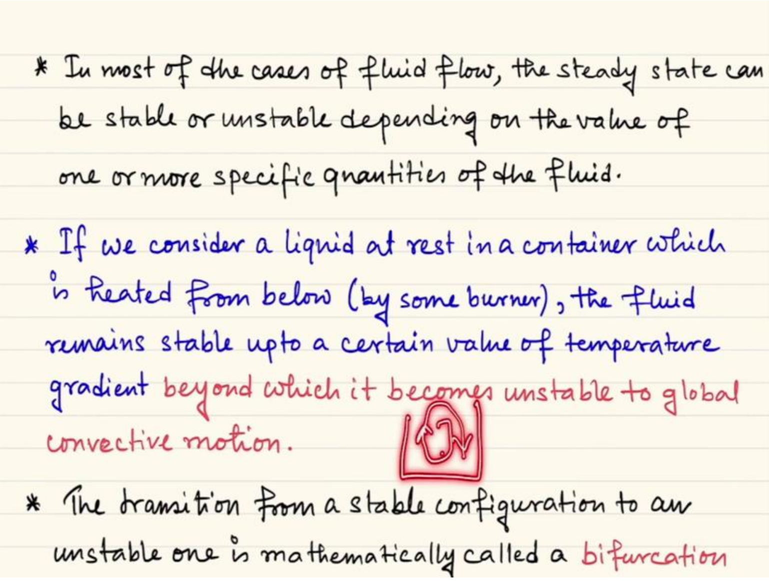

Now, it is true that most of the practical cases of fluid flow it is observed that a steady state is neither absolutely stable nor absolutely unstable. A steady state can be stable or unstable depending on the value of one or more specific quantities of the fluid.

So, a steady configuration of a fluid, if I just ask you that for example, is a steady configuration of such fluid is stable or unstable, the answer is well, it depends on the different values of different quantities of the fluid, and depending on one or more specific quantities of the fluid that is the relevant quantities for the stability of the fluid, it may change the nature of the steady state can change from stable to unstable from unstable to stable.

For example, we consider a liquid at rest in a container, and the container is heated from below by some energy agent like a burner. The fluid remains stable up to a certain value of temperature gradient. That means, the temperature is I mean the heat is transported from one layer to the other layer just by conduction, so macroscopic motions cannot be visualized.

However, beyond a critical value of the temperature gradient, the fluid becomes unstable to global convective motions and you can actually see that in this vessel, there will be a convective flow type like this. So, what is the convective flow?

Convective flow means vertically up and down flow. So, the downward fluid which is getting heated, it is getting lighter, so it flows up, and the fluid which is on the upper side this is less heated because the heat energy source is at the bottom, for this type of system, and then this is heavier, so, it comes to fill the blank part or the empty part in the downward level.

In this way when this heated fluid goes up, and it actually again loses its temperature, and finally, it becomes again heavier and comes down, and this is a cyclic process and maybe you know that one of the primary source or origin of rain falls in the equatorial region is the convective rainfall by the convective motions. It is called convective rainfalls.

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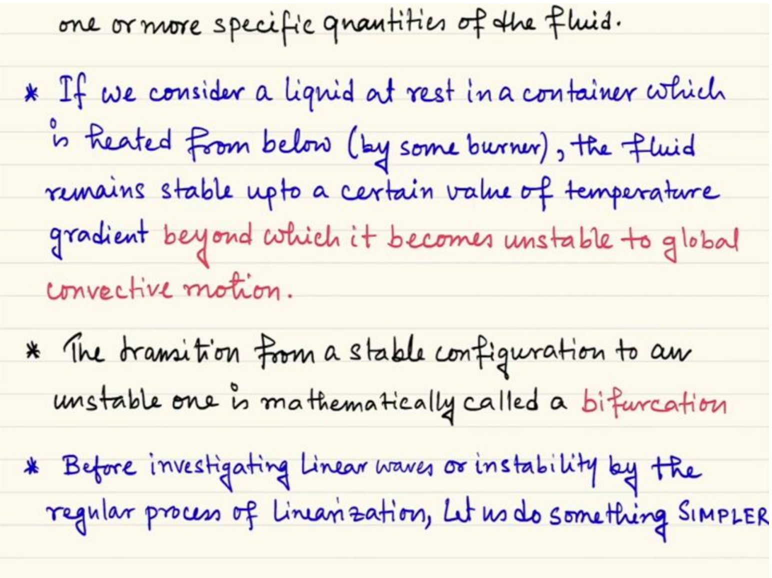

So, now, this convective motion basically designates the transition from the stable steady state to the unstable steady state. So, that means, that if your system is perturbed with respect to a state where the temperature, I mean the heat was conducted just by conduction microscopic method, then any small perturbation could have been died out because of some linear modes or something.

