Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture – 25
Effect of nonlinearity: shocks

Hello. Welcome to another lecture of Introduction to Astrophysical Fluids. Previously, we discussed that in a fluid if the fluid velocity is greater than the sound speed then any weak perturbation which is caused in that fluid, I mean of course, the fluid should be compressible, and for our convenience we also assumed that the fluid is polytropic then actually we showed that for the case where fluid velocity is greater than the sound speed that is the Mach number exceeds 1.

Any weak perturbation, which is applied to the flow field, in general cannot be repaired and eventually it gets aggravated by the fluid flow, and in the same context in this discussion we will now introduce the concept of shock, and we will see that how mathematically this thing can be treated.

One thing is, of course, true that as I said that the perturbation is no longer weak. So, we can no longer treat the perturbation part in the linear framework. So, how including the non-linear effects we can handle this type of case of aggravated perturbation that we will see in this lecture.

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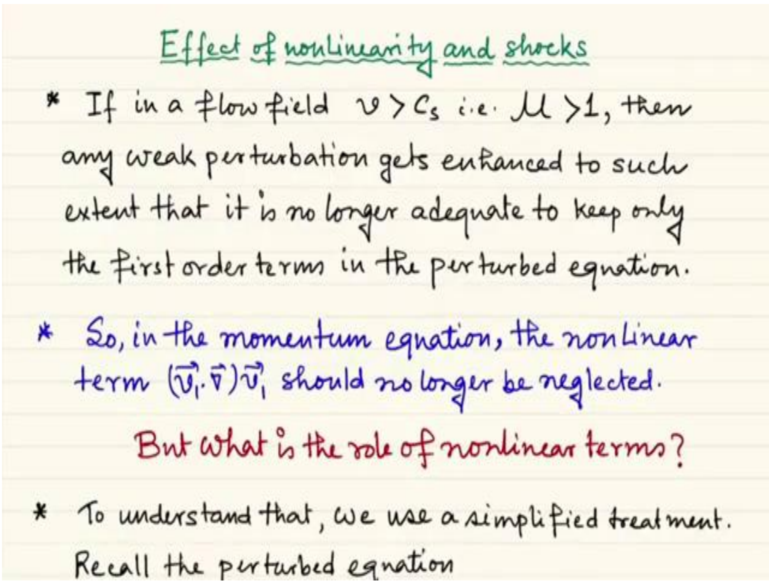

Once again $v > C_s$ simply says that the fluid velocity is much more important than the sound speed. So, any weak perturbation caused inside the fluid will be actually aggravated by the fluid flow before the sound speed reaches to the point of damage in order to repair the damage. What is the meaning of that?

That means, the Mach number is greater than 1, and at that point it is no longer adequate to keep only the first order terms in the perturbed equation. So, that means, the linear approximation is holding no longer. So, in the momentum equation as we just neglected previously the term relating to $(\overrightarrow{v_1}.\vec{\nabla})\overrightarrow{v_1}$, now, it cannot be neglected any longer. We must include this.

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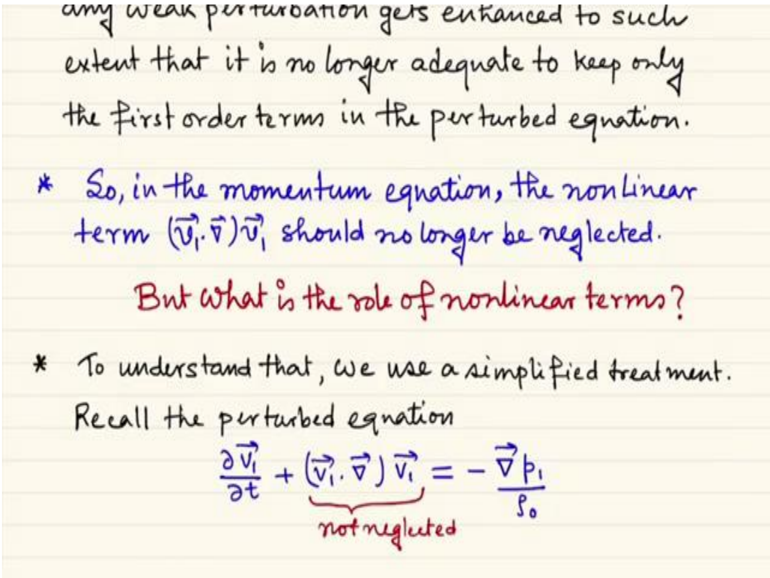

So, the question is that we must include this $(\overrightarrow{v_1}.\vec{\nabla})\overrightarrow{v_1}$, but what is the role of this non-linear terms finally. That means, what is the physics which is brought by these non-linear terms? To understand that it is true that analytically it is not evident. That means, you cannot do the general treatment in pen and paper, but we can do some simplified approach.

So, for example, we can simply now for instance to understand roughly what is the role of the non-linear equations I mean non-linear terms in the perturbation equation, we will simply write again the perturbed equation now, but including the non-linear term. So, $\frac{\partial v_1}{\partial t} + (\overrightarrow{v_1}.\vec{\nabla})\overrightarrow{v_1} = -\frac{\vec{\nabla} p_1}{\rho_0}$ and this one $(\overrightarrow{v_1}.\vec{\nabla})\overrightarrow{v_1}$ is no longer neglected. Of course, here in this case we neglect again as we neglected last time the forcing term and the viscosity term.

Just to mention, in the physics of turbulence or in the physics of fluid I mean non-linearities, and so, the non-linear phenomena in normal fluids and in plasmas very often we treat the shocks in an equation called the Burger's equation. The Burger's equation is something where it is the total equation is written in one dimension, and the term corresponding to this term and this term they survive, we neglect this pressure term, but then the dissipation term is no longer neglected there.

