Introduction to Astrophysical Fluids
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Lecture – 23
Accretion disks II

Hello, and welcome to another lecture of Introduction to Astrophysical Fluids. In this lecture, we will discuss the dynamics of accretion disks in astrophysics. Before starting, I try to tell you a very important point that here in this context of Accretion Disk we will do some modeling.

When we will do some modeling, sometimes it may happen that the steps are not mathematically rigorous. So, at that point we have to remember that we will try to match some of our results or our mathematical structure or model with some known phenomena or observed phenomena.

That is why, if sometimes something is reasonable from our practical point of view, that means, from our observational knowledge or some common sense then I mean we will not really abide by the mathematically 100 percent at that point, and we will do something offhand approximately.

But I will tell you, there will be no secret or no hidden message of this type of approximation, whenever there will be approximation, I will tell you explicitly. But finally, the goal is to find something or to recover something, which we already observed, even if not exactly at least moderately exactly, so that is our motto or our objective for this thing.

Another point is that, in case of the accretion disks, there will be a lot of mathematical steps. So, in this course, it is not our objective to learn all the detailed mathematical steps. Sometimes, when I will say in the scope of this course to go through the mathematical steps to learn, please do that.

But for example, when I will tell here some mathematical steps, and I will say that after some algebra after some steps, in the most cases this is not really mandatory for you to go through all the steps, but rather you try to understand the final result and the physical meaning of the final result, and that is something which is important for this course.

However, if you are interested you can always go through this type of detailed calculation, searching through internet, or referring to some books which I have already mentioned, and some elements of the detailed calculation will also be communicated to you by me. So, do not worry much for these things. I will try to make clear the best possible mathematical details, but of course once again the objective is not to get lost inside the mathematical details rather try to find out the meaning out of the final results.

Even for exams, we will not ask you to do calculations to derive something long, it will be mostly using the results, which will be discussed here to show something interesting or to calculate something or to ask physics questions. So, it is finally, important that the picture of accretion disk, and the corresponding dynamics, the formation, the stability, the physics behind it should be clear.

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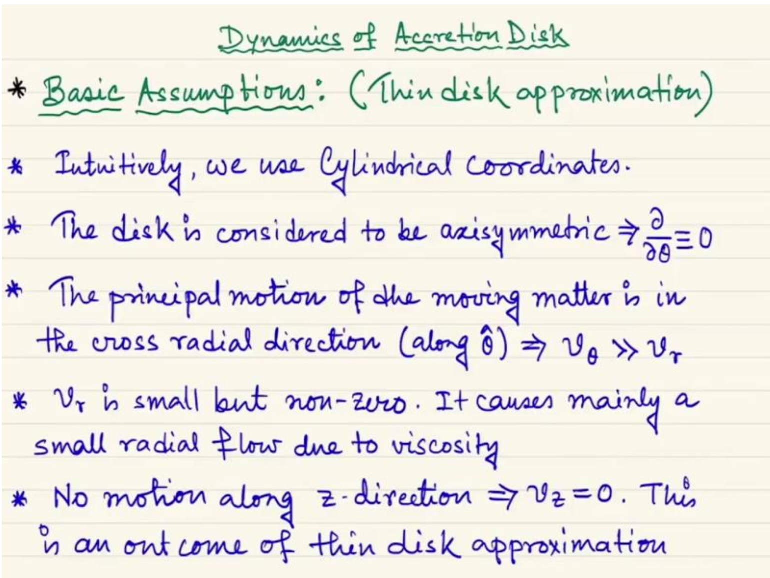

So, once again we start by capitulating the basic assumptions. As I said that for analytical convenience Sakura and Sunyaev – these two people, they propose thin disk models, and for that, we need several approximations. The first one is of course, the use of cylindrical coordinates. Then the disk to be axisymmetric, so that for any quantity $\frac{\partial}{\partial\theta}$ will be 0.

The principal motion of the moving matter will be considered mostly in the cross radial direction that is the v_θ will be the dominating component of velocity however, v_r will be non-zero, but it is very small, but v_r will cause the small radial flow due to viscosity. Now, for v_z, we will assume that to be 0.

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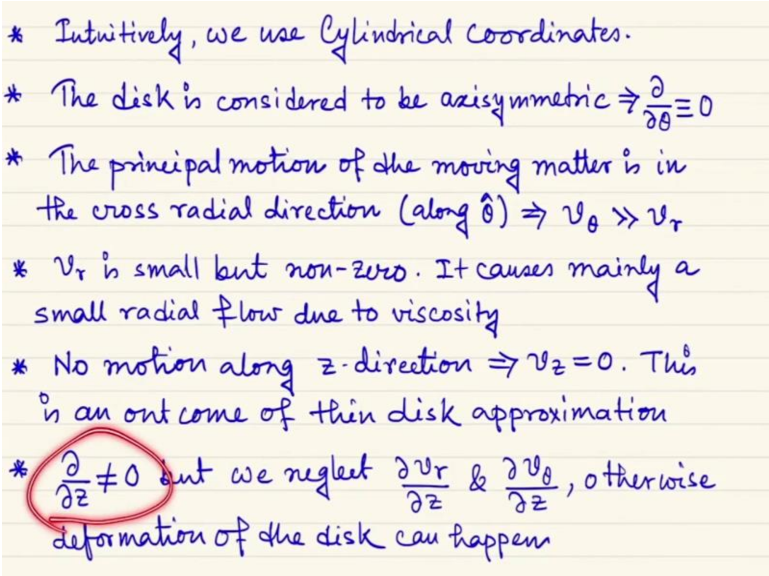

In general, we will also assume that $\frac{\partial}{\partial z}$ is not equal to 0. For example, $\frac{\partial \rho}{\partial z}$ is not equal to 0, but for the velocity components of v_r and v_z, v_θ that will be 0. Because v_r, v_θ – they are small and their changes are also assumed very reasonably to be small.

So, if it is not as I said last time then think of it. You will understand that there should be some deformation in the disk formation and that is not quite welcome in this simplified framework. Now, with all these assumptions, finally, we will go and study the dynamics of accretion disks.