But if it is not then the system is actually globally convective in nature and the any perturbation can be aggravated. Now, the transition from a stable configuration to an unstable one is mathematically called a bifurcation. Although we will not be discussing bifurcation in details, but that is just for your interest.

Once again, when the system is unstable and the perturbation grows I mean how to say the perturbation grows beyond a critical value then it is no longer possible to treat them linearly then the non-linear theory must be taken into account and the analytical scope is also very much limited, and you know that a system where the total I mean inter non-linear motion is established is called a turbulent motion. So, this is a system of utter interest for many researchers including me.

So, whenever we are talking about linear instability that means, this is the initial phase of an instability. Now, it is true that people always think and that is the correct way, of course, that whenever there is a fluid system is unstable or it conceives some linear instability then the fluid with time and when the disturbance grows with time then it has a transition from normal motion to the turbulent motion.

So, from linear instability with time, it is possible to go to turbulent motion that is the phenomenological view which looks quite evident. However, it is very interesting to note that till today there is no very formal and systematic theory to show that a linear instability can really lead to a turbulence just by using a very systematic bifurcation type of rules. So, these are very mathematical details.

Now, coming to our point here, in the scope of this discussion, we will be only talking about the investigation of linear waves or linear instability, and then we will actually investigate them using regular process of linearization. But even before doing that let us do something which is much simpler and more interesting for astrophysics. Here in this course, I am doing something which are much more interesting for astrophysics.

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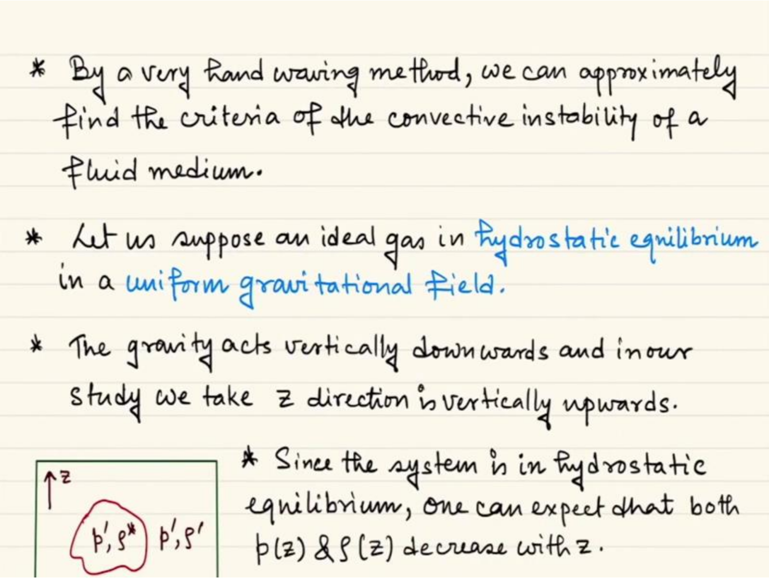

So, what we will do? Basically, we will show by a very hand waving method that we can approximately find the criteria of the convective instability of a fluid medium which was initially at rest, of course. So, now, let us suppose an ideal gas in hydrostatic equilibrium. So, it is totally at rest, so trivial steady in a uniform gravitational field.

So, in this case, we are no longer neglecting the body force because the system is in hydrostatic equilibrium and you know that the hydrostatic equilibrium is established by the interplay between the pressure gradient force and the body force most of the cases. So, of course, viscosity is there, but viscosity is actually non-important for this case.

Now, the gravity acts vertically downwards for our case. We are just now defining the system. So, gravity is acting downward, so here you can see this one.

