Introduction to Astrophysical Fluids
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Lecture – 27
Supernova explosions and blast waves II

Hello, and welcome to another session of introduction to Astrophysical Fluids. Previously we discussed the evolution of the blast wave which appear in general during a supernova explosion and we also said that interestingly this type of blast wave also appears when atom bomb explodes. Now, in this lecture we will talk about a bit more formally using some mathematics about the structure of the blast wave.

What is inside? So, starting from the center of explosion directly up to the blast wave front, how different fluid variables change that we will mostly discuss in this lecture. One simple thing is that we will use some mathematics some basic concepts of dimensional analysis and the self-similar solutions.

So, some parts or some steps of the mathematics may lo like non evident however, during this lecture I will not explain everything 100 percent in detail. Later I will communicate you some elements of calculation, so that these parts can be much clearer.

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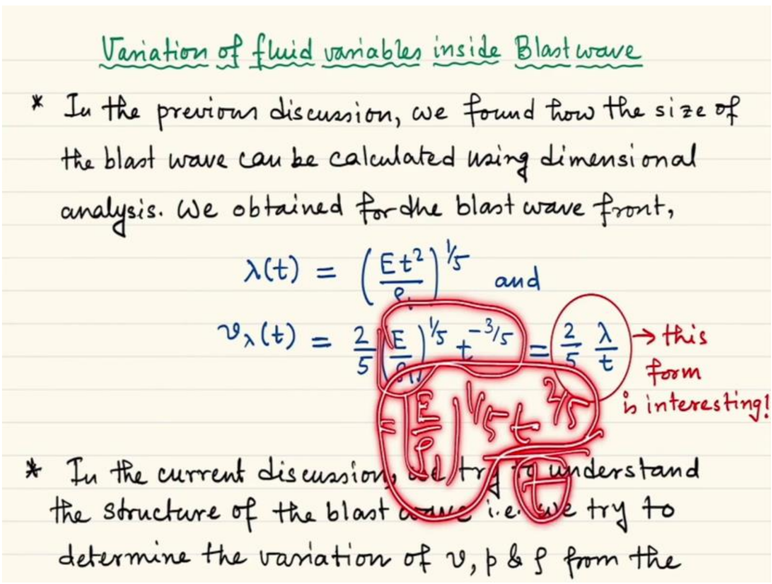

So, here for example, you can see that in the previous discussion, we actually found that just by using very simplistic dimensional analysis along with the self-similarity condition. We obtained for the blast wave front, that the size of the blast wave which is given by lambda.

This is an explicit function of time that should be equal to $(\frac{Et^2}{\rho_1})^{1/5}$ where E is the energy at the initial phase of the explosion just when the explosion takes place is the energy released, t is the time elapsed and ρ_1 is the density of the environment or the ambient medium.

Now, v_λ was the velocity of this blast wave front and that was exactly equal to $\frac{2}{5}(\frac{E}{\rho_1})^{1/5}t^{-3/5}$, when we said that the velocity is actually changing with, I mean decreasing with time, whereas λ increases with time.

But, now, in today's discussion, it will be much more useful, if we write this expression in an alternative form equivalent form, which should be written as v_λ is equal to $\frac{2}{5}\frac{\lambda}{t}$. Simply if you just write this expression as $\frac{2}{5}(\frac{E}{\rho_1})^{1/5}$, then $t^{-3/5}$ you just write as $t^{2/5}$ by t and then you just extract this whole part over here as your λ and this will be then t simple and but this form will be very much useful for us.

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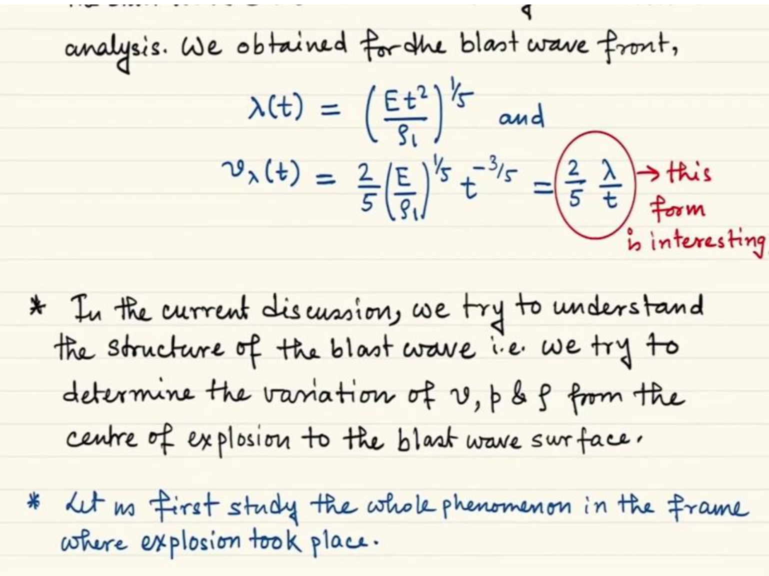

Now, in the current discussion, today we try to understand the structure of the blast wave, more precisely, we try to determine the variation of velocity pressure and density from the center of the explosion to the blast wave surface. That means, exactly what is happening to the exploded material. So, that is the part which we will study today and we will be using our previous knowledge of shocks.

Now, let us first study the whole phenomena in the frame of the explosion that means, where explosion to placed. That means it is mostly right I mean for example, if white dwarf explodes. Then you see the whole phenomena being placed on the center of the white dwarf this type of thing. So, in this frame of reference, the shock is moving so, this is not the shock frame.

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* So in the frame of study, the shock moves radially outward with a velocity $v_\lambda(t)$. Let us take v_2, ρ_2, p_2 as the fluid variables just inside the shock front.

* But Rankine-Hugoniot conditions are valid in the rest frame of shocks. What happens there?

* In the shock frame, the ambient medium appears to stream in the shock with velocity $-v_\lambda$ (radially inward) and the material inside the blast wave appears to go away of the shock with velocity $-v_\lambda + v$

* So comparing with our previous analysis, we note that

Now, in the frame of study which we are now talking the shock moves radially outward with a velocity v_λ, which is a function of t.