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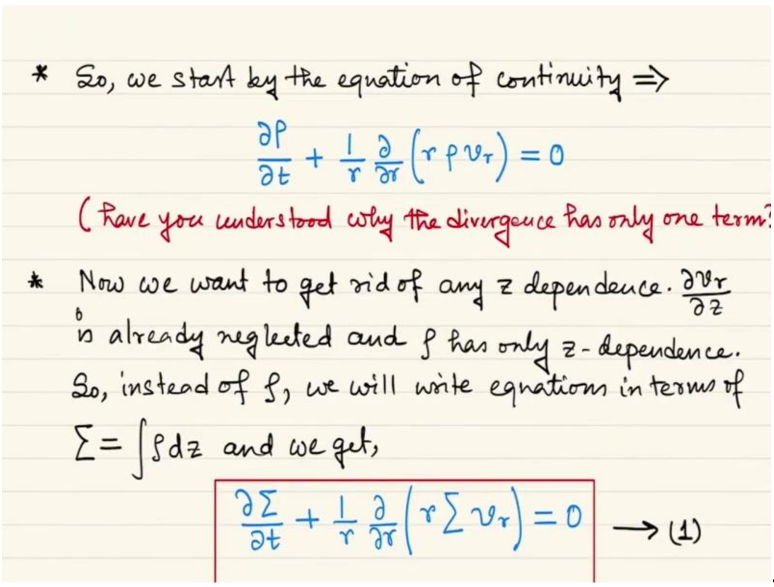

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Now, first this is nothing but the equation of continuity comes. In cylindrical coordinate, this is written as $\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)=0$. Now, this is my question to you, have you understood why the divergence term has only one term and not the del? So, the term involving $\frac{\partial}{\partial\theta}$ is 0, but the term involving $\frac{\partial}{\partial z}$ is not 0, but v_z there will be 0. So, this type of simplifications you can actually check at home.

Now, we want to get rid of any z-dependence that is because our primary assumption or the premise is that we should discuss for the dynamics of a disk whose thickness is negligible with respect to its radial extension, I mean radial dimension.

And that is why any variation about z or something is not really of our current interest, it may be interest for other research topic. But for our case, we will just try to see what happens roughly in the plane, which is perpendicular to the z-direction or that means, the $r-\theta$ plane.

For that, what we will do? We will try to get rid of any z-dependence. So, we all know that $\frac{\partial v_r}{\partial z}$ is already 0 and so $\frac{\partial v_\theta}{\partial z}$ is something, which we should be taking care of, other than that ρ has a z-dependence, and $\frac{\partial v_z}{\partial z}$ is also 0.

Now, the question is that how to get rid of the z-dependence of ρ? One very simple thing, we can do that we say that instead of ρ, we will write equations in terms of Σ. So, this Σ is not for summation that can be confusing, but just please bear with it for instance that Σ is equal to $\int\rho dz$. So, this is the density integrated along z-direction.

So, that does not have any information about the variation. Already, it is integrated over z, so any variation of ρ with respect to z will be contained in it. Finally, if we just study the evolution equation of Σ, now basically in those equations we are actually concentrating what is happening on the plane of the disk that is $r-\theta$ plane.

For that, you can actually see that this continuity equation becomes simple. So, you just multiply every term with dz, and then you integrate, and you will say you will have $\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r)=0$. So, this is a good exercise to do at home.

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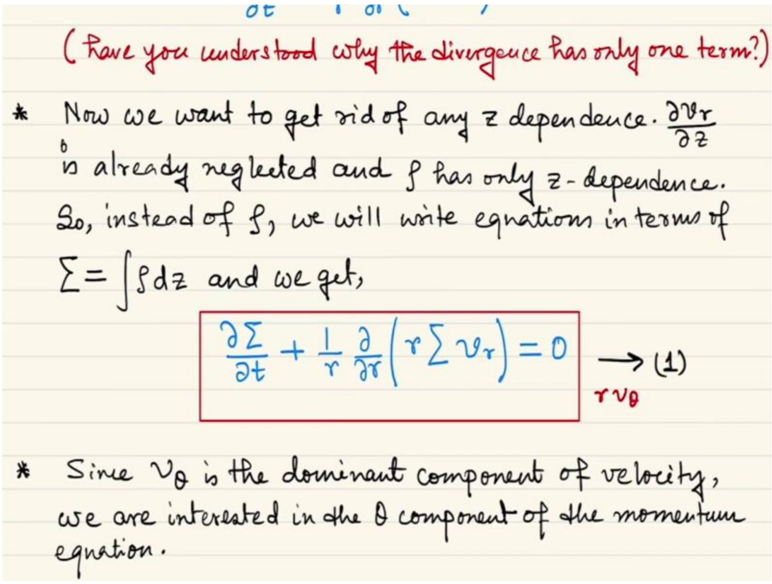

Up to this point, we are using assumptions, but we are not compromising anything with mathematically. So, up to this, this is mathematically exact. Now, this was all for the continuity equation. Now, for the momentum evolution equation, we know that v_θ is the dominant component of velocity. So, we are interested in the θ component of the momentum equation, fair enough.

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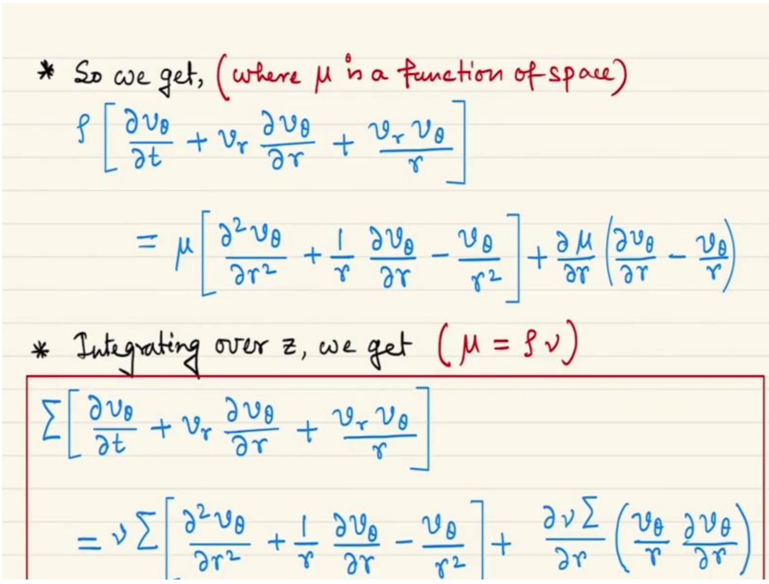

So, that is some part you can check in the books of vector algebra vector calculus and vector analysis that if you write this type of equation – Navier-Stokes's equation in cylindrical coordinate system, and where you do not use the μ to be constant.

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* So removing the part of solid body rotation, we can say that, in fact

$$\Pi_{ij} = -\mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\vec{\nabla}\cdot\vec{v})\right]$$

which one get exploiting the traceless condition.

And hence we recover the form which we obtained kinetic theory. Substituting the expression of Π_{ij}, we get,

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p + \rho\vec{g} + \vec{\nabla}\cdot\left[\mu\overline{\overline{\Lambda}} - \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{v})\overline{\overline{I}}\right]$$

where μ is not necessarily constant in space.

If you remember this, then you actually see that the θ component of Navier-Stokes equation should look like this $\rho[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}]$, that one term which comes, you can actually show is equal to $\mu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right] + \frac{\partial \mu}{\partial r}(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r})$. Now, this term $\frac{\partial \mu}{\partial r}(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r})$ is something, which comes as a result of the spatial dependence of the coefficient of viscosity.