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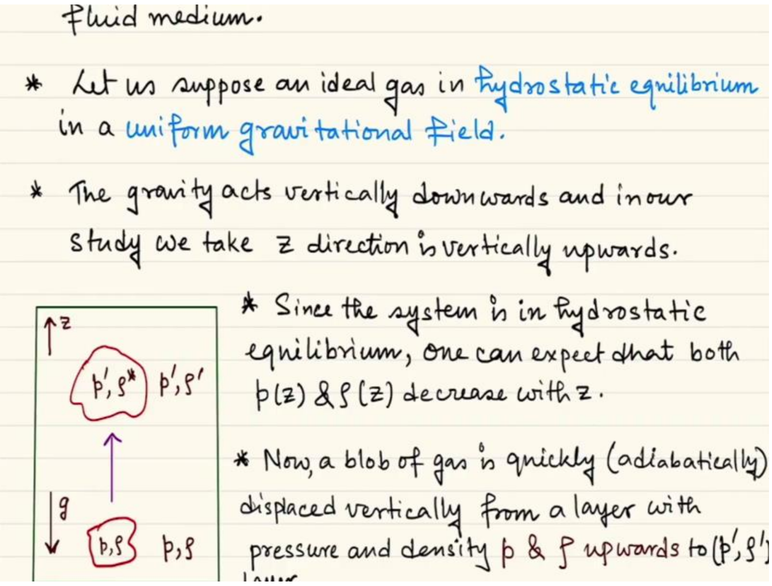

And z is taken upward vertically. So, gravity will be acting along minus z direction. Now, the system is in hydrostatic equilibrium that is our assumption. So, one can simply expect, if you remember our discussion of hydrostatic equilibrium, one can simply expect that both the pressure and density they decrease with z and actually you can see a stratification. So, it is a layer type of stratified fluid, where p and ρ both are decreasing as a function of z and we try to perturb that system.

How? We take up small blob of gas, and after taking we quickly displaced, the quickly is very important, quickly displace it vertically from a layer which was from a layer below to a layer above. So, the layer below had the density and pressure p and from the blob is now displaced to a level where the surrounding pressure and density they are given by p' and ρ'.

Now, remember before perturbing, this was everything is in equilibrium. So, for a given layer and I am actually taking the mass of the blob of the fluid which was an indistinguishable entity from the surroundings before taking it. So, both the blob and the surrounding had pressure and density, p and ρ after displacing this one, vertically upward.

So, now, the blob just has reached to a layer which was not its known layer or familiar layer. So, this layer had a pressure and density p' and ρ', and because this mass is adiabatically displaced, so this mass will also have some adiabatic change, and due to that thermodynamic change, the pressure inside of this blob is foreign entity in this layer, this blob will have a pressure which we can call as p^* and ρ^*. Here I have written p' I will explain this.

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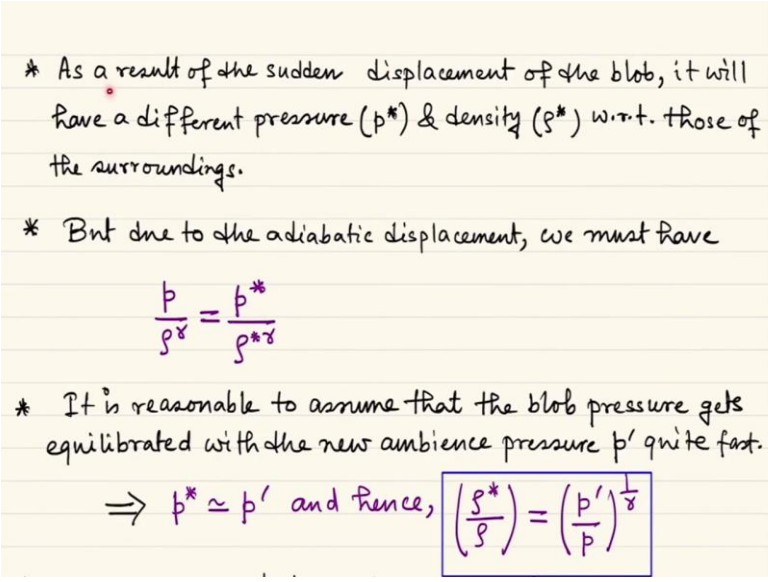

But as a result of the sudden displacement of the blob, it will have a different pressure and density p^* and ρ^* with respect to those of the surroundings. So, the blob is no longer indistinguishable with respect to the surroundings because the surrounding is a new one. Now, since the system was quickly displaced and that is why we are actually talking of adiabatic displacement.