If you do that if you solve that Burger's equation, it is possible to solve analytically or even you solve it numerically. In both cases you can see that something like I mean a discontinuity in the velocity profile of the fluid should appear and that is the case of traditional shock, and we say that the Burger's equation contains or entertains a shock type of solution. Now, coming to our case where viscosity is throughout neglected, we have this equation $\frac{\partial v_1}{\partial t} + (\overrightarrow{v_1}.\vec{\nabla})\overrightarrow{v_1} = -\frac{\vec{\nabla} p_1}{\rho_0}$.

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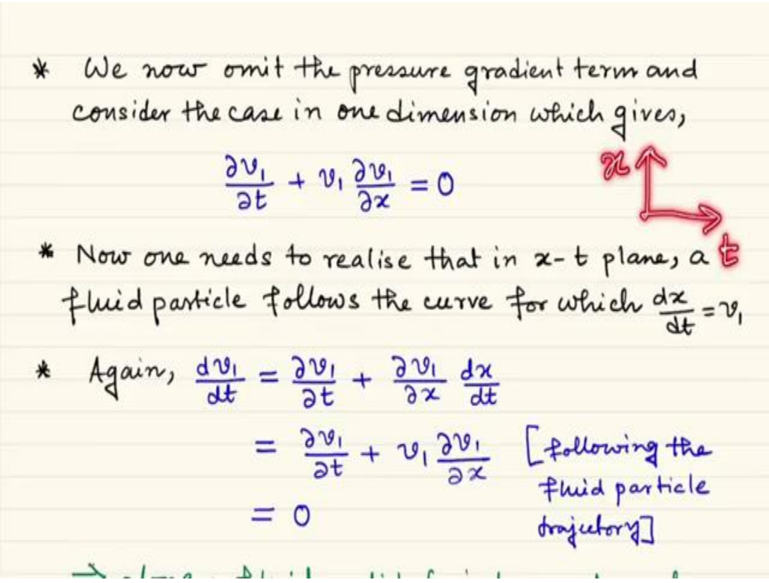

Again, we will do two more simplification. So, we will omit the pressure gradient term and we will write the whole thing in one dimension. So, finally, it simply gives us this form. So, this is a trivial form of Euler's equation. No viscosity, no pressure just the inertia term, $\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} = 0$.

Now, one needs to realize that in an $x - t$ plane, where t is along the horizontal direction and x is along the vertical direction. So, this is the $x - t$ plane. What will be the trajectory of a fluid particle? Now, remember the concept of fluid particle, which we introduced in the framework of Lagrangian approach.

Now a fluid particle, of course, in this case will follow a curve for which $\frac{dx}{dt}$ is equal to the velocity of the fluid particle and that is the fluid velocity itself at that point and that is v_1. So, this is the curve followed by a fluid particle and in other ways this defines a flow line. But you know that if we write this expression now, we know this expression.

Now, we just try to express $\frac{dv_1}{dt}$, how does it look like. So, $\frac{dv_1}{dt}$ is equal to $\frac{\partial v_1}{\partial t}$, because v_1 is a function of x and t, plus $\frac{\partial v_1}{\partial x}\frac{dx}{dt}$. We are in one dimension. So, only space coordinate is the x coordinate. And what is $\frac{dx}{dt}$? It is nothing but v_1. So, if we consider the $\frac{dv_1}{dt}$ along the trajectory of the fluid particle that means, along the flow lines real flow lines. Then we can simply say that this term $\frac{\partial v_1}{\partial x}\frac{dx}{dt}$ is nothing but this term $v_1\frac{\partial v_1}{\partial x}$.

Otherwise, this is a general statement, but at the step where we are writing that $\frac{dx}{dt}$ is equal to v_1. Then we are actually constraining to check or to study the $\frac{dv_1}{dt}$ along our trajectory of a fluid particle. That means, the following fluid particle trajectory and if we can write this from this one $\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x}$. we know by definition this one is equal to 0. So, $\frac{dv_1}{dt}$ is equal to 0 along a flow line.

So, along the flow line, there is no change in velocity of a fluid particle and that is very true. You can easily understand because in this total equation this is the momentum evolution equation. The right-hand side is identically 0. So, there is no net force. So, all the fluid particles are moving freely right.

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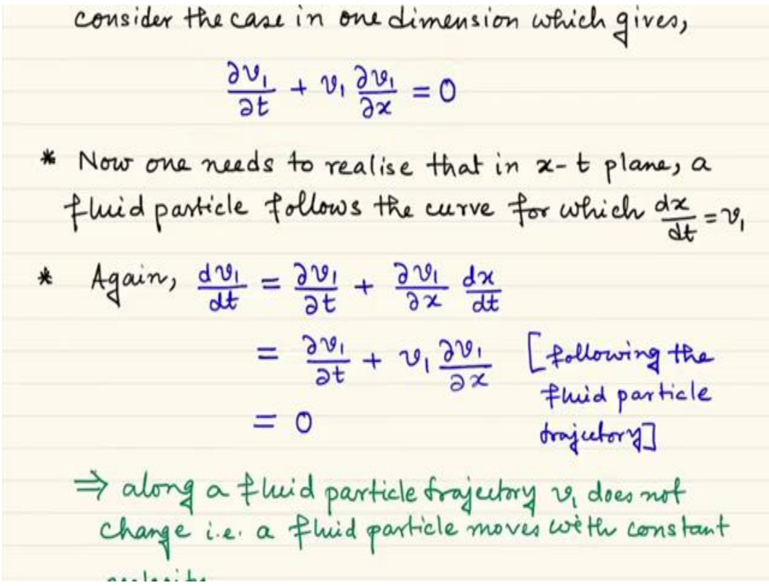

Of course, once again, this is a very simplified case, but as you will see that this will already help to understand how the non-linear terms play a role in steepening the different flow field. Now, this one what we concluded. A fluid particle always moves with constant velocity when we are talking in terms of this equation.

