Introduction to Astrophysical Fluids
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Lecture – 24
Weak perturbation in a compressible fluid: sound waves

Hello and welcome to another lecture session of Introduction to Astrophysical Fluid. In this lecture, we mainly discuss the response of a compressible fluid to a very weak external perturbation.

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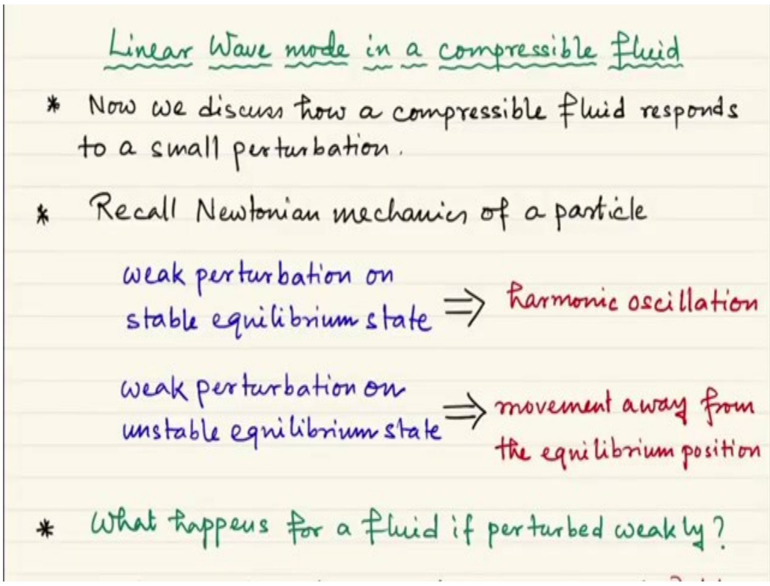

So, for that first we will remember what happens, if we perturb a particle in the framework of Newtonian mechanics. So, if the particle is perturbed very weakly with respect to some stable equilibrium state, then the particle actually performs a harmonic oscillation with respect to its mean position of equilibrium or position of equilibrium and if the particle is perturbed very weakly with respect to some unstable equilibrium state then the particle moves in such a way that it always goes farther from its original equilibrium position.

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to a small perturbation.

* Recall Newtonian mechanics of a particle

weak perturbation on stable equilibrium state $\Rightarrow$ harmonic oscillation

weak perturbation on unstable equilibrium state $\Rightarrow$ movement away from the equilibrium position

* What happens for a fluid if perturbed weakly?

stable initial state $\Rightarrow$ Linear wave mode
unstable initial state $\Rightarrow$ Linear instability

We talk later in detail.

Analogous things can be expected here as well. For a fluid if the fluid is somehow at some stable initial state and then some weak perturbation is performed or is carried out on it then we can see a linear wave mode. So, linear wave mode, if you think this is nothing but a generalization of this type of oscillatory motion which propagates in space and time.

If the fluid is perturbed with respect to some unstable initial state, then what we get is known as the linear instability. We will come to these two things in more detail when we will talk about the wave modes and instabilities in a general framework.

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* Let us see what happens if we perturb a polytropic fluid ($P = K\rho^{\gamma}$) very weakly.

* The equations of a polytropic fluid are given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho}$$

(in the absence of external force & viscosity)

* Now we perturb on a state $\rho = \rho_0$, $p = p_0$ & $\vec{v} = \vec{0}$ (the initial state is at rest)

ρ_0, p_0: constants

* After perturbation, $p = p_0 + p_1(\vec{r}, t)$

Now, for the time being we just discuss the what happens or how a polytropic fluid responds to a very weak perturbation and so, let us start with very basic polytropic equations. First one is of course, the well-known continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) = 0$$

and the second one is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho}$$

this is the momentum evolution equation. So, this equation is written for simplicity in the absence of any external force and viscosity. These two effects are neglected here. We will just see that if such a system is perturbed with respect to some initial condition, then what happens.

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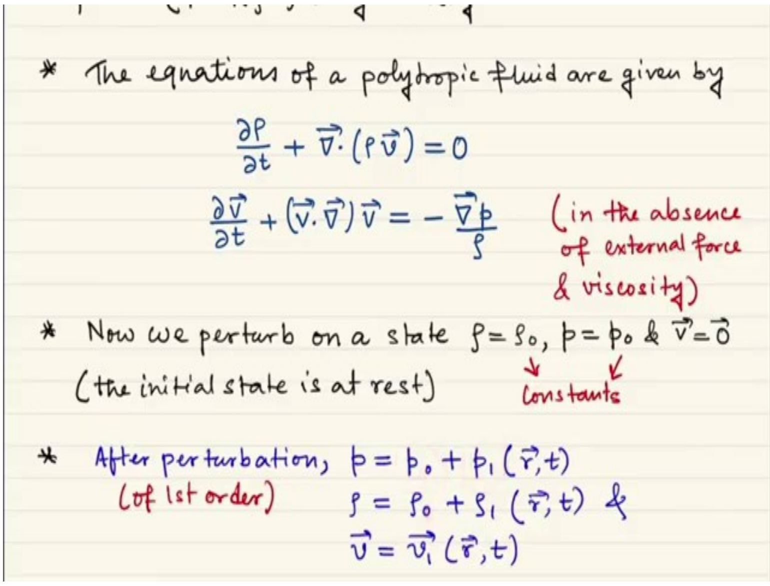