Then, we can actually think that if you just consider the initial configuration of the initial thermodynamic state of the blob and the final thermodynamic state of the blob, we can simply write this equation, that $(\frac{p}{\rho})^\gamma$. So, $(\frac{p}{\rho})^\gamma$ is equal to $(\frac{p^*}{\rho^*})^\gamma$.

Now, coming to a very important point, it is reasonable to assume, for our case that the blob pressure gets equilibrated very quickly after the blob reaches to the new ambience. So, after it reaches to the new ambience finally, the walls are not rigid in nature.

So, mechanical equilibrium will be very quickly established and actually you know maybe that the mechanical equilibrium for fluids is much fast established than the thermal equilibrium. But p^* can all almost be equal to p'. However, ρ^* and ρ', they are significant they are not equal, they are different.

So, now, in this equation, we can simply replace p^* by p' and then we can write this equation as $\frac{\rho^*}{\rho} = (\frac{p'}{p})^{1/\gamma}$. Now, once again here we are talking about adiabatic case.

So, γ is not a general polytropic index, but this is exactly equal to the $\frac{C_p}{C_v}$ or $1+\frac{2}{f}$, where f is the degrees of freedom of the system molecules, I mean the system constituents.

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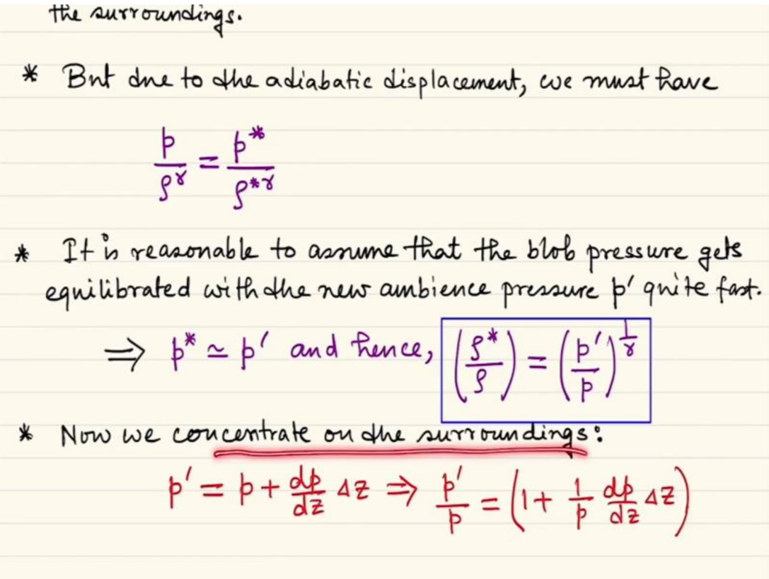

Now, we concentrate on the surrounding, that was the story for the blob. Now, what happens for the surroundings? For the surroundings, we can say that these starting from this level, one can actually go vertically upward and reaches to another pressure p'.

If we suppose that we have displace the blob very, very small amount then p' can be written as Taylor expansion just up to the first order, so, it will be $p + \frac{dp}{dz}\Delta z$. So, Δz is the amount of displacement vertically. So, $\frac{dp}{dz}$ is the normal pressure gradient in the normal fluid. So, for our case, for the surroundings.

So, $\frac{p'}{p}$ will be $\left(1 + \frac{1}{p}\frac{dp}{dz}\Delta z\right)$. Now, $\frac{p'}{p}$, this is equal to this one and this one raised to the power $1/\gamma$ will give you $\frac{\rho^*}{\rho}$, and that exactly what we will do now.