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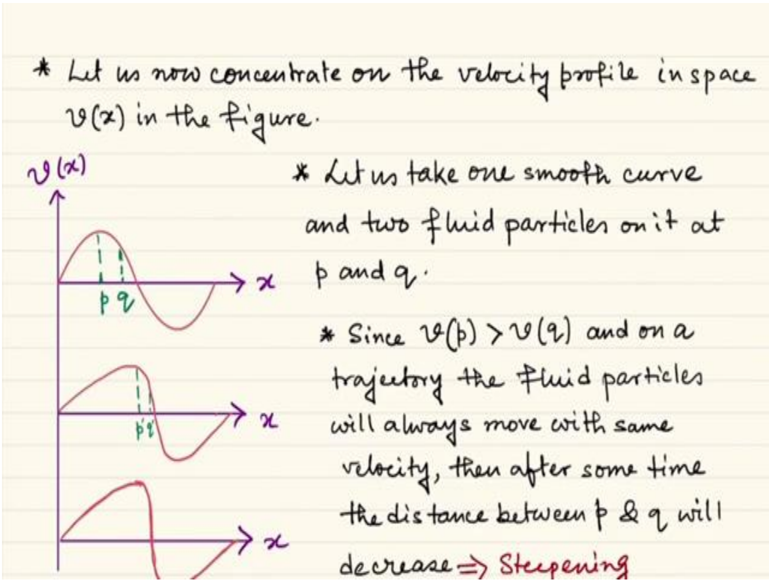

Now, the question is that how to get the essence of steepening using this concept. So, let us now concentrate on the velocity profile in a space which is v as a function of x in the figure. Now, remember there is only one space coordinate and only one velocity coordinate in the problem because this is 1-D flow. Now, this is the velocity profile as a function of space and let us take one smooth curve.

Let us say some sinusoidal type of curve and two fluid particles on it at p and q. This is taken. Now, p and q has certain distance and the velocity of the fluid particle which is situated initially at point p, has a greater velocity than that of the fluid particle which is situated at q. Now, since it has the greater velocity then it will traverse a greater distance in some given time.

So, after some given time p comes to p'. So, it traverses a greater distance, but the fluid particle which was at q having a smaller velocity. It cannot go equal amount of distance. So, it can go a distance which is smaller in compared to with the distance traversed by the fluid particle which was initially at p, and then new positions are given by p' and q', respectively.

So, you see as a result what happens when their distances decrease, so, now, the two fluid particles they are situated at p' and q'. So, they are coming closer, but the problem is that and this is exactly what it is written. So, I am just try to concentrate on the figure now.

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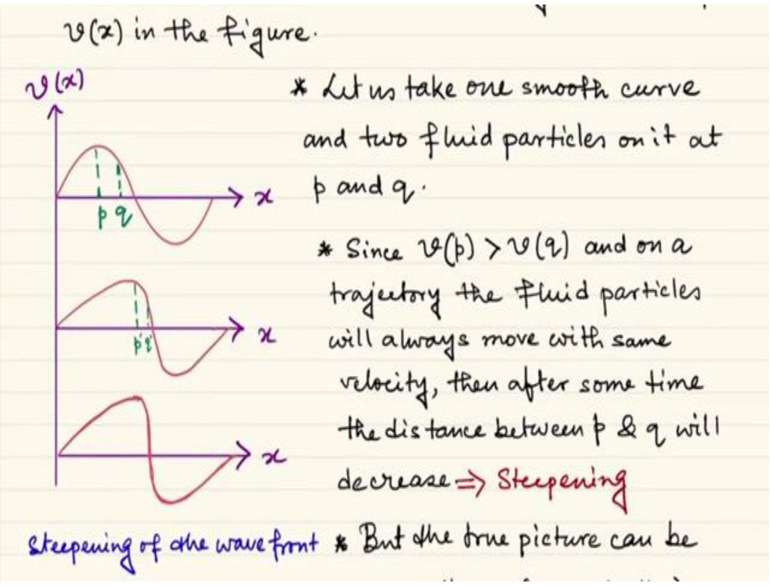

So, now the important thing is that as we have just shown that for a fluid particle along its trajectory v will be unchanged. So, the v at p and v at p' will be the same. So, for example, the value of v over here and now the value of v over here is the same.

In the similar way value of v at q and value of v at q' they are also same. So, you see that these two things and what happens then? That this velocity discrepancy is now constrained in a shorter region in space, and what is the meaning of that? That means, for example this wave type of shape gets steepened like that and it will continue steepening until you have something totally perpendicular.

This is the traditional picture of a shock. So, this is the known as the steepening of the wave front, and actually it is possible that for any smooth velocity field. But of course, we have to understand that this is not the whole picture. For obtaining the true picture one can go back to the original equation that means, again considering this pressure term and actually you can consider the dissipation term.

You have to go to more than one dimension and then you have to do treat the problem numerically. Then also people have done this. For example, people who work in the domain of turbulence, let us say compressible turbulence, astrophysical turbulence, supersonic turbulence, they actually see very evidently that shock fronts appear. So that means, the very localized steepened wave fronts I mean steepen this type of things, they are like how to say that?

A very steep velocity profile basically that appears, if you see the spatial profile of the velocity. So, that is something generalization of a traditional shock. So, traditional shock this is actually always discussed in one dimension. In three dimension that is a generalized shock.