Now, we again integrate over z to have the equations in terms of Σ, and you will see that this equation gives that Σ times the left-hand side basically does not change. So, if you just check, you will see, but the right-hand side changes a bit and we have already transformed μ is equal to $\rho\upsilon$, so that we can actually include that also in the integration.

And we will say that υ is called the coefficient of kinematic viscosity, and μ is the coefficient of dynamic viscosity. So, $\Sigma\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right] = \upsilon\Sigma\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right] + \frac{\partial\, \upsilon\Sigma}{\partial r}(\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial r})$, that is something you can actually check. So, this is also a good exercise to check.

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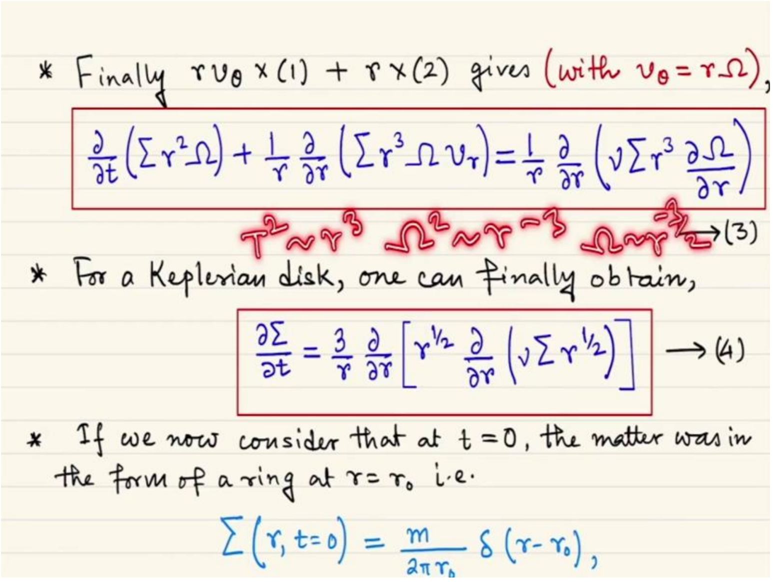

Now, if we can do something to find the evolution of the angular momentum of the disk, because the disc is somehow rotating. So, we can be interested in understanding the evolution of the angular momentum. That can be obtained if you do one trick that you multiply rv_θ, with this equation $\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0$, and you multiply simply r with this equation $\Sigma\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right] = \nu\Sigma\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right] + \frac{\partial\, \nu\Sigma}{\partial r}\left(\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial r}\right)$.

If you do rv_θ times equation 1 and r times equation 2, and you sum them up you will have $\frac{\partial}{\partial t}(\Sigma r^2\Omega)$, and what is this? This is nothing but the angular momentum density right, integrated over z, of course, plus $\frac{1}{r}\frac{\partial}{\partial r}\left(\Omega\Sigma r^3 v_r\right)$.

So, that is equal to $\frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r^3 \upsilon \frac{\partial\Omega}{\partial r}\right)$. Now, for a Keplerian disk, we know that Ω is just a function of r, and it is just you know how should it look like, that should look like $r^{-3/2}$. So, you all know that Ω should have $r^{-3/2}$, so if you remember that $T^2 \propto r^3$. So, this type of thing.

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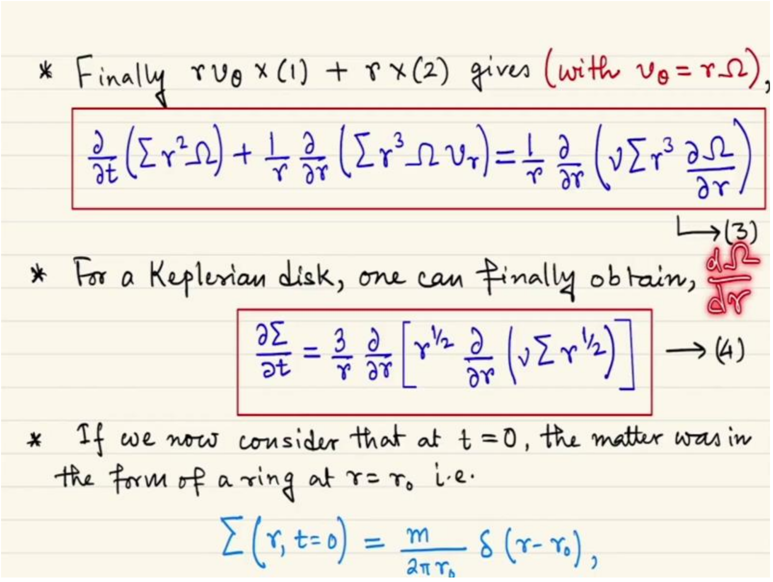

Now, if you do that, you will see that this $\frac{\partial\Omega}{\partial r}$ can actually also be written as $\frac{d\Omega}{dr}$. We also know how should it look like because of the dependence. Then for a Keplerian disk, you can further simplify this equation to get this $\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}(\Sigma r^{1/2}\upsilon)\right)$. So, from equation 3 to this $\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}(\Sigma r^{1/2}\upsilon)\right)$, I will try to communicate the analytical steps, but this is a bit cumbersome do not worry much.

If you want to show that you can enjoy the calculation. But, even in case you are lost in it, do not worry. The basic objective is to understand the final result, and the final result is not yet obtained.

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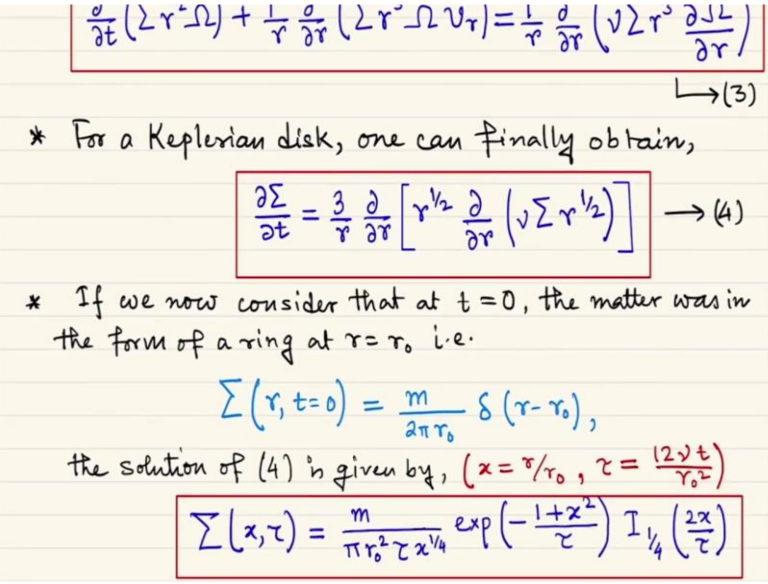

So, finally, what we can see that $\frac{\partial \Sigma}{\partial t}$. So, what we get? Try to understand what we get an equation for the Σ. So, $\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}(\Sigma r^{1/2}\upsilon)\right)$, and you see that this is an equation of evolution of the mass for a Keplerian disk. So, we needed all these things.