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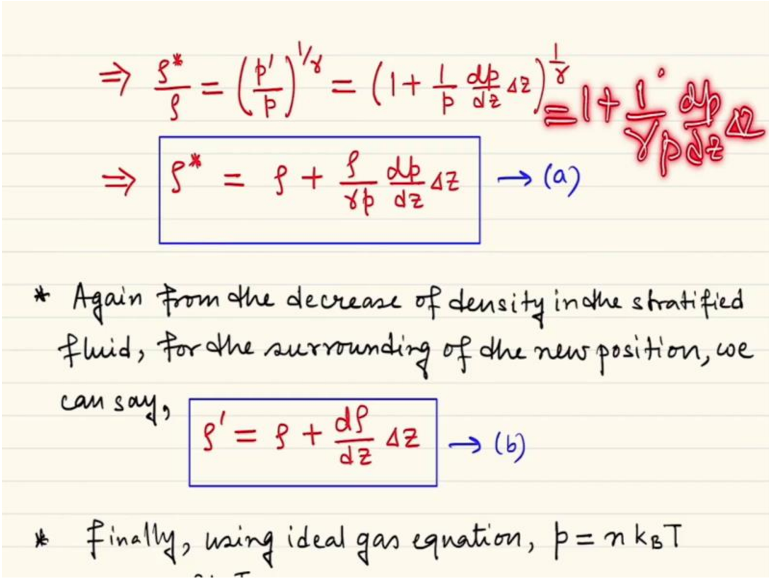

So, $\frac{\rho^*}{\rho}$ will be equal to $(\frac{p'}{p})^{1/\gamma}$. Now, we can say that this is nothing, but $\left(1+\frac{1}{p}\frac{dp}{dz}\Delta z\right)^{1/\gamma}$, and just remembering that this can be assume to be moderate where this $\frac{1}{p}\frac{dp}{dz}\Delta z$ total factor can be assumed to be sufficiently small with respect to 1.

So, this raised to the power $1/\gamma$ will simply help us expanding these things like a binomial series type of thing, and finally, you will have

$$\rho^* = \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \qquad \text{(a)}$$

Again, from the decrease of density in the stratified fluid, so that was the story for the decrease of pressure and then we have just mixed this with the relation of the densities for the blob. Now, coming to the law of decrease of density in the stratified fluid and for the surrounding of the new position, we can say that

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \qquad \text{(b)}$$

Again, the same thing because Δz is small enough. So, this is the gradient of density in the surrounding gas. So, we can write the new density with respect to the initial, I mean the new layer density with respect to the old layer density plus this term.

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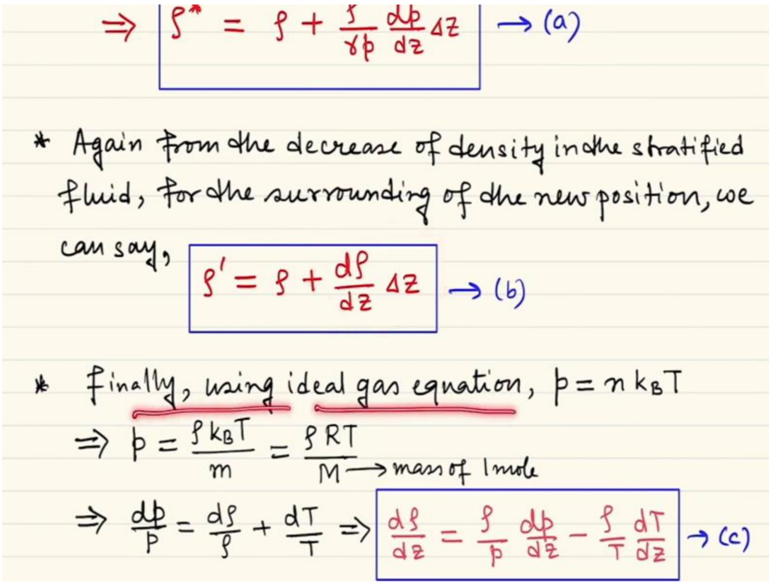

Finally, we assume that the gas is ideal. So, from kinetic theory, we can simply write p is equal to nk_BT, n is the number density, T is the temperature and this k_B is Boltzmann constant. So, p is equal to we can write $\frac{\rho k_B T}{m}$, where m is the mass of one particle then for one mole of gas or fluid, we can write $\frac{\rho RT}{M}$, and this M is constant, this R is constant, so M is the molecular mass actually or molecular weight of the constituent particles. We are taking one mole for our convenience.

So, p will be equal to some constant times ρT. So, then you just take log and differentiate both sides, you will have this $\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$ type of relation. This is a very handy relation. So, you take a log and then you get rid of log and then differentiate, so you get rid of all the constants.