Now, it is true that and as you will see that we will make the initial state in such a way that it is actually a stable initial state. So, what is the most intuitive stable initial state? That is the state is at rest. Then, we will actually decide to perturb the fluid with respect to an initial state where ρ is equal to ρ_0 which is a constant; p is equal to p_0, which is a constant and $\boldsymbol{v}$ is equal to 0; that means, the system is at rest.

A fluid for example, is confined in a container. So, throughout the fluid, it has one density, one pressure and that is the case of our normal thermodynamics. Now, we perturb a bit and we let the system flow. So, the perturbations are done very weakly, so that they are called first order perturbation.

This type of perturbation theories are also called the first order perturbation theories that means, the perturbations are so small that only the linear order is kept. Now, we just call the equilibrium quantities with index 0 and the perturb quantities with index 1, okay. So, equilibrium quantities were all constants and for $\boldsymbol{v}$, it is not only constant but is actually 0.

For pressure and density, the perturb quantities are p_1 and ρ_1, both are functions of r space and time and for v, the nonzero part comes due to its perturbation which is v_1, which is also a function of r and t.

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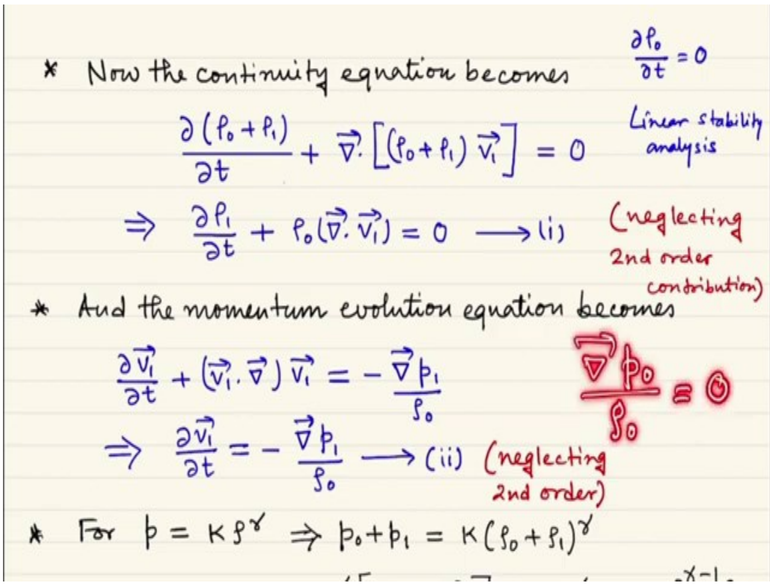

Now, if we perturb the system in this way, then the continuity equation should look like this

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \boldsymbol{\nabla}.\big((\rho_0 + \rho_1)\boldsymbol{v_1}\big) = 0$$

the total fields are now decomposed into its equilibrium value plus some fluctuation. If you do the same thing for momentum evolution equation then you will have

$$\frac{\partial \boldsymbol{v_1}}{\partial t} + (\boldsymbol{v_1}.\boldsymbol{\nabla})\boldsymbol{v_1} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0}$$

What happens for 0th order case? 0th order case v is 0. So, what was the continuity equation for equilibrium? It was simply $\frac{\partial \rho_0}{\partial t} = 0$, because the other term is 0 since v_0 is 0.

So, we can use this one $\frac{\partial \rho_0}{\partial t} = 0$ in this equation $\frac{\partial(\rho_0+\rho_1)}{\partial t} + \boldsymbol{\nabla}.\left((\rho_0 + \rho_1)\boldsymbol{v_1}\right) = 0$ and finally, we will see that we will have three terms; $\frac{\partial \rho_1}{\partial t}$ plus $\rho_0 \boldsymbol{\nabla}.\boldsymbol{v_1}$ and then $\boldsymbol{\nabla}.\rho_1\boldsymbol{v_1}$. But, $\boldsymbol{\nabla}.\rho_1\boldsymbol{v_1}$ includes two terms of first order smallness that means the composite term is of 2nd order smallest that we neglect here. We are doing here a linear analysis.