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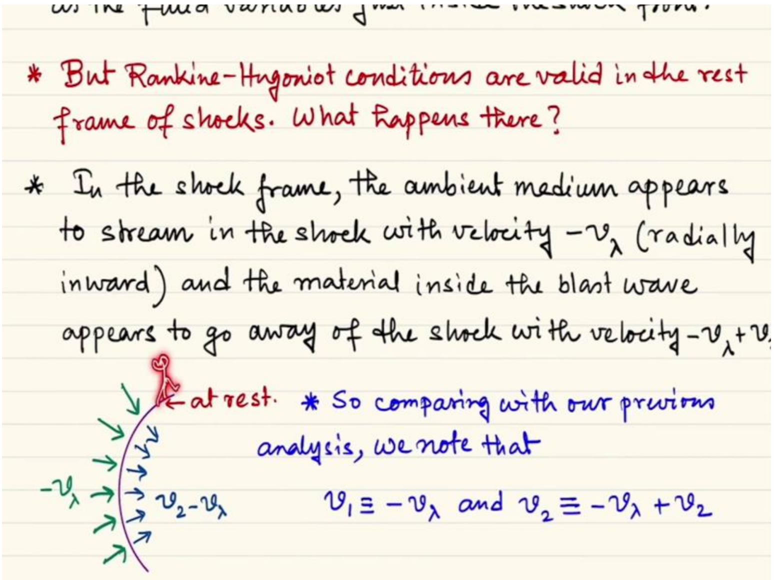

Now you can easily see here, this shock front and this is actually moving in outward direction with the velocity v_λ. So, v_λ is radially outward direction. Now, let us designate v_2, ρ_2 and p_2 as the fluid variables just inside the shock front. Now, that is completely we have characterized the medium.

But now we have to place ourself in the rest frame of the shocks, because if you remember the Rankine-Hugoniot conditions were only valid in the rest frame of the shocks. Then what happens there in the shock frame how the system is started in the shock frame, well when you place yourself on this shock now you are here.

Then, the shock is at rest for you and you will see that the shock is actually expanding outwards radially with respect to the center of explosion, with respect to the shock the ambient would appear to approach the shock with a velocity $-v_\lambda$ as simple as that.

So, in general, why it is, because in general, with respect to the center of explosion the ambient is almost at rest and so that was the story for the ambience with respect to the shock. What happens that the everything which is inside that had some velocity v_2 with respect to the center of explosion and shock had some velocity v_λ.

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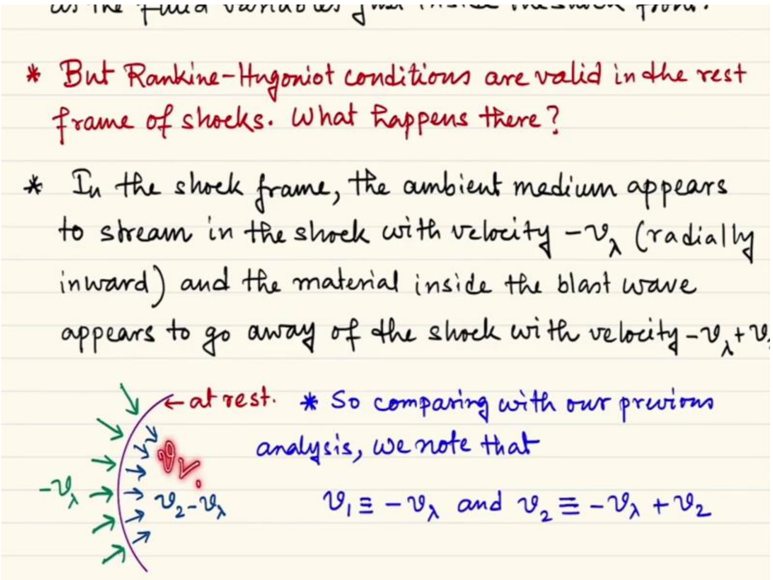

Now, with respect to the shock the relative velocity of this part will be simply $v_2 - v_\lambda$. So, with someone who is sitting on the shock, we will see the whole system as that some material from the ambient would actually stream in the shock and behind this, some material actually is flowing and is flowing farther from the shock that means, away of the shock with a velocity $v_2 - v_\lambda$.

So, if we just make a comparison with our previous v_1 and v_2, if you remember then v_1 was the velocity just in front of the shock and v_2 was the velocity just behind the shock. So, here simply you can say that $-v_\lambda$ is equivalent to v_1 and $v_2 - v_\lambda$ that is equivalent to v_2. Now, with this thing we are all set for using Rankine Hugoniot conditions.

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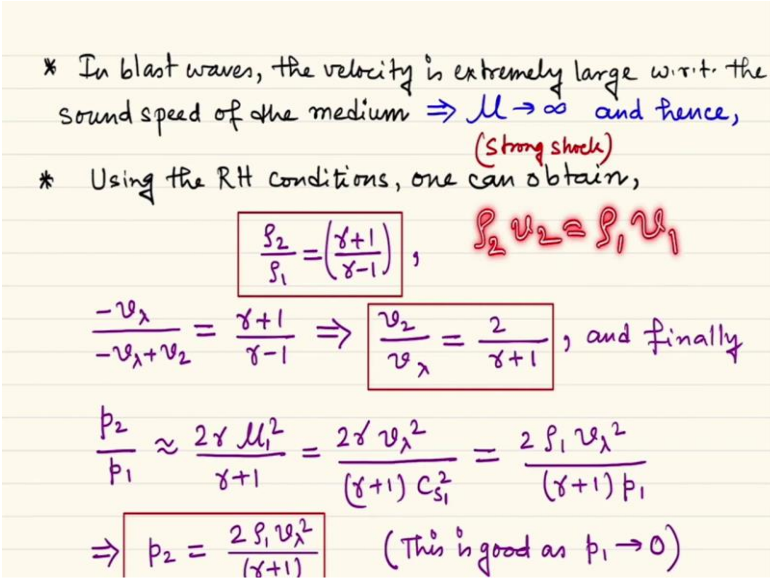

Now, in blast waves, the velocity is so large with respect to the sound speed of the medium, you can actually do some estimate and you can check, and just you can remember that if you can correctly remember the velocity. The velocity was 10^4 kilometers per second and that means 10^4 meters per second.

So, well, this is not very, very far from the relativistic domain however, for simplicity here we are taking non relativistic cases. But we are considering the system to have a very strong shock that means, the Mach number can easily be or reasonably be approximated to be very large as infinity, then this condition is known as a strong shock condition.

Now, we can use the Rankine Hugoniot condition for the strong shock, if you remember we said that even if the Mach number tends to infinity, the density jump will be simply finite and it will be given by $\frac{\gamma+1}{\gamma-1}$, the density jump between the material behind the shock to the material in front of the shock, and just by the continuity equation you know that $\rho_2 v_2$ is equal to $\rho_1 v_1$, so ρ_2 by ρ_1 will be v_1 by v_2.

Now, writing the new expressions for v_1 and v_2, we can write $\frac{-v_\lambda}{-v_\lambda+v_2}$ is equal to $\frac{\gamma+1}{\gamma-1}$. If you do the simplification, you will see $\frac{v_2}{v_\lambda}=\frac{2}{\gamma+1}$. So, once again, if you remember v_λ is nothing but the velocity of the shock front or the blast wave front with respect to the center of explosion, and v_2 is the velocity of the matter, which is just inside the shock front with respect to the center of explosion. Now, just check if your γ is 5 by 3 that is a very good approximation for mono atomic gases.