We used Keplerian disk formula and finally, we got this $\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}(\Sigma r^{1/2}\upsilon)\right)$. So, although it looks like sometimes, we thought that it can be directly derived from continuity equation, you can check. It is not that easy, because υ is there.

So, υ should come from your momentum equation. So, it is really obtained after combining continuity equation with the momentum equation. However, when you will see the calculations or if you do that at home, then you will check it yourself.

Now, if we at this point, this is the fundamental equation of evolution of a Keplerian disk. Now, if we consider that at $t = 0$, we are now imposing some initial condition, the matter was in the form of a ring at $r = r_0$. Then what happens, that Σ at $t = 0$ is nothing but $\frac{m}{2\pi r_0}\delta(r - r_0)$, that means, the mass is only concentrated at $r = r_0$. That means, this is a thin ring.

Now, with the initial condition, then this $\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}(\Sigma r^{1/2}\upsilon)\right)$ type of equation finally gives this type $\Sigma(\mathrm{x}, \tau)$ of solution, where x is nothing but process of non-dimensionalization of $\frac{r}{r_0}$, and r_0 is the position where the initial mass ring is supposed to be situated, and τ is nothing but another way of non-dimensionalization of the time. So, finally, in the solution you can see that the solution for Σ.

So, how Σ evolves in space and time that is given by this $\Sigma(x, \tau)$. So, you see whenever there is a τ, there is a υ, there is viscosity. So, the viscosity plays a very important role in determining this type of equation. So, what is this type of equation? It is an exponential type of equation which is $-\frac{1+x^2}{\tau}$, and it is multiplied by a modified Bessel function. So, once again do not get horrified by these mathematical mumbo jumbos.

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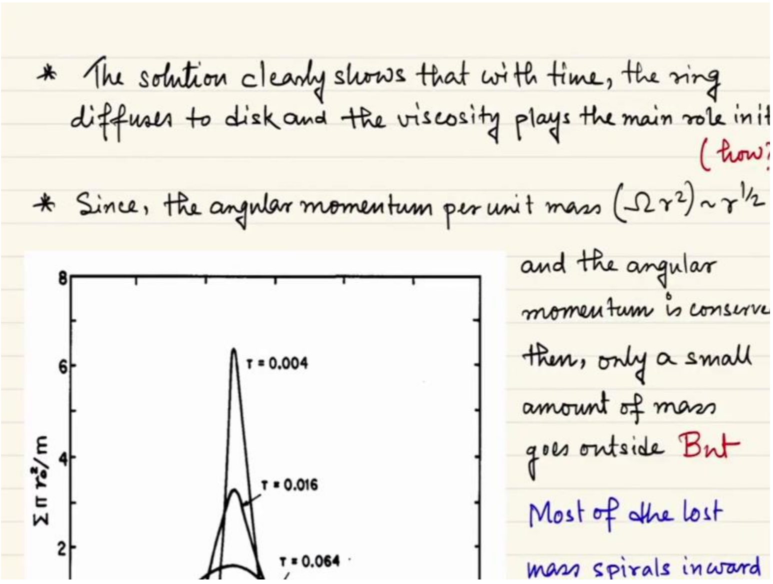

On the other hand, try to understand what it gives qualitatively, and because if you do not know how to calculate analytically, you can always calculate this type of equations numerically. So, this solution clearly shows something very interesting, and that is why we are very happy.

If you plot the solution, then you will see that at very initial position there will be a peak like this is direct peak there it starts to diffuse. So, this is 0.004, then this is 0.016, then this is 0.064, and then this is 0.256. Now, you see that finally, it happens. So, first it got started diffusing. So, basically this says that this solution clearly sees that when t increases it simply says that if this is not t, this is τ basically.

When τ increases, basically it says that this is nothing but when υ increases and t increases. It is also possible for a given t if υ increases, we can also have an increasing τ. So, that way you can see that if we just freeze the time for example, and if we just increase the I mean if we just change the viscosity coefficient for example, then we will see that this type of effect can also occur.

So, in this way, you can also think that viscosity is something which plays a very important role in transforming this peak type of mass distribution, ring type of mass distribution very localized to a diffused disk. If you do not just change here υ, υ is a constant, you just change t. You will see that exponential also includes this τ.

So, for that, basically you will see this broadening. So, this is the mathematical detailing. But, just believe, at that point the broadening with bigger and bigger τ is mainly caused by the viscosity coefficient. And not only that you will see after a certain time this the symmetric nature is already get. So, at first it was a bit symmetric this one, and then it was more or less a bit less and this one is not at all symmetric the final one – the lower one.

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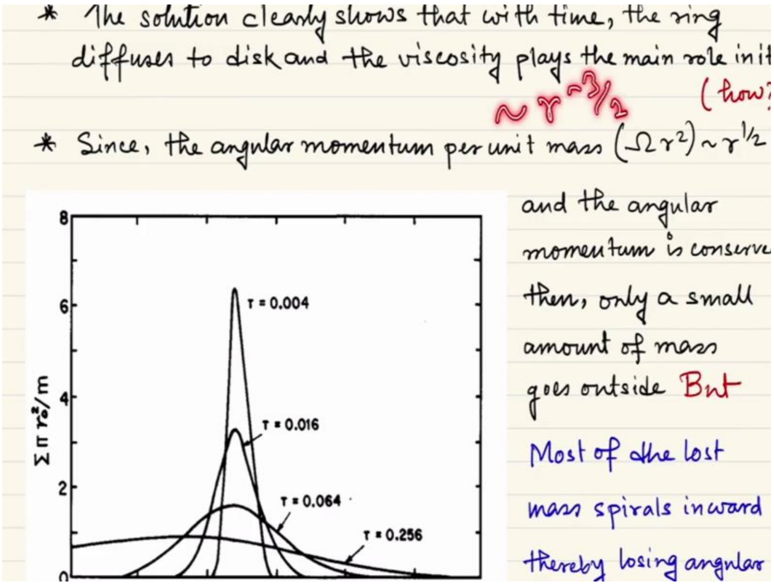

What is the meaning of that? That is because the whole system does not have any net external torque, so the angular momentum should be conserved. But we know that the angular momentum for a Keplerian disk Ω is proportional to $r^{-3/2}$, so Ωr^2 should be proportional to $r^{1/2}$.

Then the angular momentum actually should increase when some mass goes outside for larger and larger r. If the angular momentum should be conserved, then what happens? That the momentum, which is gained for some mass which is going outside should be compensated by the angular momentum lost by the mass which is spiraling inward.

As the angular momentum density or the specific angular momentum, angular momentum per unit mass, increases with r, we can actually say that only a small amount of mass can have large angular momentum, and vice versa, a large amount of mass can have small momentum, and that is why we conclude that most of the lost mass actually spirals inward thereby losing angular momentum and only a very few percent of the lost mass go outward.