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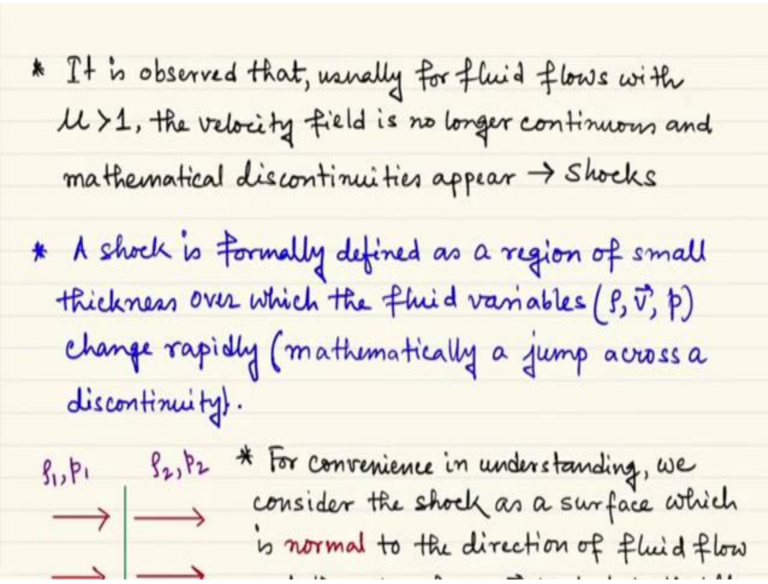

Now, it is observed that usually for fluid flows with Mach number greater than 1, the velocity field is no longer continuous that is why that is what I just mentioned and mathematical discontinuities appear. Now, in 1D this is possible to treat analytically and so, this type of discontinuities we call them shocks. So, once again the concept of shock is not constrained in one dimension, but as far as we are talking about the analytical approach for shocks, very frequently we talk in one dimension.

So, now we have to define. So, once again, shock is nothing but a steepened wave front or a steepened velocity profile basically, let us say like that velocity profile due to the non-linear effect. Now, we have to define somehow formally what a shock is.

So, a shock is a region of small thickness over which the fluid variables let us say ρ, v and p, change very rapidly across this surface. So, if you want, for example, here the velocity and here the velocity, here the density, here the density, here the pressure, here the pressure they are finitely apart from each other.

Now, mathematically this type of case can be handled using a jump condition across a discontinuity. I hope you are aware of this type of jump across a discontinuity. In case you are not aware just you can search over internet. So, normally how we can treat analytically shocks and discontinuities in fluids and plasmas and you will come across these things.

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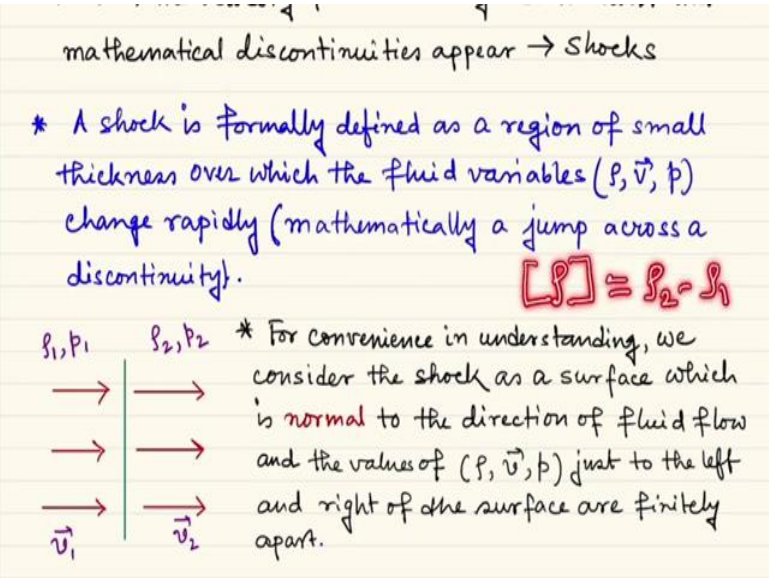

But, here for this course even if you do not know that very in detail that is not a problem. Just think that its jump means that roughly the difference of these two values. So, let us say this green color surface is the so-called shock front, the shock surface, and just the value of the density of the fluid which is very near to this surface, but to the left of the surface and very near to the surface and, but to the right of the surface they are different, and the difference between them is called the jump.

So, a jump is actually designated by a symbol like this. So, let us say jump in ρ or density is nothing but equal to $\rho_2 - \rho_1$, so, this type of things. Now, for convenience in understanding we use this figure and we consider the shock. So, a shock is nothing but a region in flow field with very small thickness, but for our analytical convenience we consider shock as a surface with actually zero thickness here.

For example, it is simply a straight line. There is no width, for example now. This is idealized shock for mathematical treatments, and this one is considered to be normal to the direction of the fluid flow. So, if the fluid is flowing in this direction horizontal direction, then the shock is in the vertical direction, and the values of ρ, v, p for example, just to the left and right of the surface are finitely apart. So, once again just to tell you clearly what this is. This means, that two points which are infinitesimally close, but situated to the two sides of a shock surface have finite difference in fluid variable values and this difference is known as the jump.