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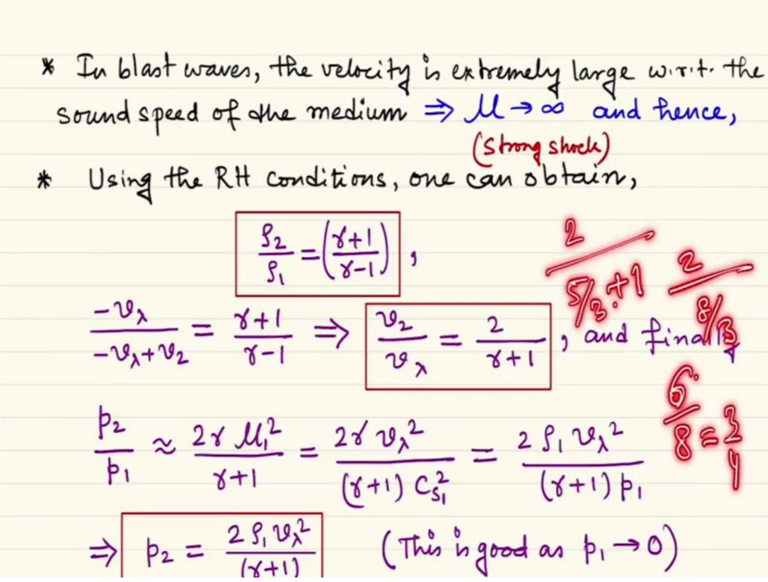

So, if it is 5 by 3 then $\frac{v_2}{v_\lambda} = \frac{6}{8} = \frac{3}{4}$.

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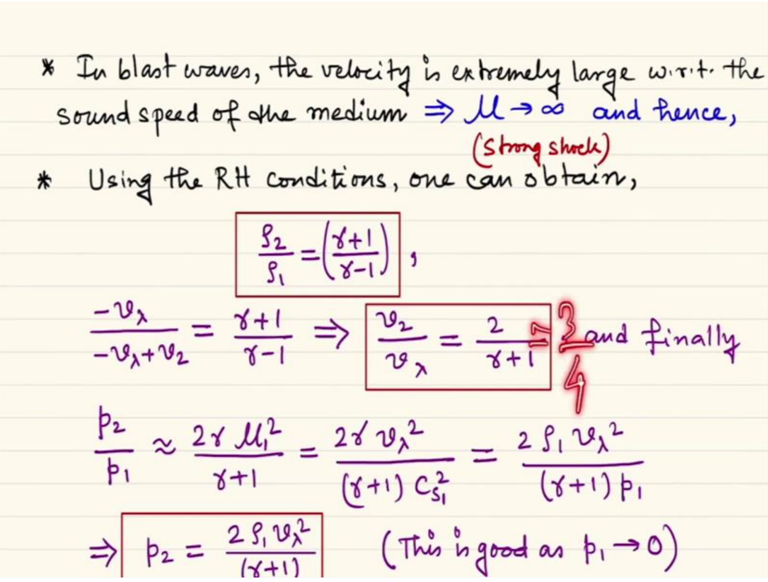

So, this one has a velocity which is greater than the v_2, that you have to understand always. That the matter which is just inside the blast wave actually has a less velocity with respect to the velocity of the blast wave itself I mean blast wave front itself. Now, if you remember the ratio of $\frac{p_2}{p_1}$, for again the case where μ tends to infinity you can write.

So, of course, μ goes to infinity, but now we are taking not infinity but very large then this ratio, you can just refer to your previous lectures, becomes simply $\frac{p_2}{p_1}$ almost will be equal to $\frac{2\gamma\mu_1^2}{\gamma+1}$, and this μ_1^2 is the Mach number of the ambient and then this is nothing but $\frac{v_\lambda^2}{C_{s1}^2}$.

So, v_1^2 is nothing but v_λ^2. So, this ratio $\frac{p_2}{p_1}$ is $\frac{2\gamma}{(\gamma+1)}\frac{v_\lambda^2}{C_{s1}^2}$. Now, C_{s1}^2 is nothing but $\frac{\gamma p}{\rho}$, if we are talking about a polytrophic gas, actually we will take it later we will see the adiabatic simply.

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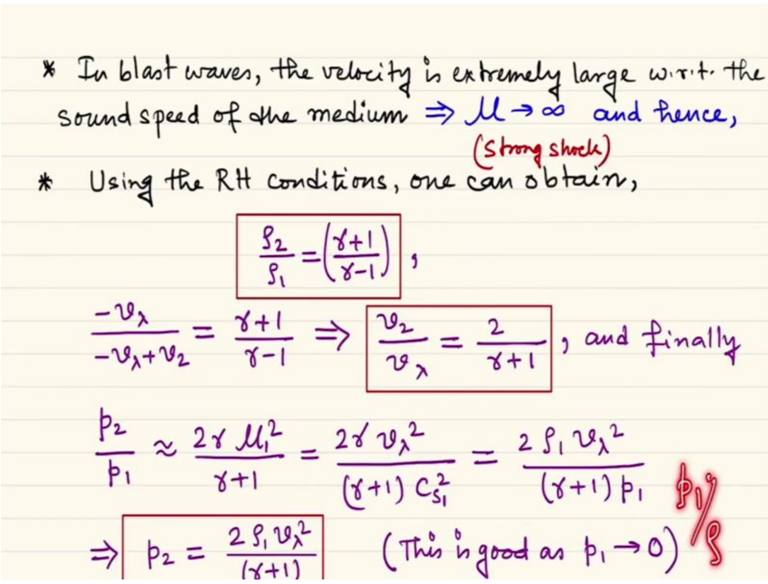

So, then p is nothing but $\frac{p_1}{\rho}$.

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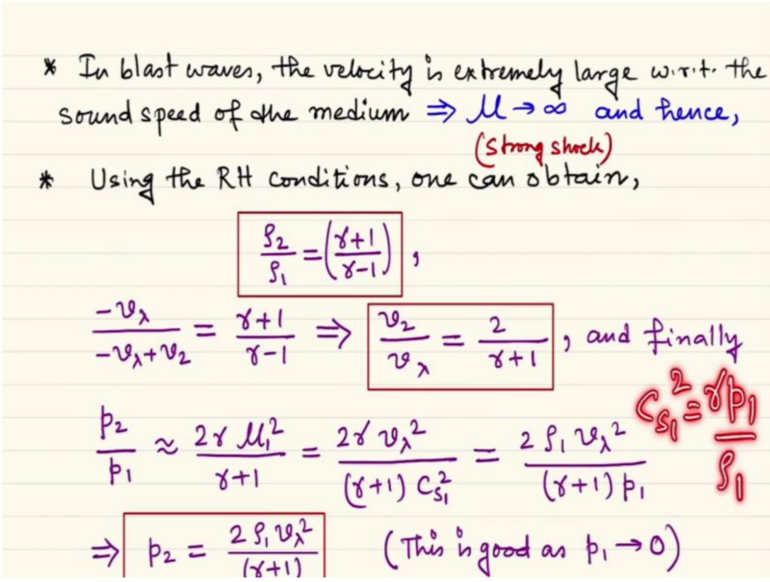

So, C_{s1}^2 will be equal to $\frac{\gamma p_1}{\rho_1}$, if this is true then you just put it here and you will see that p_2 will be then equal to $\frac{2\rho_1 v_\lambda^2}{(\gamma+1)}$.