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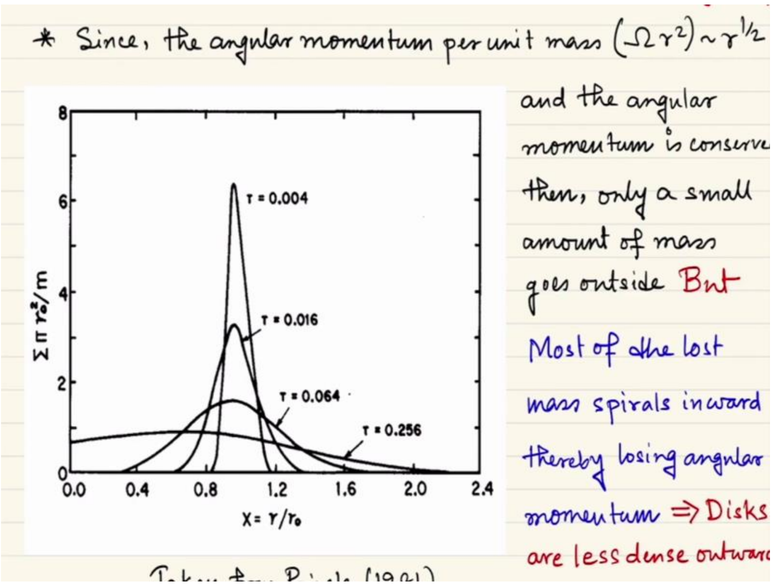

And that is why the disks are less dense outward, and that is exactly what you can see here. So, this is somehow the Σ is getting less and less when you are going to more and more away of the accretion disk center. So, now, you see that is the picture you have to understand. It is not really that you have to understand every single functional form or something at this point. If you are interested you can of course, go into the detail.

But what I am saying that this is interesting. So, there are two things to understand at this point. The first thing is that you have to see that whenever at initial point of time, you are giving some ring like very locally concentrated mass, I mean distribution. Then with time, it is the viscosity which diffuses the whole thing. Because if there is no viscosity, then there is no angular momentum transfer that we discussed last time, and that actually diffuses the whole thing from ring to disk like type of thing.

Then there is another point is that some masses are going inside and some masses are going outside. So, now, is it the same amount of mass? There is another one point that the disk should be maintaining its Keplerianity, and that is why the angular momentum should be increasing with r as $\sqrt{r}$.

That is why, if the total angular momentum should be conserved and that is the case because the system does not have any net external torque that we have already seen. There is no external torque in the system. Otherwise, we should have mentioned.

Then the total angular momentum lost by the inward motion of the mass should be exactly compensated by the angular gain in the angular momentum by the outward motion. Now, the angular momentum per unit mass is much important for the outward mass. So, in order, the two parts to be equal the amount of mass involved in outward going motion should be much less than the amount of mass which is spiraling inward.

So, you want to say simply that $ab = cd$. Now, if b is greater than d, then a must be less than c, as simple as that. So, if you understand these two aspects of accretion disk dynamics, then this is already a good start. So, at this point I will stop this discussion, and in the next discussion, I will discuss a little bit about the steady disks.

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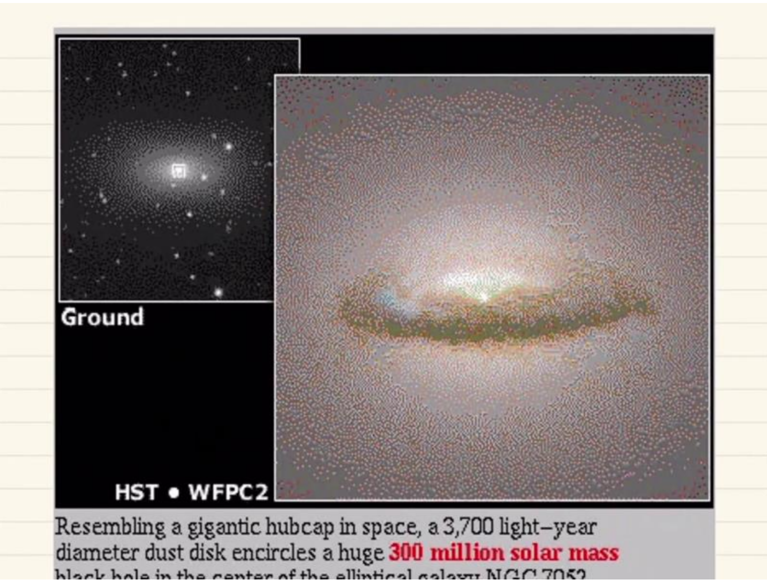

But before that, I will just show you a very interesting picture which is taken by Hubble's Space Telescope. So, this is the ground picture. Here, you can see something very prominent occurs and when you just like zoom and take proper picture. You will see that this is nothing but a dust accretion disk which has a diameter of can you imagine 3700 light-years. This is the diameter, and here, at the very center you can see something like a diamond, something is glittering.

And what is that? This is nothing but a huge 300 million solar mass black hole. Of course, you can say that we cannot see the black hole that is true. So, this is just the radiation from the horizon of the black hole which we can see. So, this simply says that the existence of black hole at the proximity of this thing, I mean this is the horizon of the black hole which we can see. They are not the black hole of course.

So, this is a real image of an accretion disk around a black hole, and you can see that what we are just discussing the image which was developed in our mind is somehow very close to what we see in reality.

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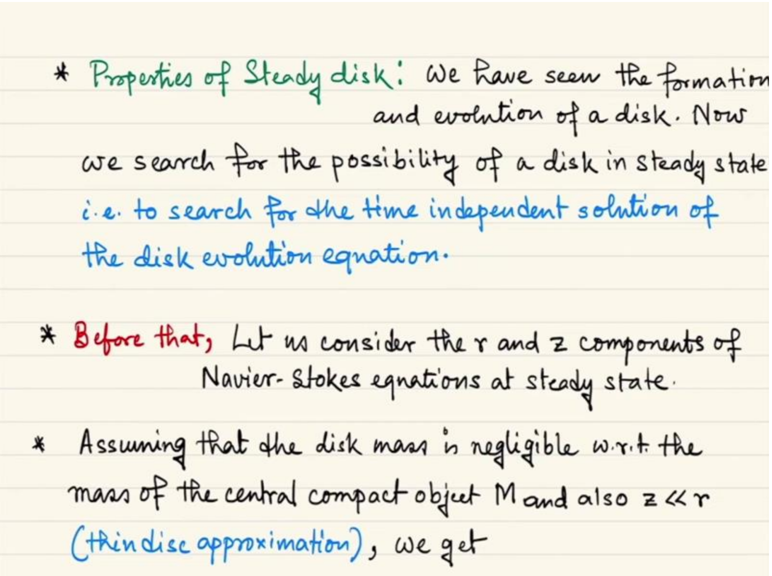

Now, next part is the properties of steady disk. Now, it is true that till now we have been discussing the evolution of disk. So, if we assume that was also an assumption now that initial mass was in the form of a ring, then how it got diffused by the virtue of viscosity? Now, we will see that if the disk reaches to some steady state, then what happens? So, that what I mean we discuss now.