Finally, you can write

$$\frac{d\rho}{dz} = \frac{\rho}{p}\frac{dp}{dz} - \frac{\rho}{T}\frac{dT}{dz} \qquad \text{(c)}$$

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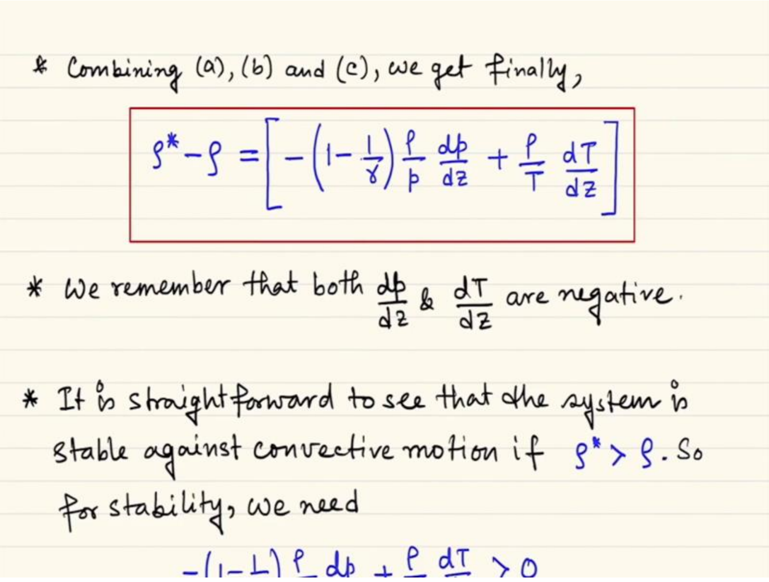

Now, combining, so first we will subtract equation (b) from (a) and then we actually replace this value of $\frac{d\rho}{dz}$ in this subtracted result, and finally, we have

$$\rho^* - \rho = [-\left(1 - \frac{1}{\gamma}\right) \frac{\rho}{p} \frac{dp}{dz} + \frac{\rho}{T} \frac{dT}{dz}]$$

Now, you remember that now try to understand one simple thing that for an ideal gas and in a hydrostatic equilibrium, in general, $\frac{dp}{dz}$ and $\frac{dT}{dz}$ they are negative. Combining all these things finally, what we get is the difference between the density of the blob in its new position minus the density of the surroundings of the new position. It makes sense.

Now, at this point we can think of stability only when the density of the blob is much larger, I mean not much larger, but it is larger than the density of the surroundings. That means, the surrounding will consider this one as heavier with respect to this layer and that is why this blob will try to come back downward.

But if ρ^* is less than ρ that means, the blob density is less than the surroundings density then the blob will find itself lighter in its new position, and then it will be simply dominated by the buoyance force which will help it go further, I mean go vertically upward further.

So, we started by displacing it to vertically upward direction and if ρ^* is greater than ρ', this motion is actually counter balanced by its downwards motion, so, this gives us the condition of stability. Whereas, if ρ^* is less than ρ', then the vertical direction displacement in the upward vertical direction will be supported somehow and it will go upward further and further.

So, that simply says that the system is stable against convective motion, if ρ^* is greater than ρ. Of course, when the particle is displaced a bit vertically upward and it continues going upward and upward then the system will be becoming convective, right.

But if the system is just displaced infinitesimally and it comes back then the system is convectively stable, and that is the condition where ρ^* should be greater than ρ. So, for stability we actually need the whole thing to be greater than 0.

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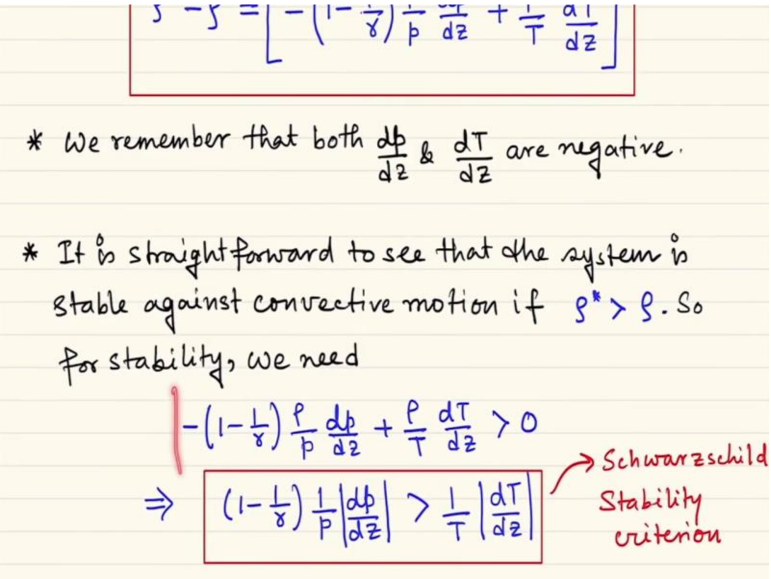