So, this expression finally, gets rid of p_1 and this is completely very good because p_1 is negligibly small. So, something which is negatively small p_1, this is the pressure of the ambient medium. So, if that is small then actually there is no concrete interest to include this in.

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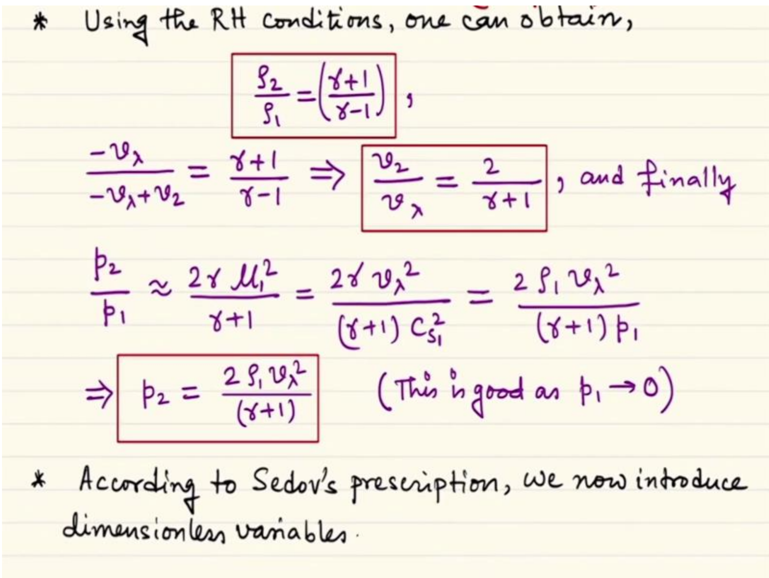

Now, we follow the Sedov's prescription and we first introduce dimensionless variables. We are doing this thing because at the end we will try to see that how much or how long we can use the self-similarity conditions in order to analytically treat the whole problem.

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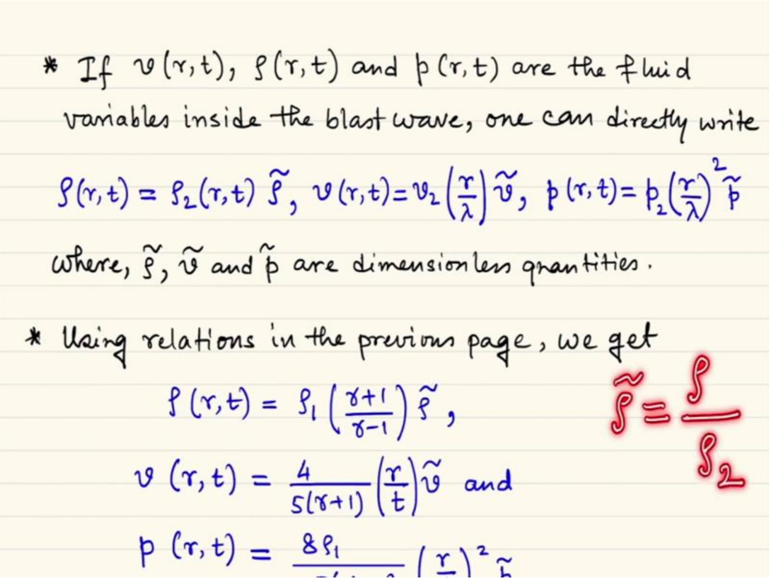

So finally, we will try to solve the equations of dynamics, there is nothing new. So, if v which is a function of r and t, and ρ, p all are functions of r and t which are the fluid variables inside the blast wave, then one can directly write $\rho(r,t)$ is equal to $\rho_2(r,t)\tilde{\rho}$, this $\tilde{\rho}$ is a dimensionless quantity. So that means that the $\tilde{\rho}$ is constructed just by dividing ρ by ρ_2.

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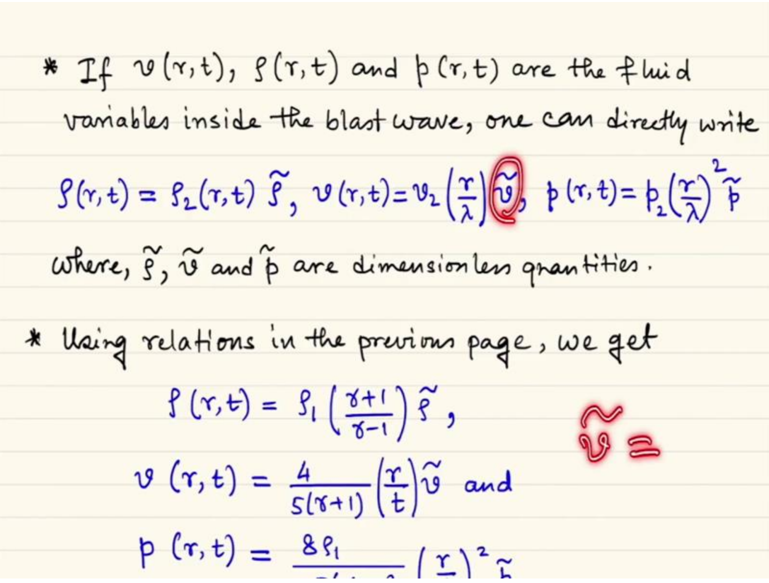

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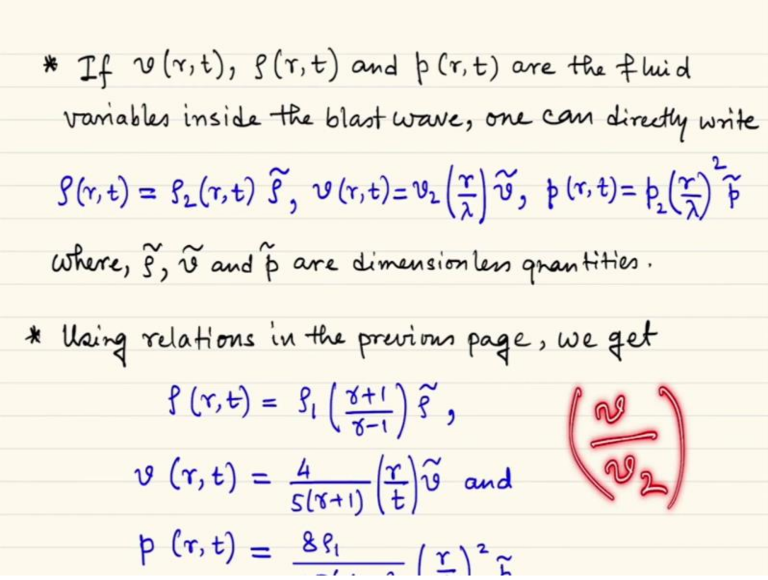

Similarly, $\tilde{\rho}$ is also constructed by dividing v by v_2, but there is one quantity $\frac{r}{\lambda}$ which comes into play.