That we have seen the formation and the evolution of a disk, now we search for the possibility of the disk in steady state. So, that means, we would like to search for the time independent solution of the disk evolution equation. There is a routine process for physics.

But before that let us consider the r and z component of the Navier-Stokes equation at steady state, and we will see something very interesting from that. Last time we neglected those two components just by saying that v_r and v_z they are very small actually v_z is 0. So, we really do not have anything very interesting for the evolution equation of the corresponding momentum. But here we will again fetch them both and we will see that something very interesting may be concluded.

So, assuming that the disk mass is negligible with respect to the mass of the central compact object M, for example, the black hole mass or the neutron star mass which is at the center of an attraction and also that z, the height of the disc or the thickness of the disc is very very less than the radial dimension of the disc that is the classical thin disc approximation.

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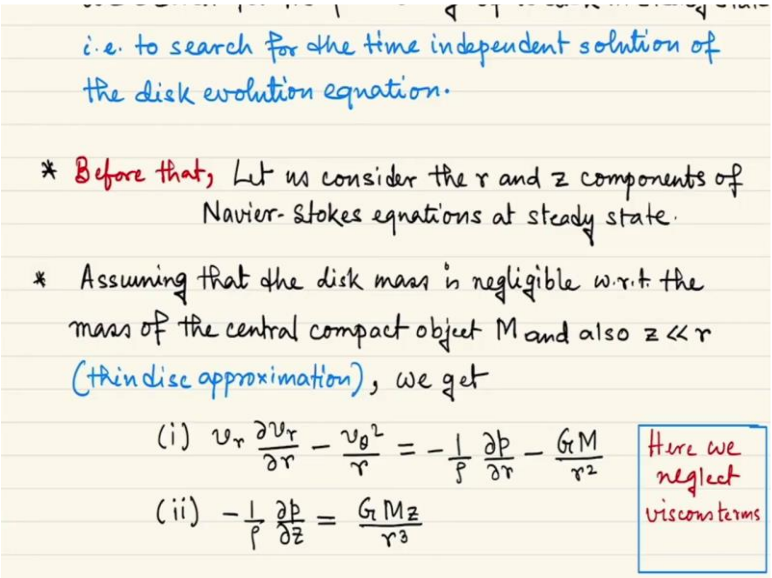

Finally, we get that $v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}$. Of course, the $\frac{\partial}{\partial t}$ terms term goes away, and also the z-component gives us simply this $-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{GMz}{r^3}$. So, these two expressions are actually obtained.

You have to just check yourself the calculation, and I think that is quite an easy calculation. So, this one $v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}$ was the r^{th} component, radial component of the momentum equation and this one $-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{GMz}{r^3}$ was the z component of the momentum equation.

And in this one $-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{GMz}{r^3}$, as the v_z is 0, this simply says that the gradient pressure force is exactly balanced by the gravitational force. Although, here the gravitational intensity is calculated under this approximation $z \ll r$. In these both equations, we have neglected the viscosities.

Although viscosity plays a very crucial role in the radial motion of the system but this is only becoming prominent when we are talking about the evolution of momentum in the $\hat{\theta}$ direction.

For the $\hat{r}$ direction and for the $\hat{z}$ direction really the viscous terms are not of importance, that is something very important in modeling. We have always kept an eye open to understand at which level, which terms are important, and which terms are not.

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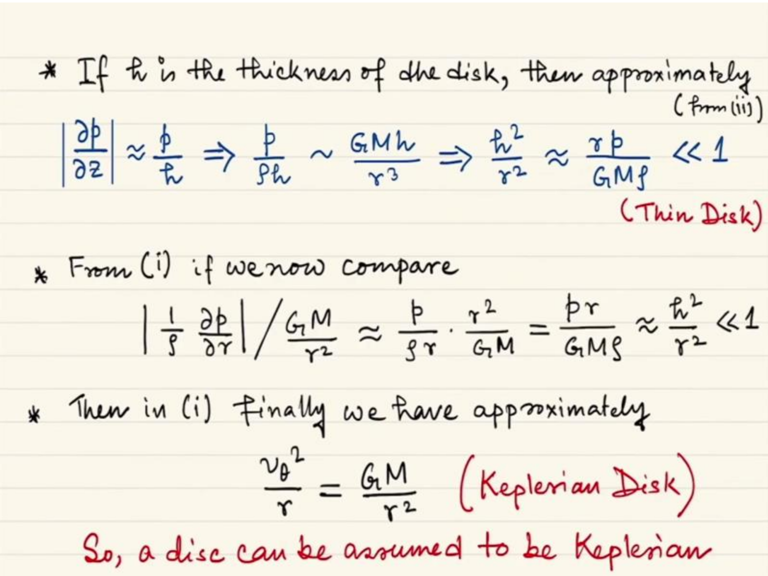

Now, we assume that h is the thickness of the disk, which is also a very small quantity, then approximately we can say that the $\frac{\partial p}{\partial z}$ that is the change of pressure a linear function of z. This is a fair assumption, so that is nothing, but $\frac{p}{h}$.

Then you can say using this one $\frac{\partial p}{\partial z} = \frac{p}{h}$ and using this $-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{GMz}{r^3}$ last equation, you can say finally that $\frac{p}{\rho h}$ almost of the same order of $\frac{GMh}{r^3}$, and if you do the correct calculation, one line calculation, you will see $\frac{h^2}{r^2}$ is of the approximately equal to $\frac{rp}{GM\rho}$. And this is very less than 1 because this is nothing but $\frac{h^2}{r^2}$, and $\frac{h^2}{r^2}$ is very very less than 1 by thin disk approximation.

Now, from $v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}$, we can compare two terms. Here, we have two terms in the right-hand side, one is the pressure gradient force, another is the gravitational force. If we do that, you will see that the ratio of these two is roughly also equal to $\frac{rp}{GM\rho}$ and exactly equal to the above calculated.

This is then also very less than 1 in thin disk approximation. That is why we can actually assume that this term $\frac{GM}{r^2}$ is dominating over this term $\frac{1}{\rho}\frac{\partial p}{\partial r}$. So, we just neglect this term $\frac{1}{\rho}\frac{\partial p}{\partial r}$. We can forget this term $v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r}$ because v_r is very small with respect to v_θ. So, finally, the r component or radial component of the momentum evolution equation simply gives us this $\frac{v_\theta^2}{r} = \frac{GM}{r^2}$, and this is nothing but Keplerian Disk approximation.