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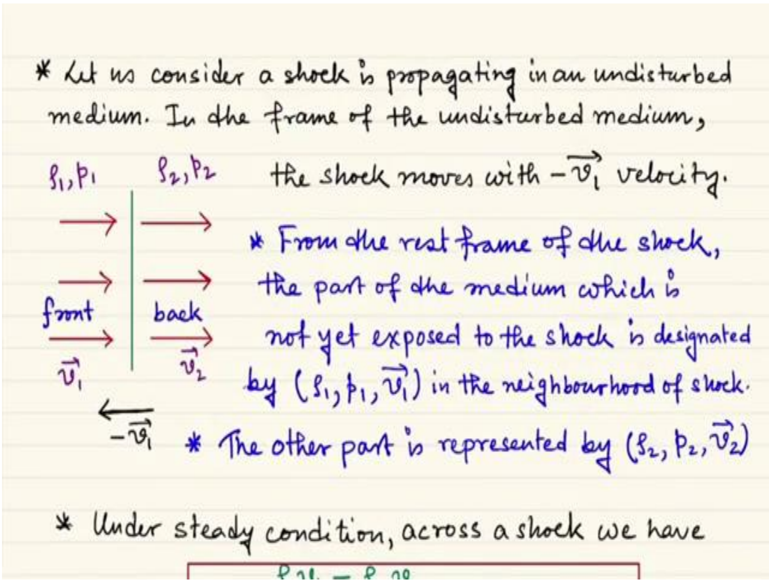

Now, let us consider a shock is propagating in an undisturbed medium. Now, let us do something mathematical and to see analytically how much we can go. So, again I have given the same picture.

So, a shock is propagating in an undisturbed medium let us say the media had nothing, so no perturbation. So, let us say we can just see. So, someone is in the frame of the undisturbed medium and let us say some fluid, and we see that the shock basically moves with $-v_1$ velocity.

So, we see that this surface of discontinuity perpendicular to the fluid flow propagates along this negative direction with the velocity v_1. So, its velocity will be $-v_1$ then, if the original v_1 is in this direction.

Now, if this is the case, we can see from the frame of the undisturbed medium that the medium is at rest. It is the surface of discontinuity which is moving in this direction.

Now, we have to think that we place ourselves on the shock that means, we are now here in the frame of the shock. Then what we will see? The person will see that which is in front of the person. So, this is the fluid which is in front of the person and that is the front part of the shock and this is the back part of the shock.

So, for the person who is sitting on the shock we will see the fluid here is coming towards him or her with velocity v_1, density ρ_1, p_1, of course, when we are talking about these values ideally. So, they are just the general quantities, but their values can change from every point in this fluid.

So, but in general what I am saying that these are general names of those variables in the front part of the shock, and the general symbols for the ρ, p and v for the back part of the shock is known as they are known as ρ_2, p_2 and v_2.

Of course, as I said that the for the fluid which is very, very close to this shock front but being inside the front part of the shock and in the back part of the shock they have a finite difference in values although they are very, very close. Now, this is true for all ρ, p and v.

Now, it is also true that in the rest frame of the shock what we will see that the front part fluid is coming in this direction with velocity v_1, and the back part of the fluid is going farther from the shock surface, with the velocity v_2. You can simply imagine that.

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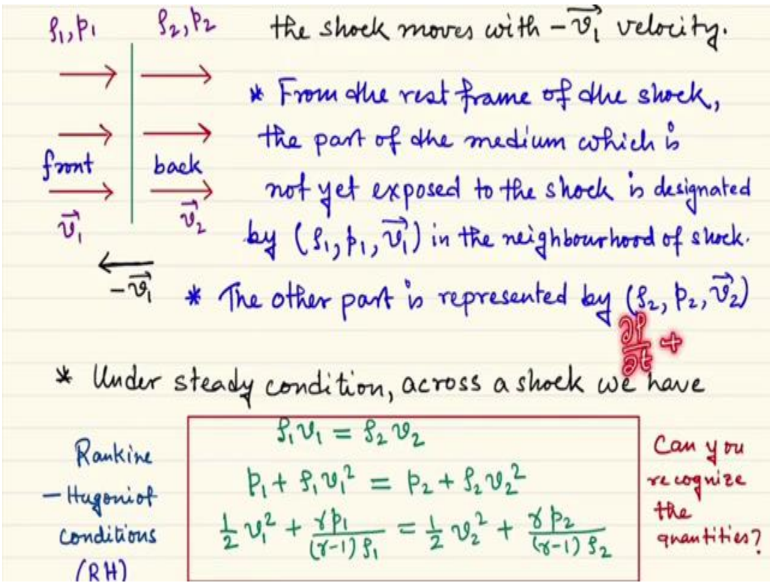

Then it can be shown by the virtue of all three basic equations of hydrodynamics that is the continuity equation, the momentum evolution equation and the internal energy evolution equation. One can actually show that at steady state if we consider that the whole system the flow is steady. If it is not steady then these conditions are no longer true, but at steady state these three conditions are true.

That means, although ρ, p and v they have jumps across the surface of shock there are some quantities which are continuous across the shocks. One is the ρv that is $\rho_1 v_1$ will be equal to $\rho_2 v_2$. So, although ρ_1 is not equal to ρ_2, v_1 is not equal to v_2, but their products will be the same.

Again, p_1 is not equal to p_2, but $p_1 + \rho_1 v_1^2$ is equal to $p_2 + \rho_2 v_2^2$. This is the equality, and finally, although we know that v_1 is not equal to v_2, p_1 is not equal to p_2, ρ_1 is not equal to ρ_2, but finally, $\frac{1}{2} v_1^2 + \frac{\gamma p_1}{(\gamma-1)\rho_1}$ is equal to $\frac{1}{2} v_2^2 + \frac{\gamma p_2}{(\gamma-1)\rho_2}$, this is another equality.

So, these three quantities, they are continuous and these conditions are known as Rankine Hugoniot conditions, RH conditions. But my question to you is that can you completely recognize these quantities?

So, these quantities are nothing but the quantities which appear. Now, you remember when we discussed about the conservative form these quantities appear.