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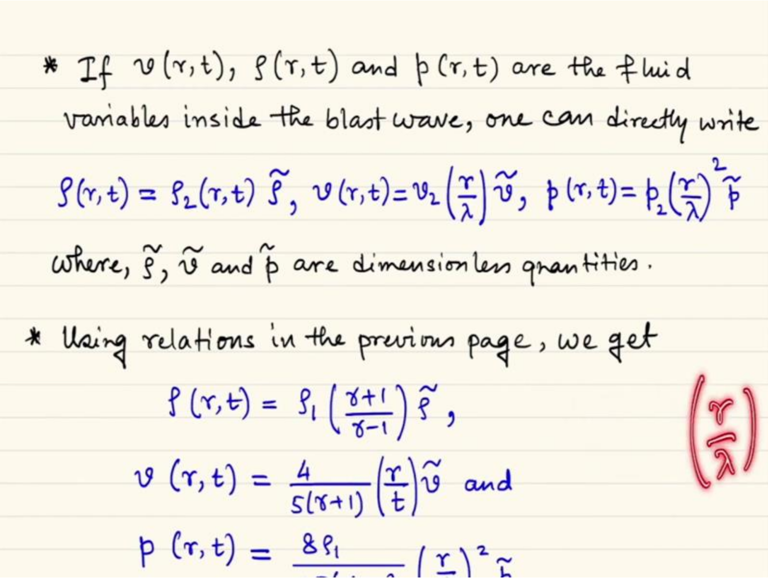

So, if you remember what $\frac{r}{\lambda}$ is this is nothing but the ξ_r.

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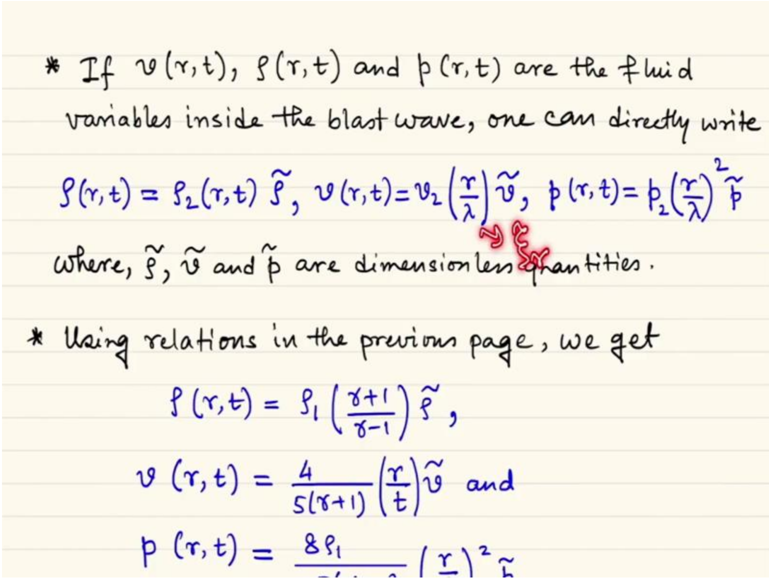

ξ_r is a quantity which is a constant over time if you choose 2 shells. For example, with r and λ, this ξ_r will be constant over time. So, if your system is self similar then the ξ_r will be totally uncoupled or independent of time. So, that is something very important and this is a famous trick of writing self-similar solutions. So, just by thinking, now, if you just follow this writing for $p(r,t)$, we write $p_2(\frac{r}{\lambda})^2\tilde{p}$.

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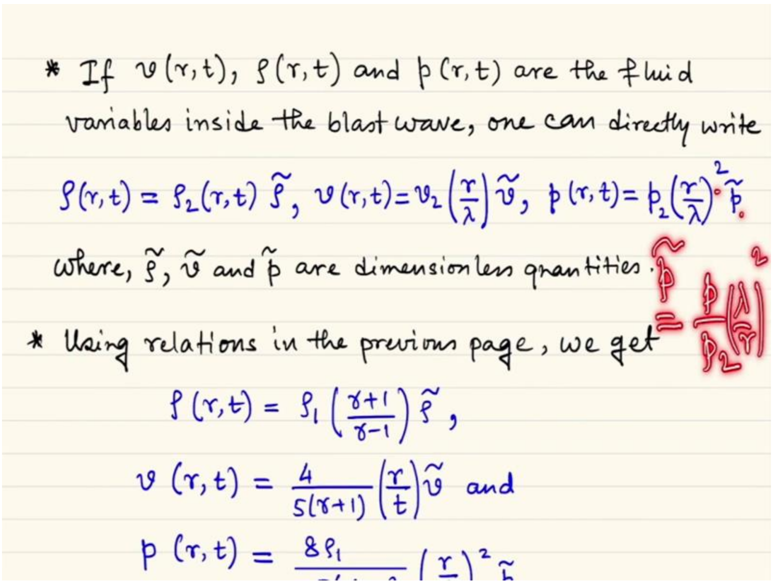

So, $\tilde{p}$ basically is constructed by $\frac{p}{p_2}(\frac{\lambda}{r})^2$. Now, why it is $(\frac{r}{\lambda})^2$ they came though. So, for clarifying this part I will communicate some steps of calculations, but you can also check over internet or in very good books. For example, one of the books Clark and Creswell which I recommended in the reference you can go through that.

But very roughly speaking here, you can see that if you just try to be convinced roughly. ρ has the order of moments of these macroscopic quantities, ρ is the 0^{th} order moment of velocity v, which is the first order moment of velocity, p is the second order moment of velocity.

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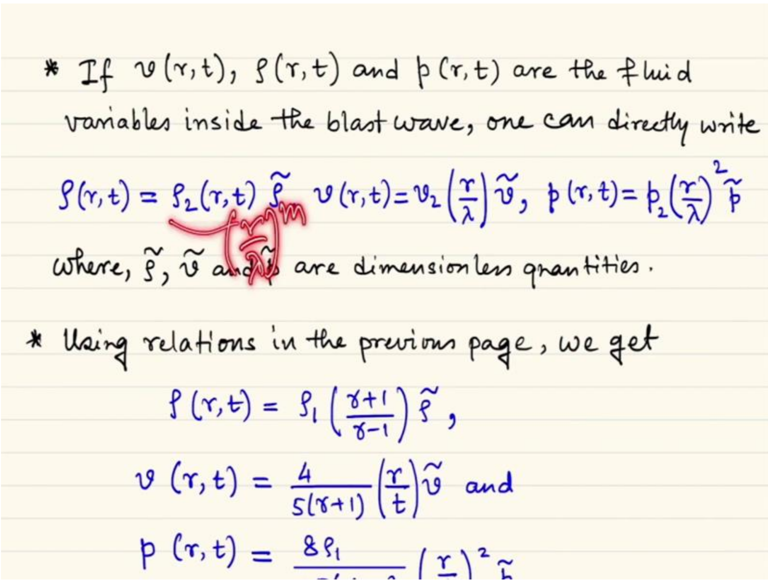

So, exactly, their order of moment, if it is m then there will be a multiplication of $(\frac{r}{\lambda})^m$. So, this is the 0^{th} order that is why there is nothing.