Linear analysis; linear sometimes we said linear stability analysis. In several literatures you can see this type of vocabularies. When you do that, you will only have two terms

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla}.\boldsymbol{v_1} = 0 \qquad (i)$$

If you linearize the momentum evolution equation, then you will have

$$\frac{\partial \boldsymbol{v_1}}{\partial t} + (\boldsymbol{v_1}.\boldsymbol{\nabla})\boldsymbol{v_1} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0}$$

that is simply because v_0 is identically 0, then momentum evolution equation in steady state is given by this equation $\frac{\boldsymbol{\nabla} p_0}{\rho_0} = 0$.

So, that exactly we incorporate here and we will have this equation $\frac{\partial \boldsymbol{v_1}}{\partial t} + (\boldsymbol{v_1}.\boldsymbol{\nabla})\boldsymbol{v_1} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0}$, but this is not yet linearized because we have a term $(\boldsymbol{v_1}.\boldsymbol{\nabla})\boldsymbol{v_1}$, containing two terms of 1st order smallness which is a 2nd order smallness again, we neglect that and we finally, have an equation which has a smallness of order 1.

$$\frac{\partial \boldsymbol{v_1}}{\partial t} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0} \qquad (ii)$$

So, the advection term $(\boldsymbol{v_1}.\boldsymbol{\nabla})\boldsymbol{v_1}$ which we call the advection term for Navier-Stokes equation does not appear in the linearized form. So, now we have these two linearized equations (i) and (ii).

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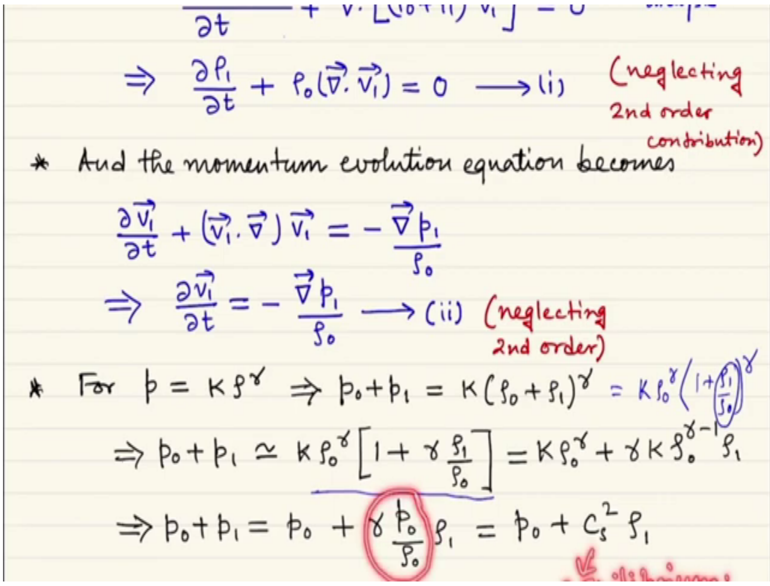

Here we are considering a polytropic fluid, for polytropic fluid you can have $p = K\rho^\gamma$ and p you can write as $p_0 + p_1$ and ρ you can write as $\rho_0 + \rho_1$. So, K into $(\rho_0 + \rho_1)^\gamma$ and then since ρ_1 is very very small with respect to 0; one order smaller, then you can just take ρ_0^γ to outside of the bracket and inside the bracket you will have $\left(1 + \frac{\rho_1}{\rho_0}\right)^\gamma$. So, this is actually, $K\rho_0^\gamma \left(1 + \gamma \frac{\rho_1}{\rho_0}\right)$, since this $\frac{\rho_1}{\rho_0}$ is a very small quantity.

$$p_0 + p_1 = K\rho_0^\gamma + K\gamma\rho^{\gamma-1}\rho_1$$

If you see that what this is? This $K\rho^{\gamma-1}$ this is nothing but $\frac{p_0}{\rho_0}$ and $K\rho_0^\gamma$ is p_0.

So, finally, your thing will come up

$$p_0 + p_1 = p_0 + \frac{\gamma p_0}{\rho_0}\rho_1$$

And what is this $\frac{\gamma p_0}{\rho_0}$? This is nothing but C_s^2, it is called the equilibrium sound speed $C_s^2 = \frac{\gamma p_0}{\rho_0}$? So, this total thing will then come down to be $p_0 + C_s^2\rho_1$.