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(Thin Disk)

* From (i) if we now compare

$$\left|\frac{1}{\rho}\frac{\partial p}{\partial r}\right| \Big/ \frac{GM}{r^2} \approx \frac{p}{\rho r}\cdot\frac{r^2}{GM} = \frac{pr}{GM\rho} \approx \frac{h^2}{r^2} \ll 1$$

* Then in (i) finally we have approximately

$$\frac{v_\theta^2}{r} = \frac{GM}{r^2} \quad \text{(Keplerian Disk)}$$

So, a disc can be assumed to be Keplerian when the pressure gradient is neglected w.r.t. the gravity term. Consistent with Thin Disk.

So, you see the conclusion is very interesting that a disk can be assumed to be Keplerian when the pressure gradient is neglected with respect to the gravity term, and this is consistent with the thin disk approximation. So, if the disk is such that the pressure gradient force cannot be neglected with respect to the gravitational force, then Keplerian disk is an approximation is a non-reasonable or unreasonable approximation.

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* Now we consider the steady state condition of the basic equations of disk evolution [(1) & (3)], which gives

$$\text{(a)} \quad r\Sigma v_r = C_1 \quad \text{and}$$

$$\text{(b)} \quad r^3\Sigma\Omega v_r - \nu r^3\Sigma\frac{d\Omega}{dr} = C_2$$

(C_1 & C_2 are constants)

* Note that the mass loss rate $(\dot{m}) = -(2\pi r\Sigma)\frac{dr}{dt}$

$$= -2\pi r\Sigma v_r \Rightarrow \dot{m} = -2\pi C_1 \quad \text{(constant)}.$$

* Again we assume that at the surface $(r = r_*)$ of the gravitating body, matter is dragged into a rigid rotation and hence $\left.\frac{d\Omega}{dr}\right|_{r=r_*} = 0 \Rightarrow C_2 = -\frac{\dot{m}}{2\pi}(GMr_*)^{1/2}$

Now finally, we consider the steady state condition of the basic equations (1) and (3) of disk evolution, that means the continuity equation and the evolution for the angular momentum equation. So, now, if you just say that the explicit time dependent terms are 0, then you will directly can show that $\Sigma r v_r$ is equal to C_1.

So, I can just show you once again, there is no problem, in case you forgot. So, in this equation (3) at steady state this $\frac{\partial}{\partial t}(\Sigma r^2 \Omega)$ is 0. So, $\frac{1}{r}$ goes away. So, this thing $\Sigma r^2 \Omega$ within bracket is constant.

So, you have this $\Omega \Sigma r^3 v_r - \Sigma r^3 v \frac{\partial \Omega}{\partial r}$ equal to constant, and that is exactly our conclusion over here. We call the first constant as C_1 and the second constant as C_2. Now, note that the mass loss rate is nothing but minus, because that is a loss rate, of course, $-(2\pi r \Sigma)\frac{\partial r}{\partial t}$.

This is because, if you just think that the mass is lost like this. So, the cylindrical shells like this, then you just take an infinite extended cylindrical shell like this, and you will see that the mass inside this will be nothing but $(2\pi r \Sigma)$dr and mass per unit time, the change in mass will be $(2\pi r \Sigma)\frac{\partial r}{\partial t}$, and $\frac{\partial r}{\partial t}$ is nothing but v_r.

So, the mass loss rate is nothing but $-(2\pi r \Sigma)v_r$, and this is $\dot{m}$. Then we know $\Sigma r v_r$ is constant C_1. So, $\dot{m}$ is equal to $-2\pi C_1$ which is also constant. So, the first thing is that for our consideration, at steady state for an accretion disk, the mass loss rate is a constant.

Secondly, if we assume that at the surface r is equal to r_* of the gravitating body, that means, at the radius of the central body, for example, the neutron star or the compact star of the black hole, the matter is dragged into a rigid rotation process, and then $\frac{d\Omega}{dr}$ is constant there, that is a fair reasonable approximation.

Then using this $\Omega \Sigma r^3 v_r - \Sigma r^3 v \frac{\partial \Omega}{\partial r} = C_2$ expression you can just express C_2 is equal to $-\frac{\dot{m}}{2\pi}(GMr_*)^{1/2}$. Now, you see again both C_1, and C_2 are proportional to $\dot{m}$.

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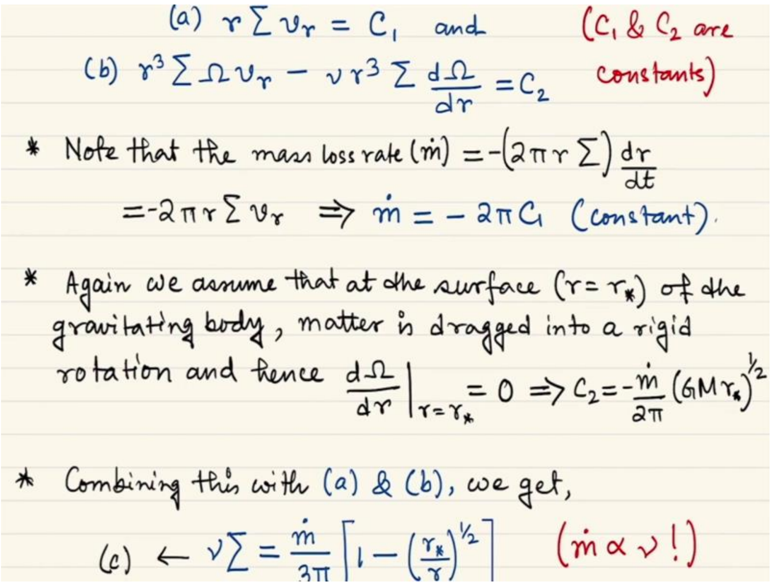

Finally, you can find an expression relating $\dot{m}$ and $\Sigma\upsilon$. So, you see $\Sigma\upsilon$ is equal to $\frac{\dot{m}}{3\pi}[1 - (\frac{r}{r_*})^{1/2}]$. So, this is one result which you can show. These are very good home works.

Now, even more interesting part is that $\dot{m}$ is proportional to υ. So, if υ is increasing, then your $\dot{m}$ is increasing, that means, the system viscosity is higher the mass loss rate is higher.

But finally, even before starting the dynamics of the accretion disks, we said that how really one detects the accretion disks that is by detecting their radiated energy in the form of X-ray.

So, now, we try to calculate although approximately, we try to rather estimate the energy loss rate by dissipation, and here you will see that we calculate the rate of energy dissipation by viscosity, and we will see what is that. So, for weakly compressible fluid, analytically one can actually show that the rate at which energy is dissipated from a fluid by virtue of its dissipation is given by $-\frac{dE}{dt}$ which is equal to $\int \mu r^2 (\frac{d\Omega}{dr})^2 dz$.