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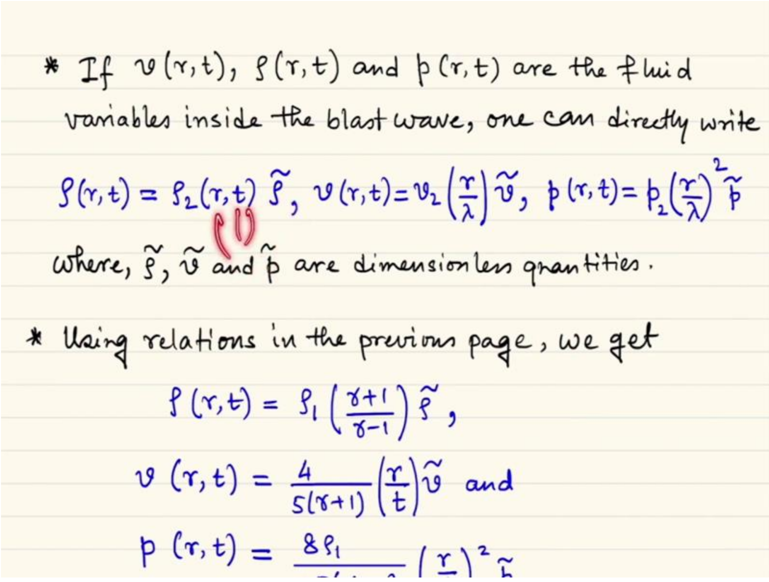

So, $(\frac{r}{\lambda})^2$ is multiplied and here this is second order. So, this is a good trick to remember, of course, the basic reason which comes that I will give you in an element of calculation by which it will be much clearer.

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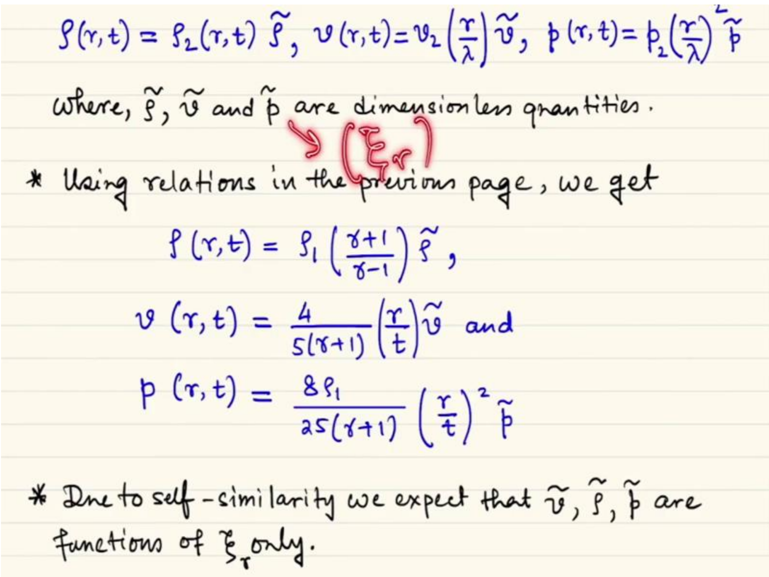

So, if you admit for the time being this type of solutions where $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$ they are dimensionless quantities and they are not only that they are functions of ξ_r, I am coming to that point.

So, they are dimensionless quantities. Then using Rankine Hugoniot conditions, we finally can write

$$\rho(r,t) = \rho_1 (\frac{\gamma+1}{\gamma-1})\tilde{\rho},$$

$$v(r,t) = \frac{4}{5(\gamma+1)}\frac{r}{t}\tilde{v},$$

$$p(r,t) = \frac{8\rho_1}{25(\gamma+1)}(\frac{r}{t})^2\tilde{p}.$$

So, this part I recommend you to check at home, maybe I am not sure but maybe I have also can communicate this part of calculation as well. But this is highly recommended that you can do it at home, you start from this and then you use all these relations over here and finally, you should reach to that.

This type of elementary calculations, I expect that you can do at home. Now finally, this is very interesting and important point that due to self-similarity now we can expect that this $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$ are functions of ξ_r only.

So, if we can write the equations of dynamics in terms of these variables and which are functions of ξ_r only, that is the best way to check that whether the system dynamics is self-similar in nature or not.

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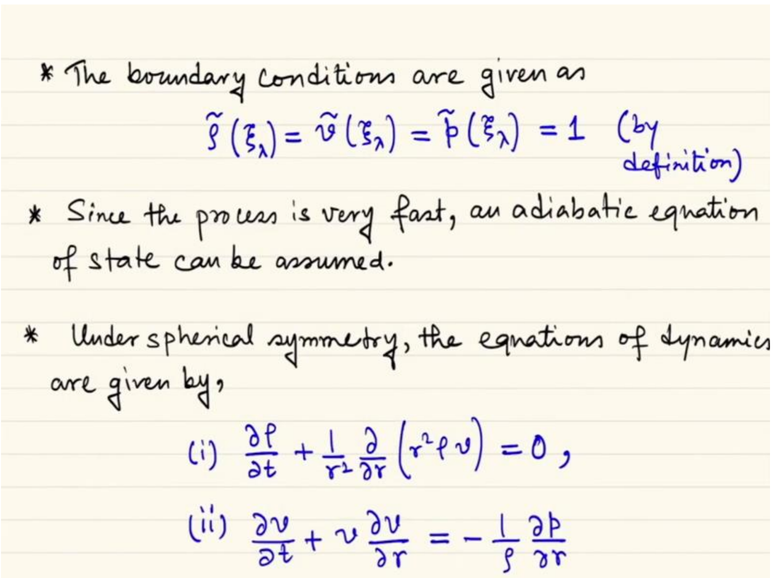

Now, boundary conditions are given as this, all these 3 things $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$, they are giving 1 at ξ_λ what is ξ_λ, it is also 1. According to our definition, if you remember, we are studying this quantity at the blast wave surface, then v will be equal to v_2.

So, $\tilde{v}$ will be 1, same thing you can do here and same thing you can do here, ρ will be simply equal to ρ_2, and of course, ρ will be not exactly equal to ρ_2, but for instance we can write this. So, there is something to say ρ will be equal to ρ at the surface when we are just moving to the surface.