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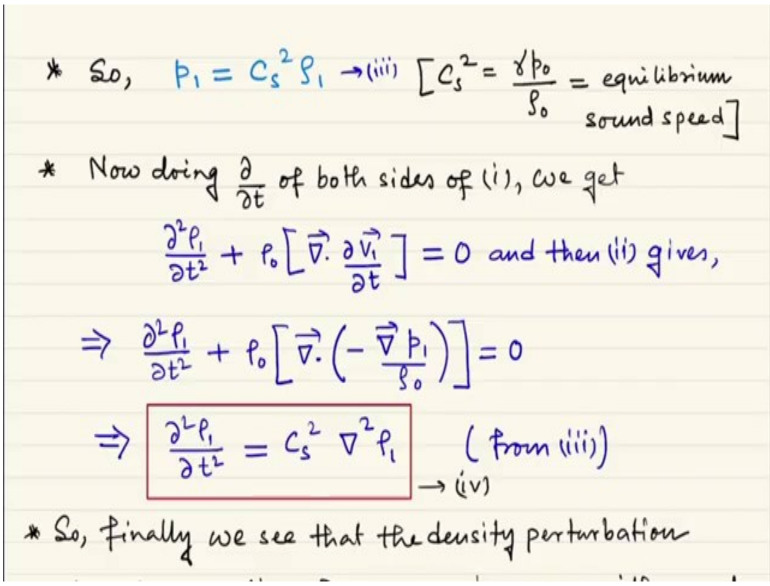

And now we can write this p_1 in terms of ρ_1 as

$$p_1 = C_s^2 \rho_1 \qquad (iii)$$

So, that is something you have to understand and why this is interesting? Because you can simply see that although normal p and ρ they were not proportional, they were actually satisfying some polytropic condition but their first order perturbations are actually proportional to each other and the proportionality constant is nothing but the equilibrium sound speed square. Now, if we take $\frac{\partial}{\partial t}$ of both sides of (i), you will have

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \boldsymbol{\nabla}.\frac{\partial \boldsymbol{v_1}}{\partial t} = 0$$

$\frac{\partial}{\partial t}$ and divergence will commute. $\frac{\partial \boldsymbol{v_1}}{\partial t}$ you can substitute from equation (ii), and you will get, $\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \boldsymbol{\nabla}.\left(-\frac{\boldsymbol{\nabla} p_1}{\rho_0}\right) = 0$. So that is the circuit of logic okay.

Now, you will have an equation for ρ_1 which is nothing but

$$\frac{\partial^2 \rho_1}{\partial t^2} = C_s^2 \nabla^2 \rho_1 \qquad (iv)$$

where C_s^2 comes due to this fact that p_1 is now substituted in terms of ρ_1 .

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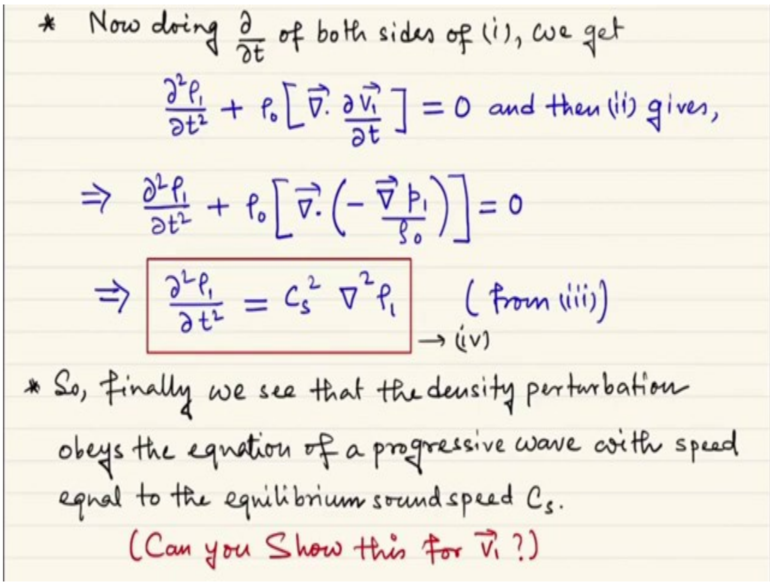

So, you see that ρ_1 satisfies an equation of a progressive wave with the speed of the wave equal to C_s. So, we see that the density perturbation obeys the equation of a progressive wave with speed equal to equilibrium sound speed. My question is, can we also do this for perturbation of velocity and perturbation of pressure?

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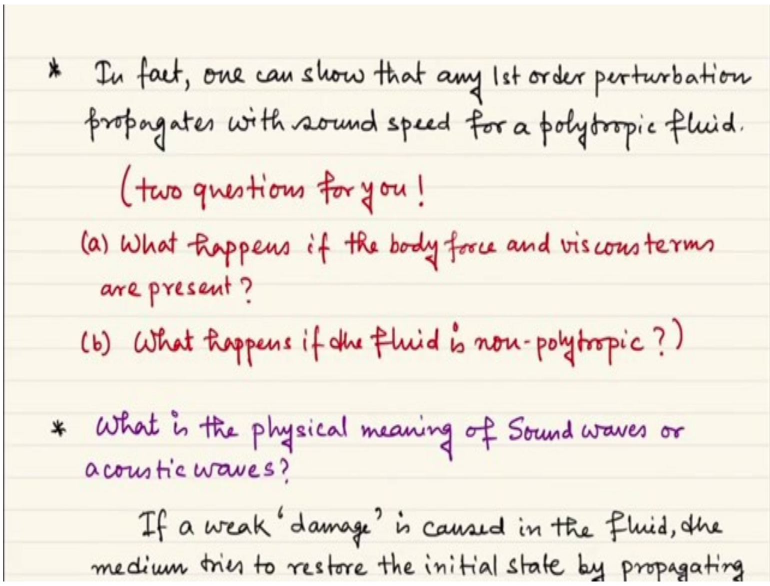