Now, since $\frac{dp}{dz}$ and $\frac{dT}{dz}$, they are negative that means, that this thing should be greater than this thing and the mode of this thing should be greater than the mode of this thing, and that is exactly which we have written over here. That is

$$\left(1-\frac{1}{\gamma}\right)\frac{1}{p}\left|\frac{dp}{dz}\right| > \frac{1}{T}\left|\frac{dT}{dz}\right|,$$

and that is the condition of stability against convective motion for a system which was ideal gas type of thing, but it was at hydrostatic equilibrium initially, and this condition is known as famous Schwarzschild stability criterion.

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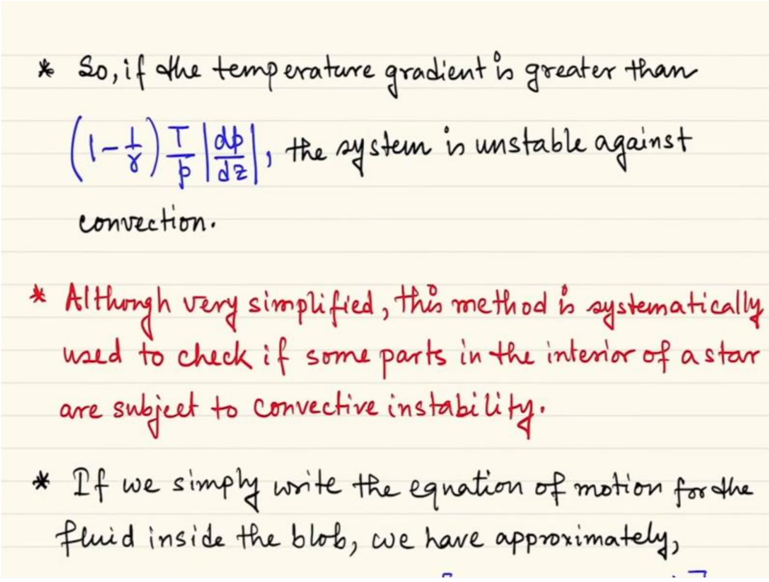

Now, you see that basically this is an interplay between the $\frac{dp}{dz}$ and $\frac{dT}{dz}$. When $\frac{dp}{dz}$ overshoots $\frac{dT}{dz}$, then we have stability when $\frac{dT}{dz}$ overshoots $\frac{dp}{dz}$ then we have an instability. So, that is what I said that if the temperature gradient is greater than this $\left(1-\frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{dp}{dz}\right|$ value, this is the critical value of the temperature gradient for which the system just can retain its stability, but beyond this $\left(1-\frac{1}{\gamma}\right)\frac{1}{p}\left|\frac{dp}{dz}\right|$, the system is unstable against convection.

So, you see an important temperature gradient that is now coming back to our initial example of heating of container of fluid or liquid from below, so when the temperature gradient exceeds this critical value then the system is unstable.

Although, our system I mean analysis is very much simplified, we have not really perturbed systematically the equations of dynamics. But at the end, our method is very much giving some conclusion which is quite intuitive and believe me this type of method actually is used to check systematically if some parts in the interior of a star for example, sun are subject to convective instability or not, this type of simplified treatment.

Of course, simplified in which sense that systems mechanical equilibrium will be very well established, but not the thermal equilibrium, this type of things are bit rough although reasonable. But you can do something much more systematic, but then also the conclusion will be the same.

There are several books which contains this type I mean the systematic approach of this problem. Now, finally, the good news is that the Schwarzschild stability criterion is remains the same even with this systematic and much more rigorous treatment.