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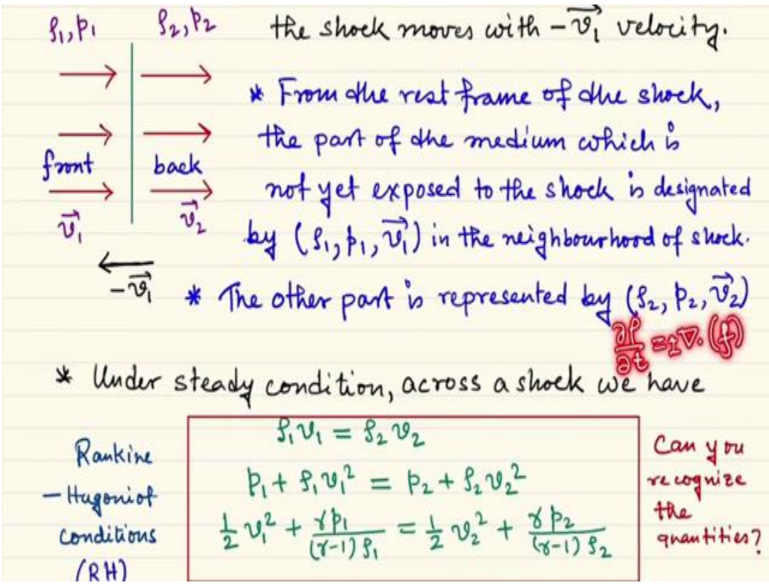

So, $\frac{\partial \rho}{\partial t}$. So, ρ is the density of some quantity is equal to somehow you can show that plus or minus divergence of some flux term.

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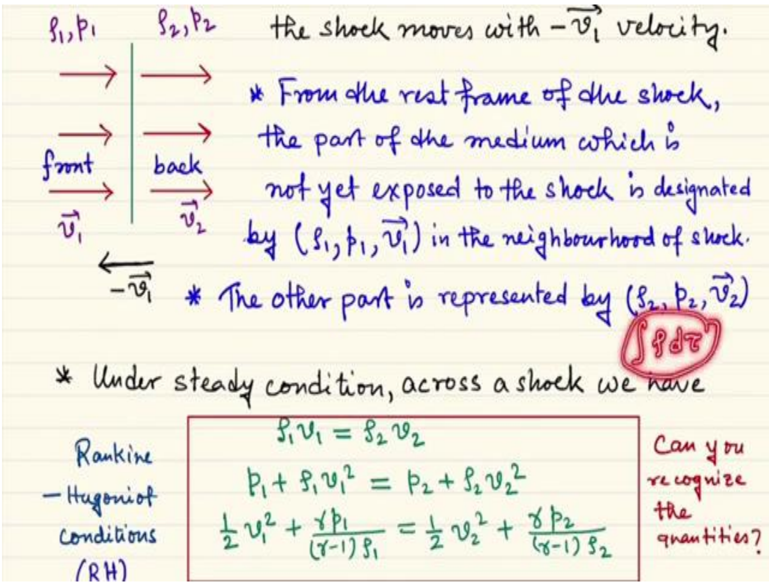

Then you can show the conservation for the quantity $\rho d\tau$ that means, in volume this quantity is conserved in time.

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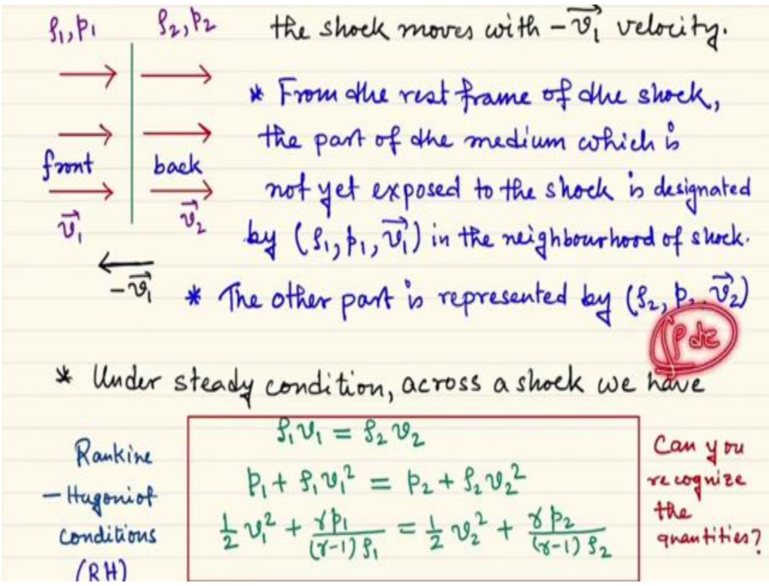

If you integrate this ρ in the volume then this quantity is a constant of motion. So, these quantities are nothing but those quantities which appear inside the divergence and what are they? So, this is known as mass flux right or mass current as I said. This one is nothing but momentum flux you remember and this one should be the energy flux.

So, this term $\frac{\gamma p_1}{(\gamma-1)\rho_1}$ comes due to the polytropic term and this term is called the enthalpic contribution. Maybe you can relate it to your knowledge of thermodynamics.