That is roughly equal to for our case $\Sigma r^2 \upsilon \left(\frac{d\Omega}{dr}\right)^2$. Finally, putting this value into this thing $\Sigma\upsilon = \frac{\dot{m}}{3\pi}\left[1 - \left(\frac{r}{r_*}\right)^{\frac{1}{2}}\right]$, so $\frac{dE}{dt}$ you get. So, finally, you will see that $-\frac{dE}{dt} = \frac{3GM\dot{m}}{4\pi r^3}\left[1 - \left(\frac{r_*}{r}\right)^{\frac{1}{2}}\right]$.

So, once again you have this expression for $\Sigma\upsilon$, and here you have an expression for $\frac{dE}{dt}$. So, you just substitute this value of $\Sigma\upsilon = \frac{\dot{m}}{3\pi}\left[1-\left(\frac{r}{r_*}\right)^{\frac{1}{2}}\right]$ over here $-\frac{dE}{dt} = \Sigma r^2 \upsilon\left(\frac{d\Omega}{dr}\right)^2$, and you will have this expression $-\frac{dE}{dt} = \frac{3GM\dot{m}}{4\pi r^3}\left[1-\left(\frac{r_*}{r}\right)^{\frac{1}{2}}\right]$.

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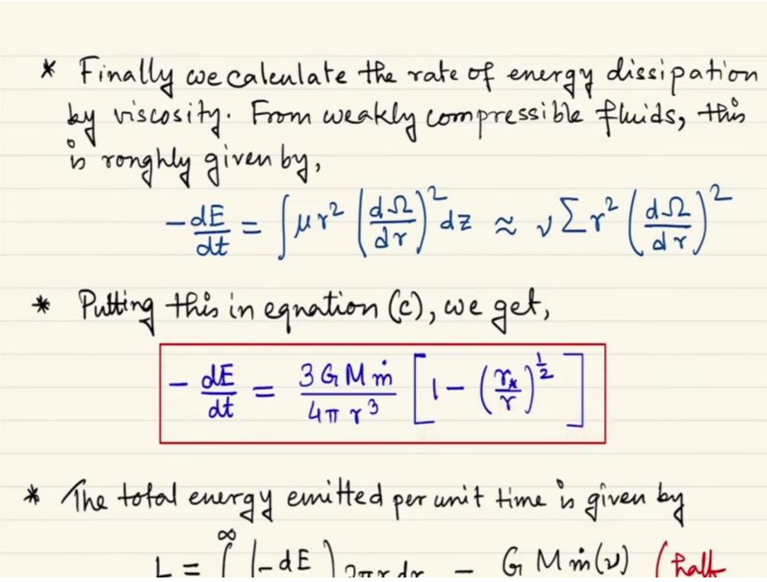

This is quite easy to understand.

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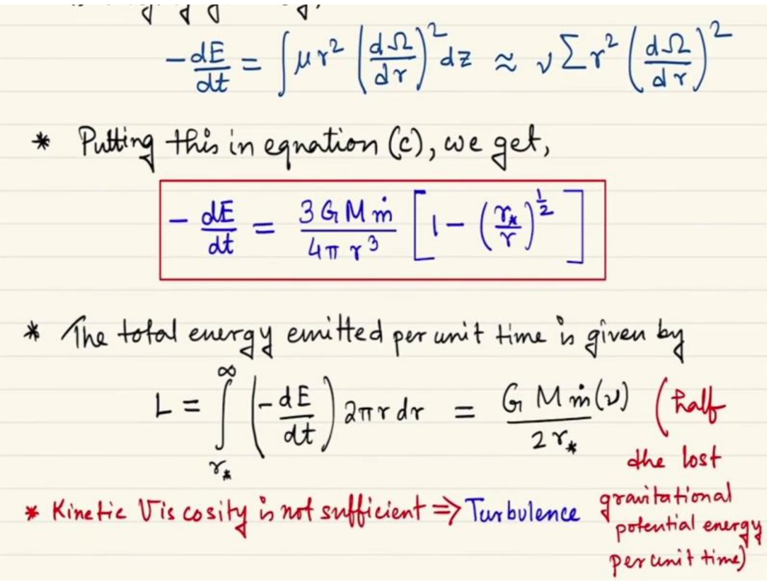

And what is the marvel of this expression? Can you see anything interesting? Well, I can see. You see this is proportional to $\dot{m}$, and $\dot{m}$ is proportional to υ – the viscosity coefficient. So, it means that the viscosity coefficient is higher. So, the viscous energy dissipation rate is also higher which is intuitive.

Now, the total energy emitted per unit volume by viscosity is given by then this L, and that is just integrated from r_* to infinity, r_* means that is the surface of the small compact star on which finally all the mass are getting accreted, $\int_{r_*}^{\infty} -\frac{dE}{dt}\, 2\pi r dr$, and finally, if you calculate you will find a fantastic result which is $\frac{GM\dot{m}}{2r_*}$, $\dot{m}$ is a function of υ.

So, L is a function of υ again, and you see what this is, this is nothing but the half of the lost gravitational potential energy by the mass lost from the less dense star. So, half of the potential energy, which is lost by the mass, which is now accreted to the compact star is converted to heat by dissipation.

What happens to the other half? Well, this is also very easy to understand. This basically contributes to the kinetic energy and you can actually calculate that kinetic energy to the disk. If you can understand that, you can simply see that the total gravitational potential energy lost now gets converted into two classes; one is the kinetic energy of the disk, one is the heat energy or the radiated energy.

And that is exactly the energy what one can see as a compact source of X-ray, for example. Now, an interesting thing is that people when they start estimating this type of energy, for example, if they use this type of formula L and then use this kinematic viscosity, then they estimate that from normal kinetic theory, they see that the L or the energy emitted is much lower than what is really observed.

That means, the kinetic viscosity or the viscosity coefficient which is calculated from kinetic theory is not sufficient to account for the energy, and that is why it is proposed that it is a resultant viscosity coefficient, which is the kinematic viscosity coefficient plus the turbulent viscosity coefficient, is actually sufficient to account for the observed energy dissipated from an accretion disk.

This thing then opened a new domain of research that is the efficiency of transport of angular momentum by turbulence. That means, turbulence what it does? It adds to the efficiency of the viscosity, resultant viscosity of the medium thereby making the process the angular momentum transfer much more efficient.

So, that was all about the accretion disks, which lies in the scope of this course. The accretion disc is a matter of ongoing research, of course, and if you are interested, you can check out different papers, works, videos over internet, nowadays you have millions of resources of a accretion disk. Of course, if you always have good physics questions, do not hesitate to share with me. I will be happy to answer you. If I do not know the answer, I will also think.

Thank you, once again for your support, and I always think that astrophysical things are best understood when you will see the images and the videos. So, do that, and that was all about this accretion disks in astrophysics. In the next discussion, we start about the behavior of a compressible fluid, and how it reacts or how it responds to a perturbation to this medium, and thereby introducing sound waves, shocks, and other things.

Thank you very much.