And actually, the good news is that one can indeed show that for any 1st order perturbation, in this case for a polytropic fluid normally, in the absence of any forcing or any viscous dissipation term; any 1st order perturbation propagates with the speed of sound, but this speed is the equilibrium sound speed that is true to understand.

Now, I have two questions for you: the first question is that what happens if the body force and the viscous terms are present? then what happens to the wave? That you would to think and search ok that is the research outlook, I want that you develop okay. The second one is that what happens if the fluid is non-polytropic?

So, you know there is a broader class which is called barotropic and there can be actually more general class where the pressure is not at all function of density only. So, then we have to use some more general consideration. So, if you think that how can one treat this type of thing that means, if some weak perturbation is done to a non-polytropic fluid, then how the system would respond okay?

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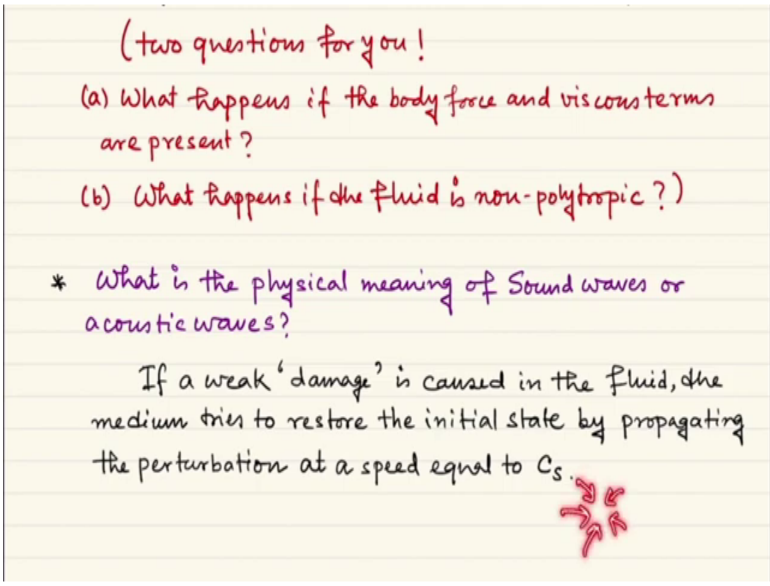

Now, coming back to our original discussion ok, what is the physical meaning of sound waves or acoustic waves? Sometimes the sound waves they are called acoustic waves as well. So, it simply means that if a weak perturbation or damage is caused inside the fluid, by altering its velocity or its density or its pressure actually, all are related, then if it is done with respect to a stable equilibrium state then the medium tries to restore the initial state by propagating the perturbation at a speed equal to C_s. So, for example, if you just reduce the density of one fluid at rest at some point in space, then the other places which have the higher density because initially equal density at all parts.

Now, if you reduce the density very locally, then the surroundings will feel that I mean inside then there is a position or there is a space where the density is somehow reduced by some external cause or however.

Then, the matter from them would come and to try to equal equilibrate the density discrepancy and that will be done at a speed equal to C_s and that is only true when the perturbation amount is very weak. So, if the perturbation amount is not very weak, then this is not evident.

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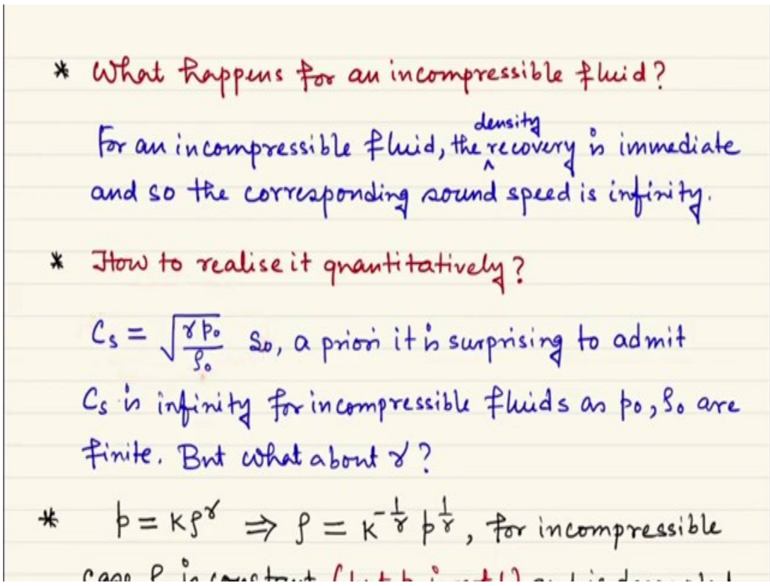

Now, one simple question is that what happens for an incompressible fluid? So, can we somehow from this polytropic fluids, can we somehow do some appropriate limit by which we can reach to incompressible fluids? Now, for an incompressible fluid the recovery is immediate that is true because for an incompressible fluid, you cannot change your density.