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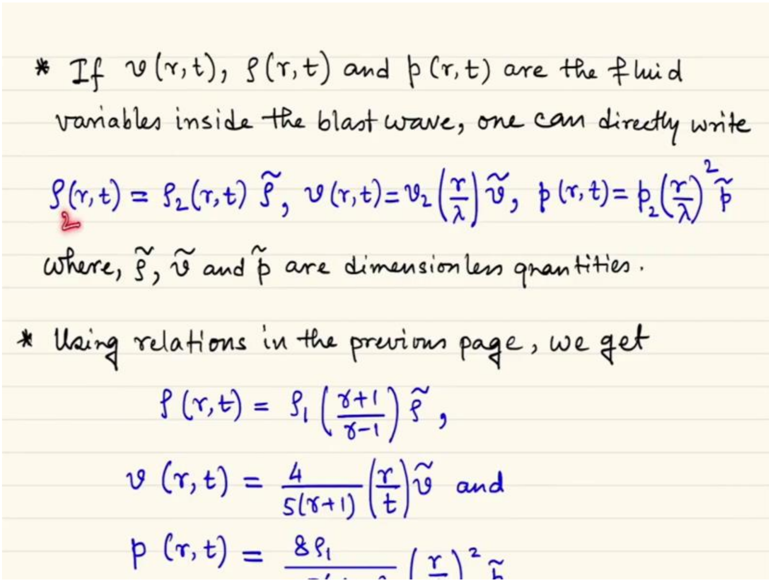

So, of course, you know that because this is a shock surface there is a discontinuity. So, proper limit cannot be achieved like that. So, what I was just saying one second ago is not a mathematically correct statement. That means, just by using these relations, you cannot check this, but if you put these things then in the reverse way you can check the consistency, that is the best thing you can do.

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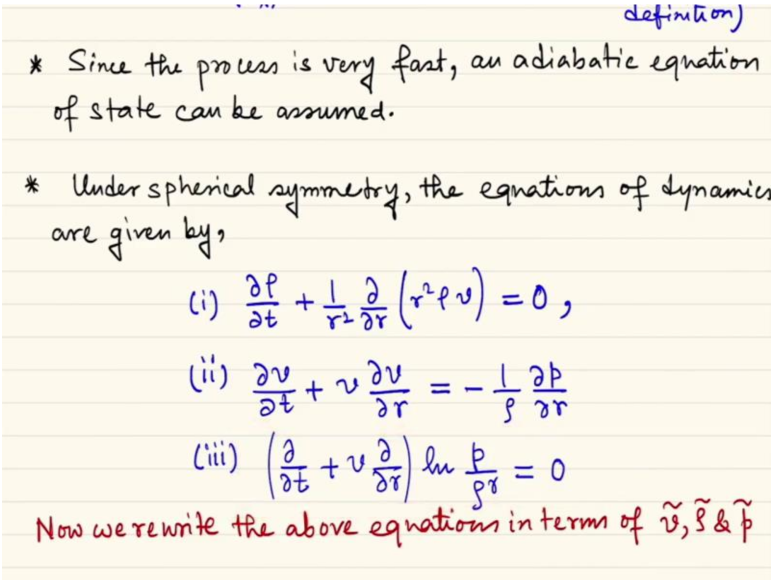

Now, it is also true that the process is very fast, that is why an adiabatic equation of state for the gas or the fluid inside the blast wave is not a bad assumption. Now, finally, we are writing the equations of dynamics of the fluid inside the blast wave. So, spherical symmetry is there, adiabatic equation of state will be there.

We have this continuity equation, this is the Navier stokes equation, of course, forcing and viscosity are neglected once again for simplicity and this is the fancy way of writing that $p\rho^{\gamma}$ is equal to constant. So, if you use these 3 things and we write the all these 3 equations in terms of this (~) quantities that means, dimensionless quantities not only dimensionless they are functions of ξ_r.

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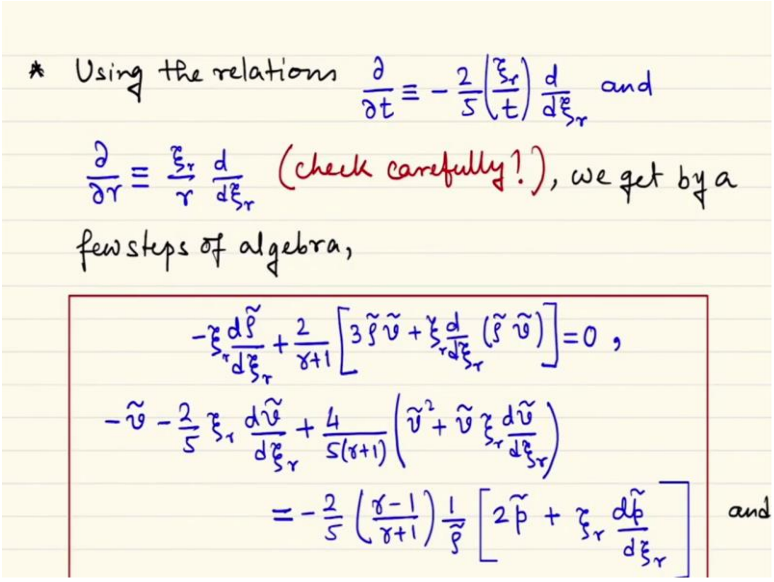

Then you can replace $\frac{\partial}{\partial t}$ by $-\frac{2}{5}\frac{\xi_r}{t}\frac{d}{d\xi_r}$, this is very, very easy, this is basic calculations of calculus and you please check it carefully, and if you are stuck, please refer to any simple book of calculus with partial derivatives and functions of multiple variables and partial derivatives total derivative this type of thing.

So, if you replace this thing then finally $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$ both are now translated to $\frac{d}{d\xi_r}$ and that is exactly the method by which one can cross check whether the system is really self-similar or not.

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$\frac{\partial}{\partial r} \equiv \frac{\xi_r}{r}\frac{d}{d\xi_r}$ (check carefully!), we get by a few steps of algebra,

$$-\xi_r \frac{d\tilde{\rho}}{d\xi_r} + \frac{2}{\gamma+1}\left[3\tilde{\rho}\tilde{v} + \xi_r \frac{d}{d\xi_r}(\tilde{\rho}\tilde{v})\right] = 0 ,$$

$$-\tilde{v} - \frac{2}{5}\xi_r \frac{d\tilde{v}}{d\xi_r} + \frac{4}{5(\gamma+1)}\left(\tilde{v}^2 + \tilde{v}\,\xi_r \frac{d\tilde{v}}{d\xi_r}\right) = -\frac{2}{5}\left(\frac{\gamma-1}{\gamma+1}\right)\frac{1}{\tilde{\rho}}\left[2\tilde{p} + \xi_r \frac{d\tilde{p}}{d\xi_r}\right]$$ and

$$\xi_r \frac{d}{d\xi_r}\left(\ln \frac{\tilde{p}}{\tilde{\rho}^{\gamma}}\right) = \frac{5(\gamma+1) - 4\tilde{v}}{2\tilde{v} - (\gamma+1)}$$

Now finally, although these equations look like a bit bulky, but you see this is the continuity equation just you do not have to go through each and every part of the equation, but just to check that only we have $\frac{d}{d\xi_r}$. Here as well, we have the differentiations are only with respect to ξ_r.