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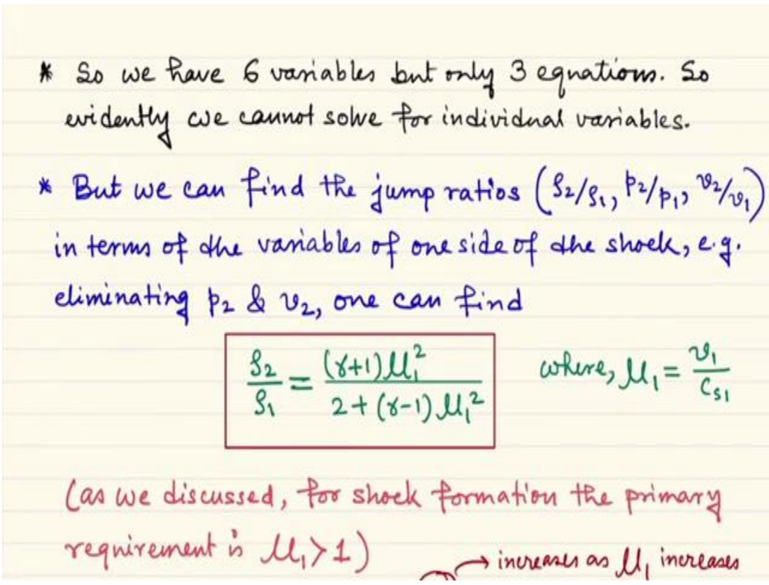

But if you just can see what how they look like there, so, 3 equations and we have 6 unknowns; ρ_1, ρ_2, v_1, v_2, p_1, p_2. So, we cannot solve for individual variables, but what we can do? We can find the jump ratios; $\frac{\rho_2}{\rho_1}$, $\frac{p_2}{p_1}$ and $\frac{v_2}{v_1}$ in terms of the variables of one side of the shock.

For example, if you eliminate p_2 and v_2 and you do some steps of straight forward algebra although. Then you can show, that $\frac{\rho_2}{\rho_1}$ is nothing but $\frac{(\gamma+1)\mu_1^2}{2+(\gamma-1)\mu_1^2}$.

So, although this is a ratio of two things, but this ratio is now expressed only in terms of the variables of the first medium. What is the meaning of first medium? First medium just means that the part of the which is in front of the shock. Because μ_1 is nothing but $\frac{v_1}{C_{s1}}$ and this C_{s1} contains the information about p_1 and ρ_1. Because this is a polytropic medium.

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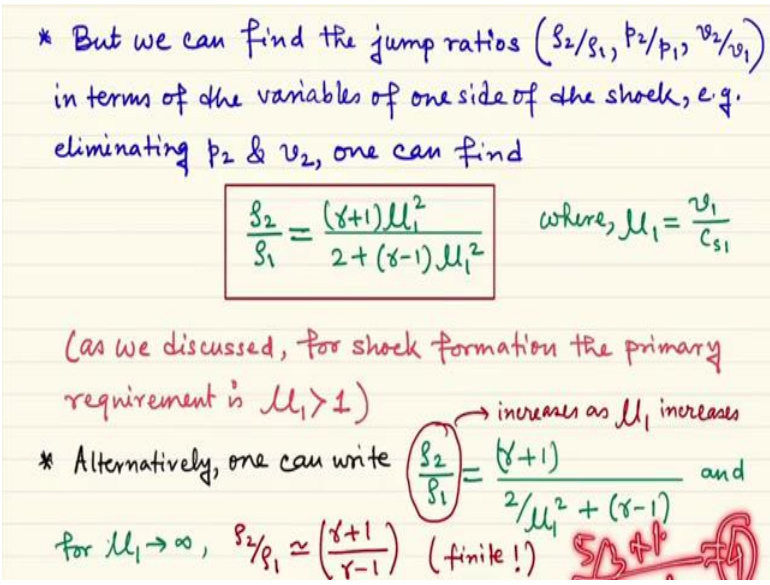

We also assume that for the both sides of the shock which is completely intuitive that polytropic index is actually same because they are the same fluid finally. So, as we discussed that for shock formation the primary requirement is that Mach number should be greater than 1. Otherwise, there will be no accumulation of disturbances or perturbation.

So, this is the supersonic case. For subsonic case, there is no provision of having a shock because every damage is sufficiently repaired by sound waves. Now, alternatively, in this from of equation we can write $\frac{\rho_2}{\rho_1}$ is equal to just by dividing both sides I mean numerator and the denominator by μ_1^2. You can write this is equal to $\frac{(\gamma+1)\mu_1^2}{2+(\gamma-1)\mu_1^2}$.

Now, there are two or three very interesting inferences. The first one is you see that this one is somehow some quantity which is μ_1 greater than 1. Now, if μ_1 is greater than 1, so, this is true that you can easily see that as μ_1. So, I mean it does not have to be μ_1 increase.

But if μ_1 is greater than 1, one simple thing you can see that μ_1 can go to some values I mean let us say large enough values like infinity, and if μ_1 tends to infinity, then the interesting thing is that sometimes we can think that if μ_1 goes to infinity maybe, from this type of expression it may happen that some of us can think that $\frac{\rho_2}{\rho_1}$ will also be infinity, but that is not the case.

Then you have to actually come to this expression because there is an μ_1^2 here and here as well. So, we have to make divide both the numerator and denominator by this so that you have only μ_1^2 over there, and when μ_1 is tending towards infinity this term will be getting smaller and smaller approaching 0, and then this limiting jump will be simply given by $\frac{(\gamma+1)}{(\gamma-1)}$. So, this is finite.

So, now, if you have some mono atomic gases and when we are considering that the idea of the system of the total flow field is now in an adiabatic process, and then what is γ? γ is $\frac{5}{3}$, it simply says that.

Now, there is another thing. If μ_1 increases this term will decrease and if this term will decrease the whole fraction will increase. So, with μ_1 this jump is also increased.