So, whether the system is in equilibrium state or not; the ρ is always constant. Whether the system is at rest or it is flowing irrespective of that the systems ρ is constant. So, if you change the ρ somehow, so it is not any recovery, but it is I have to be much more careful, it is the density recovery is immediate okay.

If the density recovery is immediate, then the corresponding sound speed is given by infinity because the time required to repair the damage is zero. But, how to realize this quantitatively?

Quantitatively, if we say that the sound speed is infinity, we just know that this is nothing but from our definition as we have seen over here that $C_s^2 = \frac{\gamma p_0}{\rho_0}$ is the equilibrium sound speed So, how can C_s be infinity?

So, p_0 is somehow finite and ρ_0 is a constant but finite and then how can C_s be infinity? At first look it may be very surprising, but if you think carefully, you will see that the mystery is lying under this γ. Now, this γ basically has a role in making the sound speed infinity. Why?

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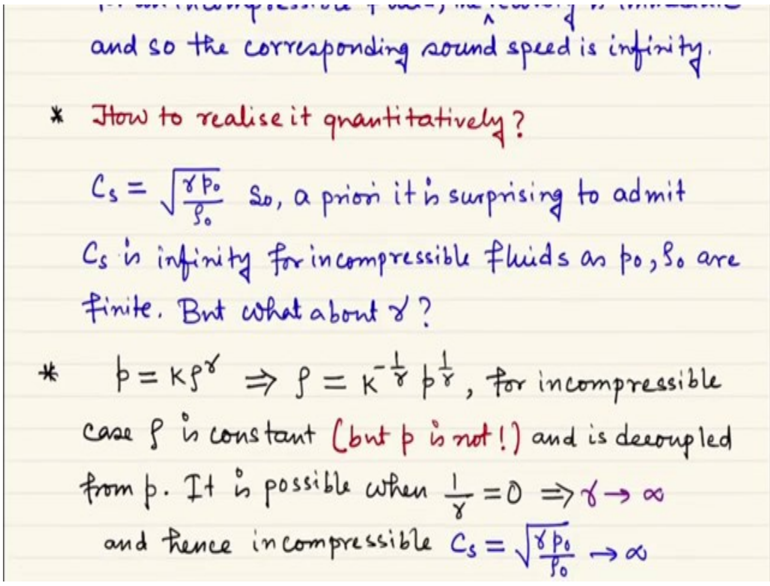

Just think that we have a polytropic fluid $p = K\rho^{\gamma}$. Then, γ, we can write is equal to $K^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$. Now, for incompressible case $\frac{1}{\gamma}$ is always constant but p is not. So, if p somehow changes and K is a constant, what will be the value of γ, so that ρ is always unchanged? That is only possible when you have p to the power 0. So, then

$$\frac{1}{\gamma} = 0 \Rightarrow \gamma \to \infty$$

and this γ while it is sitting on the numerator makes this equilibrium sound speed to be infinity.

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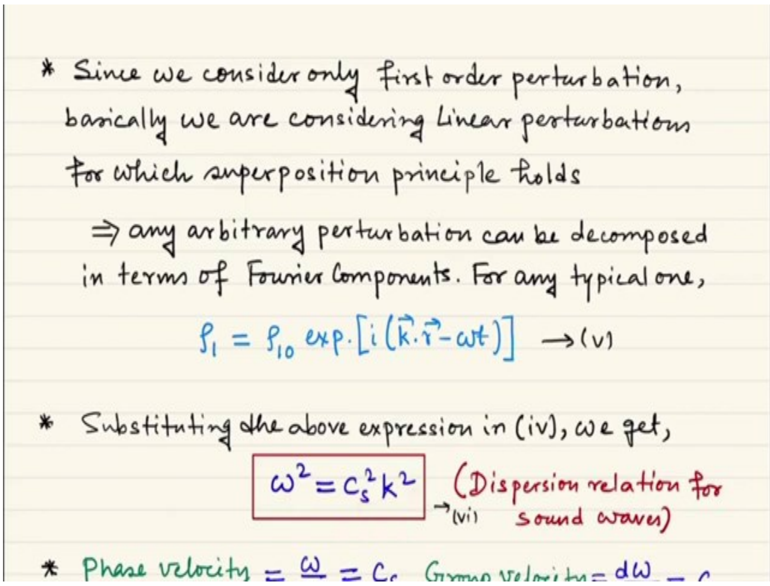

Now, coming to another interesting topic that is we are considering here only first order perturbations ok. So, these perturbations are linear in nature, I mean that as we just saw that in the equation there were no non-linear terms and for that then the superposition principle holds good and that is why we all know that any arbitrary perturbation can be decomposed in terms of Fourier components.