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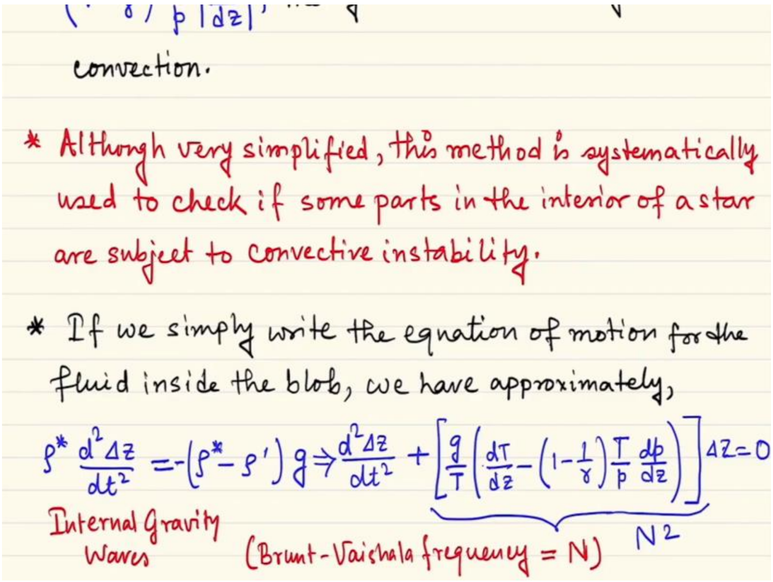

Now, we simply write the equation of motion for the fluid inside the blob. So, we would like to see what happens for the dynamics or for the blob. So, of course, you can see that for the blob we can approximately write ρ^*, so when the blob is displaced already then after being displaced the force density equation for the blob is nothing, but $\rho^* \frac{d^2\Delta z}{dt^2}$.

So, its instantaneous acceleration will be equal to $-(\rho^* - \rho')g$. So, that is the effective gravity field which is acting on the blob, and if you simply just replace this value of this $(\rho^* - \rho')$, you can simply see that the equation is looks like this

$$\frac{d^2\Delta z}{dt^2} + \left[\frac{g}{T}\left(\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dT}{dz}\right)\right]\Delta z = 0$$

So, now this one $\frac{g}{T}\left(\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dT}{dz}\right)$ is some quantity which looks like square of a frequency, and you see that is another point we can make here that if we have this type of equation, then we will just check that whether this one is positive or not. If this one is positive, then this $\frac{g}{T}\left(\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dT}{dz}\right)$ is the square of frequency, and if this one is negative then this is not a square of any real frequency then the instability comes into play that we will check.

But here you will see that this quantity can be positive that you have to think further and just tell me, I mean what I would like to make here that this quantity is positive only when this Schwarzschild stability criterion is satisfied that means, they are equivalent. This is easy to see.

So, if this is now positive then this is nothing, but the square of a frequency and we will see that this $\frac{g}{T}\left(\frac{dT}{dz}-\left(1-\frac{1}{\gamma}\right)\frac{T}{p}\frac{dT}{dz}\right)$ is nothing but the frequency of the wave oscillation. So, you can actually check that this is known as Brunt-Vaishala frequency.

What does that correspond? This corresponds to the coming back of the mass blob to its original position when this is displaced vertically upward initially, and it actually leads to linear wave modes. It actually leads to in linear wave modes that you can see. So, for the particle you can see this is an oscillation type of thing, but the whole fluid medium.

So, here we have just written for the mass blob, but for the whole fluid medium if you do the correct linearization and then you can actually see that you can have linear wave mode with the same type of frequency which is root over this thing $\frac{g}{T}\left(\frac{dT}{dz}-\left(1-\frac{1}{\gamma}\right)\frac{T}{p}\frac{dT}{dz}\right)$, and that wave is known as internal gravity wave.

So, remember internal gravity wave somehow says that your system is thermally stable or your system is convectively stable, with respect to some temperature gradient. Of course, if the temperature gradient is more and more enhanced then your system can switch from a stable system to the unstable system. So, that was a very simplistic treatment with convective instability.

In the next discussion, we will try to enter in the formal processes and we will discuss of many other types of instabilities and possible wave modes and instabilities. Because at the end, we have to remember that most of the cases the astrophysical phenomena, they are associated with different type of wave modes and instabilities.

Thank you very much.