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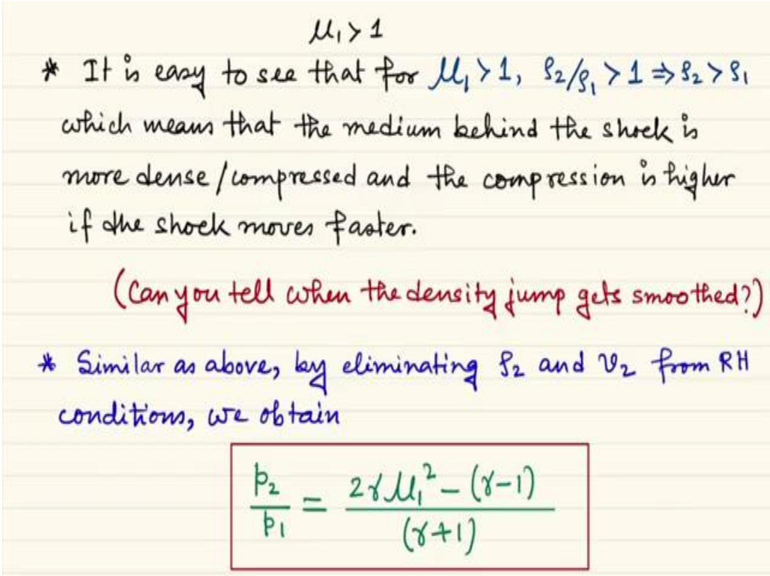

Now, you see that is also another important point. It is easy to see that for μ_1 greater than 1, $\frac{\rho_2}{\rho_1}$ is actually greater than 1. Why is that? If μ_1 is greater than 1, actually one can show that. So, just have a look. You can see that this $\frac{2}{\mu_1^2}$ that will be less than 2, and when μ_1 is exactly 1 then this $\frac{\rho_2}{\rho_1}$ is exactly equal to 1.

So, when μ_1 is greater than 1 this fraction is less than 2 and that means that this will be less than $(\gamma + 1)$. So, something divided by less than this will always be greater than 1. So, that is the reason.

So, ρ_2 will be then greater than ρ_1, and it means that the medium behind the shock is much denser or compressed and the compression is higher if the shock moves faster. Because once again, so, $\frac{\rho_2}{\rho_1}$ increases as μ_1 increases, and what is the meaning of μ_1 increases? That means, the μ_1 means this is $\frac{v_1}{C_{s1}}$, of course, and the shock speed from the undisturbed medium is nothing but $-v_1$. So, the magnitude is v_1. So, that is the thing that μ_1 increases means v_1 increases. So, in other ways, μ_1 can increase due to various reasons, but if shock moves faster then μ_1 increases. Because v_1 increases and then you can say that this ratio increases as well.

Now, it is my question to you that when the density jumps get smoothed? I have already said that the density jumps get smoothed when this ratio is simply 1 and that is the case where μ_1 is nothing but equal to 1, and when μ_1 is less than 1 this formula is not at all valid then that is actually not possible, and that is what I am saying that means, μ_1 greater than 1 is the physically possible case.

So, it is never possible that ρ_2 should be smaller than ρ_1. Once you have ρ_2 is equal to ρ_1 then the shock is vanishing, and there is no way to go to the possibility that ρ_2 is less than ρ_1. Because once the shock is vanishing then the fluid flow is smooth then there is no need for the fluid to make any again a discontinuity in the reverse sense.

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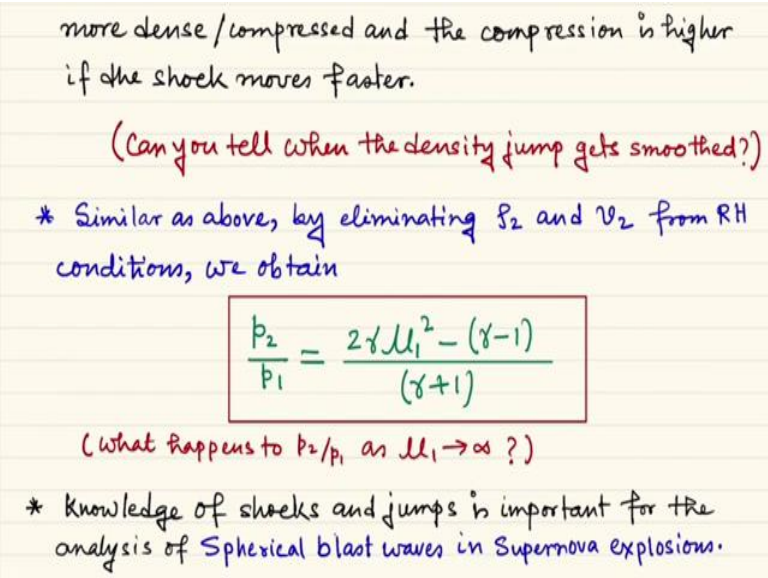

Now, similarly as above, there is one very striking result. Similarly, as above by eliminating ρ_2 and v_2 from Rankine Hugoniot conditions one can obtain

$$\frac{p_2}{p_1} = \frac{2\gamma\mu_1^2 - (\gamma-1)}{(\gamma+1)}.$$

This exercise I somehow suggest you to do at least try at home. I will send you the calculations later, but before that try at home. These are very I mean two elegant exercises.

But if you do that now, you see that when μ_1 tends to infinity, what happens to $\frac{p_2}{p_1}$? Now, you can see that μ_1 tends to infinity simply says that this ratio will go to infinity.

Whereas, for infinite Mach number, the density has jumps have a finite value, the pressure jumps become infinity. Why? Please think at that point. What is happening? Why pressure jump is infinity and what is the meaning of that?

Now, for our case, for our astrophysical fluid context, it is much more important rather than deriving all these conditions, and it is much more important to apply these conditions.

Also, you will see that to apply this type of things for analyzing different interesting astrophysical phenomena, and in the scope of our course one such phenomena we will discussed. We will discuss in the next lecture that will be the analysis of the spherical blast waves, which are observed during the explosion of very much energetic supernova. Because you will see that the shocks and discontinuities will be very much prominent there.

So, even using this one dimensional elementary analysis, you can have very interesting result and very interesting analytical advancement in explaining the splendid phenomena of supernova explosion . So, that I will do in the next session.

Thank you very much.