Like any other linear function and for any typical one Fourier mode where we are talking about ones single $\boldsymbol{k}$, then, what happens, we can write the perturbation like a plane wave solution with some amplitude as

$$\rho_1 = \rho_{10} \exp\,[\mathrm{i}(\boldsymbol{k}.\boldsymbol{r} - \omega t)]$$

and if you now substitute this expression in equation (iv) which is nothing but given here $\frac{\partial^2 \rho_1}{\partial t^2} = C_s^2 \nabla^2 \rho_1$, this one, you will simply see that this equation gives you an algebraic equation

$$\omega^2 = C_s^2 k^2$$

So, a differential equation is reduced to an algebraic equation.

There is a relation between ω and k ,where ω is the frequency and k is the wave number. This type of relation between algebraic relation between ω and k is known as the dispersion relation. I mean it is true that in complicated cases, there may not be algebraic relations that can be a transcendental relation.

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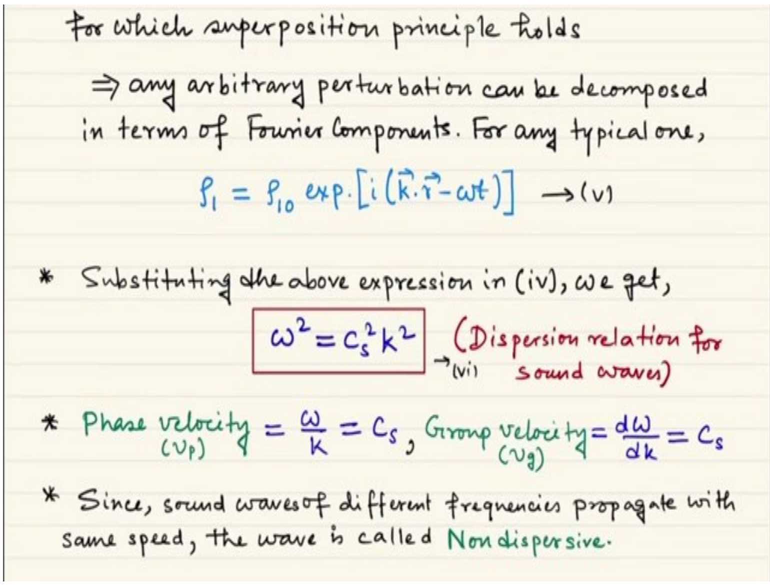

Now, if we have this type of relation, then what happens? Then the phase velocity which is nothing but the ratio of ω by k, so these vocabularies, I expect that you have heard because I expect that all of you have physics background. So, phase velocity $\boldsymbol{v_p}$ is nothing but that is just the velocity given by $\frac{\omega}{k}$ for a given mode. For this case phase velocity is sound velocity C_s itself and group velocity is given by $\frac{d\omega}{dk}$ for 1 D case and is also the sound velocity.

This is not always true for any arbitrary wave mode because for arbitrary wave mode, the dispersion relation can be much more complicated than this. Now, for very simple case here we have both the phase velocity and the group velocity they are equal. Now you see that finally, the speed of the sound does not depend on frequency.

Because if ω is different ω, then k will be different to make $\frac{\omega}{k}$ constant and this is the nothing but the equilibrium sound speed C_s. So, that is why we say that sound wave is such a wave for which the phase speed does not depend on the frequency of the wave and that is why this is known as the non-dispersive wave that means the dispersion does not take place.

What happens, for example, in a prism when you have different colors, you have different velocities of light and then what happens? Then for different wavelengths, you have different velocities. Then you have dispersion in a prism, so you can see the colors split. Here for sound wave, that cannot split because of this relation $\frac{\omega}{k} = C_s$. So, this type of way is called a non-dispersive wave ok.

Now, it is also true that this non dispersive effect only comes because we now, just try to connect with my previous question that what will happen if we consider the forcing and the viscous dissipation into effect. Maybe, the dispersion relation is not the same ok, maybe the wave is no longer dispersive, this type of thing.