The same thing over here the only thing is like $\frac{d}{d\xi_r}$, and all the quantities variables are ~variables. Finally, this is true for this one as well again $\frac{d}{d\xi_r}$. So, finally, we are convinced that the system is reasonably well approximated by self similarity.

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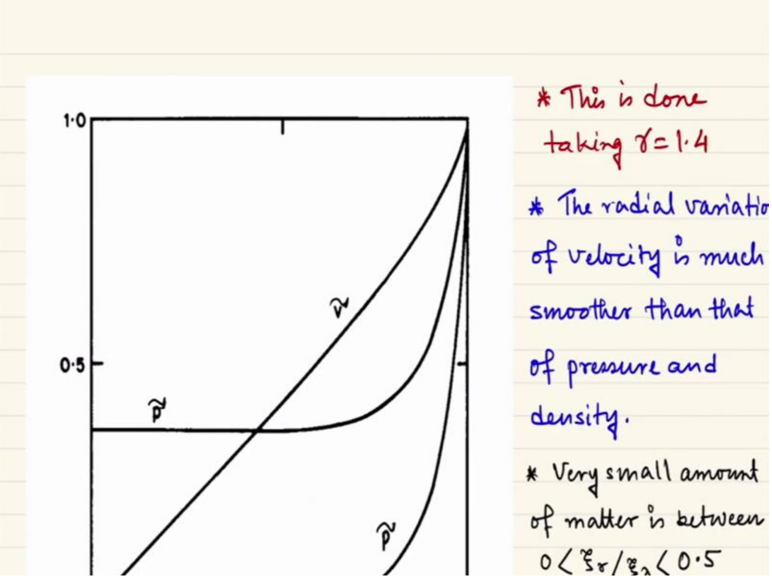

But what that gives to the structure of the system, can we have some interesting information about the structure of the system?

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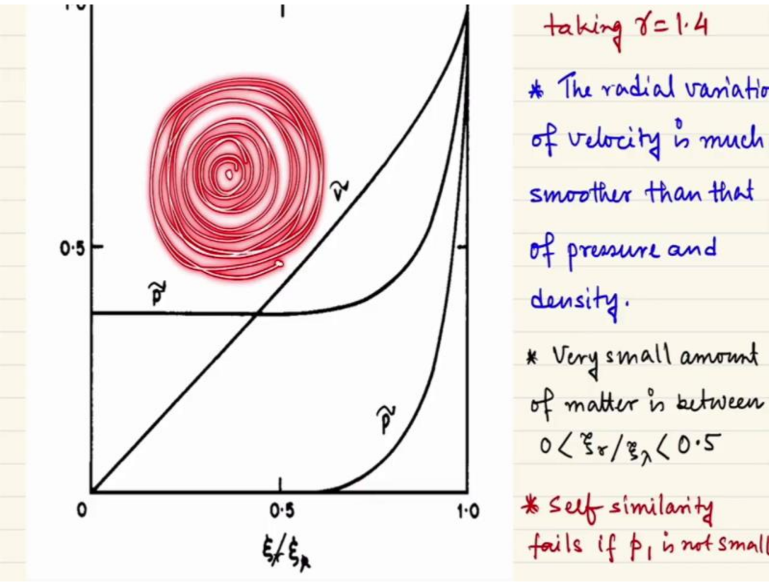

So, here we have plotted $\frac{\xi_r}{\xi_\lambda}$ along x and y axis. So, this quantity $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$ along y axis we have plotted 3 different things for 3 different plots one is $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$. So, you see, this is done, so this one is done for γ is equal to $\frac{1}{4}$. So, which is not a very practical case for astrophysics, however but this is a very practical case for atom bomb because this was done for air. But this type of behavior actually is very much similar for that of a supernova.

Now, the first thing one can see that the radial variation of velocity is much smoother than that of the pressure and density. So, this one basically changes gradually, whereas pressure and density both they do not change up to this and suddenly they change very sharply. Not only that if you just notice this attentively a very small amount of matter is between this $\frac{\xi_r}{\xi_\lambda}$ from 0 to 0.5.

So, the amount of matter very near to the center of explosion is very, very small. So that means, if you think that this is the center of explosion and just the center of explosion you have almost homogenous type of matters spherical shell totally is like a solid spherical type of thing. So, I mean full of fluid that is not a very good picture.

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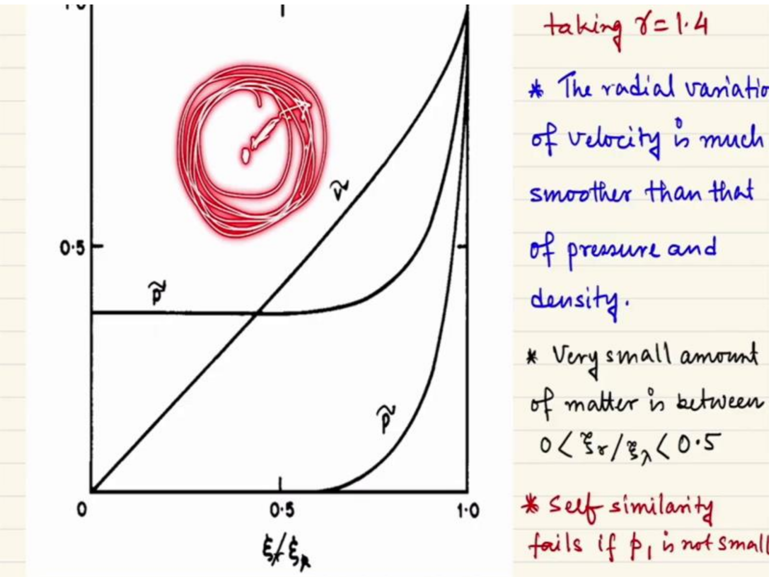

Actually, what happens after the explosion, most of the thing it basically gets detached very, very quickly from the center of explosion and goes to a certain distance and this part you have very, very small density. However, the velocity is somehow reasonable, that is only because the very small amount of matter which is also there, they have a considerable velocity.

So, in terms of velocity, the change is very, very gradual. So, this is something, we can infer for the structure of the blast wave. So, it basically gets detached like an envelope plop and then it gets off. So, one last thing is that it is true that in our whole analysis we somehow discussed the thing using self-similarity and we actually saw that this was quite a good approximation.

But it is true that please remember the self-similarity thing is no longer valid for a system, where the explosion finally expands in a medium with considerable pressure. So, for example, some supernova explodes in the neighborhood of much denser astrophysical medium. Then, this type of problem cannot be handled using self-similarity. So, that was all about a brief overview of supernova and blast waves, which are nothing but a very good application of shock waves.

Here we also discussed two important concepts, first of all dimension analysis which is very simplified, but yet very useful and another is self-similarity. Although, for some tricks of self-similarity, I did not go into the detail, because in the scope of this course, we cannot do everything. But I will try to give you some elements of material, so, the style of writing self-similar solutions gets much clearer.

Thank you very much.