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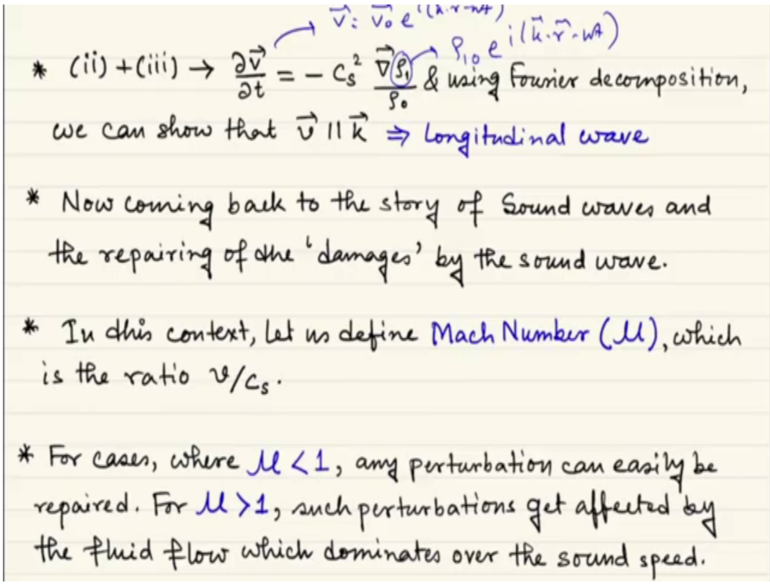

Another point about the sound wave or acoustic wave is that if we just add (ii) and (iii)

$$\frac{\partial \boldsymbol{v_1}}{\partial t} = -\frac{C_s^2 \boldsymbol{\nabla} \rho_1}{\rho_0}$$

Here $\boldsymbol{v}$ and $\boldsymbol{v_1}$ are same since mean velocity is zero. Now, if you do the Fourier decomposition, if you just write this also in terms of its plane wave solution using the Fourier mode, we can show that $\boldsymbol{v}$ will be proportional to $\boldsymbol{k}$ because $\frac{\boldsymbol{\nabla}\rho_1}{\rho_0}$ will be will be something directed $\boldsymbol{k}$ vector.

That you can just do; just for here you have to use $v_1 = v_{10}\exp[\mathrm{i}(\boldsymbol{k}.\boldsymbol{r} - \omega t)]$ and $\rho_1 = \rho_{10}\exp[\mathrm{i}(\boldsymbol{k}.\boldsymbol{r} - \omega t)]$ some according to our convention. It is a small exercise; you will see that $\boldsymbol{v}$ will be simply proportional to or parallel to $\boldsymbol{k}$ vector and that says that acoustic wave is a longitudinal wave okay.

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we can show that $v \parallel k$ → longitudinal wave

* Now coming back to the story of Sound waves and the repairing of the 'damages' by the sound wave.

* In this context, let us define Mach Number ($\mathcal{M}$), which is the ratio v/c_s.

* For cases, where $\mathcal{M} < 1$, any perturbation can easily be repaired. For $\mathcal{M} > 1$, such perturbations get affected by the fluid flow which dominates over the sound speed.

* The perturbation is no longer Linear then and the superposition principle does not hold any longer.

Now, coming back to the story of sound waves and the repairing of the damages by the sound wave. Before going to that let, me introduce a number dimensionless number called the Mach number M which is the ratio of the fluid speed by the sound speed $M = \frac{v}{c_s}$.

And for cases where this Mach number is less than 1, any perturbation can easily be repaired and for Mach number greater than 1 such perturbations cannot be repaired easily. Why? Because when Mach number is less than 1, then the sound speed is much more efficient. So, if let us say if at some point some density perturbation occurs, then the sound speed will repair it very fast and after that the fluid will reach there, ok, so further flow will be there okay.

So, in case of M greater than 1, before the repairing is done; some fluid comes over there to aggravate eventually this perturbation. A very easy and simple example of that let us say you are on a seashore and you are making some design; let us say you have you have made some design or you have written your name on the sand. Now, the sea wave came and you see that some part of your name was damaged.

Now, you immediately you tried your best to repair those damaged part, but before you complete your work if you are not fast enough ok, the second wave came and it again made your effort in vain. So, you have to be quick enough so that between two successive waves every time you can repair the damage made. If Mach number is less than 1, then you can efficiently repair the damages ok.

In case, you are not fast enough that is the Mach number is greater than 1 there is the fluid matter is moving actually from one point to the other without caring of the damage ok. For example, when the sea wave if came it does not care about which damage it has made or something ok you are caring. So, sound wave cares for repairing the damage not the fluid itself, the fluid matter just flows and at that point, any such weak perturbation can actually be aggravated and then the assumption that the perturbation is weak and can be treated linearly does not hold any longer. Then you can simply see that the perturbation is no longer linear and the superposition principle does not hold ok.

At this point one has to feel that part; there will be some accumulation of perturbations inside the fluid ok. One very simple treatment of addressing this accumulation of perturbation is the treatment of shocks and that we will discuss in the next discussion ok.

Thank you very much.