Introduction to Astrophysical Fluids
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Lecture – 22
A small digression: Newtonian fluids

Hi, welcome to another session of introduction to astrophysical fluids. So, last time we stopped just after discussing different approximations or basic assumptions in the theory of thin accretion disks.

Today before addressing the equations of dynamics of a thin accretion disks, I would like to digress a bit and to share with you a very important physical concept that is the concept of Newtonian fluids.

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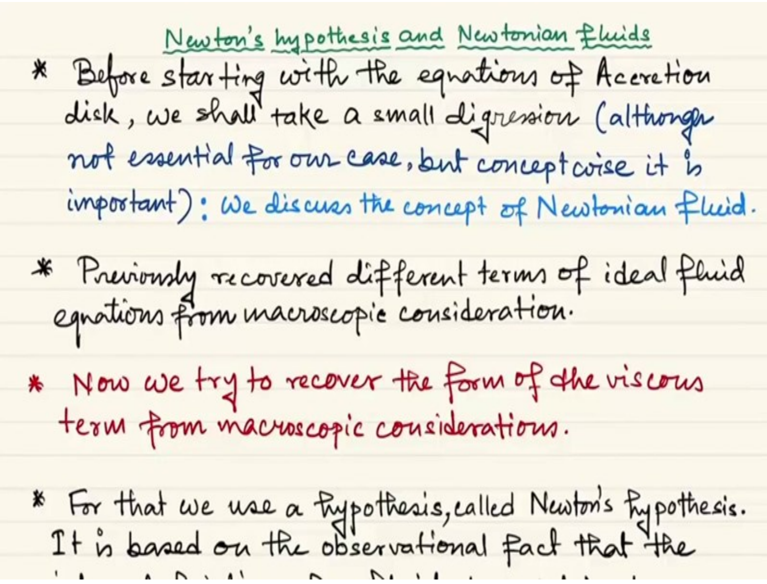

So, here basically you can see that if you remember when we were talking about the ideal fluid equations then after deriving it or rather after I mean attempting the equations to be derived from kinetic equations, we also showed that the different type of forces: body forces and surface force these can be also derived from macroscopic considerations.

If you remember and we also said that the bulk force or the body force acts equally on all points of the volume and the body forces are in general conservative in nature whereas the dissipative forces are surface force; that means, they work along a surface mostly on the surface of contacts.

Then we said that the only type of surface force which arises in the case of ideal fluids are nothing but the pressure gradient force. Now here we will try to recover the form of viscous term which was absent for the ideal fluid equations of course and which is constituted by the off-diagonal part of the pressure tensor. We shall try to recover that part from macroscopic consideration.

Now, what we in general called a Newton's hypothesis is very intuitive and this is based on the very basic level definition of viscosity. It is usually found that the internal friction or the viscous effect of a fluid always tries to reduce the relative velocity between two layers of the fluid.

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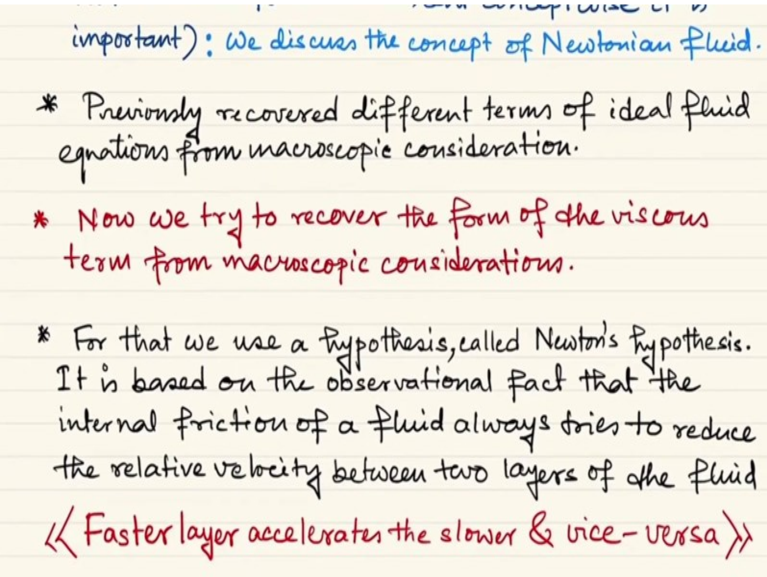

It simply means that if you have two layers of fluid then the faster layer always will try to accelerate the slower layer and the slower layer will always try to retard or the decelerate the faster one. Thereby in both cases as you can easily understand that there will be a decrease in the relative velocity and finally, they want to make a compromise between the two for becoming to a I mean intermediate velocity for example, okay and sometimes actually the global thing even comes to a rest that is also possible that differs from case to case.

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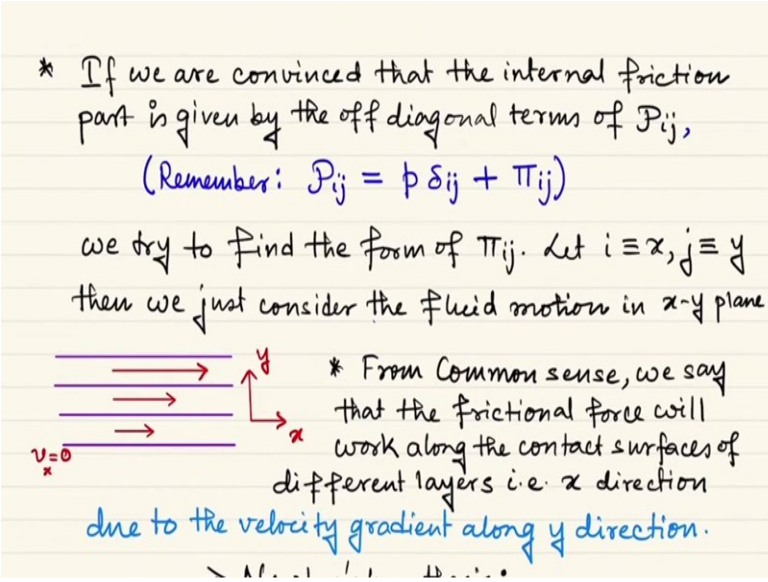

Now, in general we say from our everyday experience that internal friction of a fluid or viscous effect is something which always tries to prevent the development of a velocity gradient. Now, try to understand much more schematically and much more like systematically that we all know that the internal friction part should arise from the off-diagonal term of P_{ij} right.

That is because when we try to derive the real fluid equations from kinetic equation, we actually showed that the viscous terms are non-zero when we consider a perturbation or a departure from the equilibrium Maxwell Boltzmann distribution and this is this part π_{ij} which represents this internal friction, I mean effect of the internal friction or the viscous effect. Now we would try here to find the form of π_{ij} , already we have done something from kinetic theory but here you will see this macroscopically without using kinetic theory.

Historically, once again these equations were derived much before the development of this passage from kinetic to fluid equations ok.

So, let us take let us consider one any one component of this type of tensors let us say i is x and j is y , then we just consider the fluid motion let us say in the XY plane so, this is the XY plane and we actually consider that the z direction is perpendicular to the plane of the paper.

So, y direction is the vertically upward direction along which the different layers are situated and like the layers are basically they are flowing in the XZ plane, but now we are just neglecting the z direction we just say these layers are flowing along x directions.

Let us say this is the ground at which the flow is at rest, the fluid it at rest because of the contact with the static floor and then it starts, I mean the velocity increases gradually. You can see the different sizes of the arrows simply shows that the velocities are increasing when one goes away from the from this floor.

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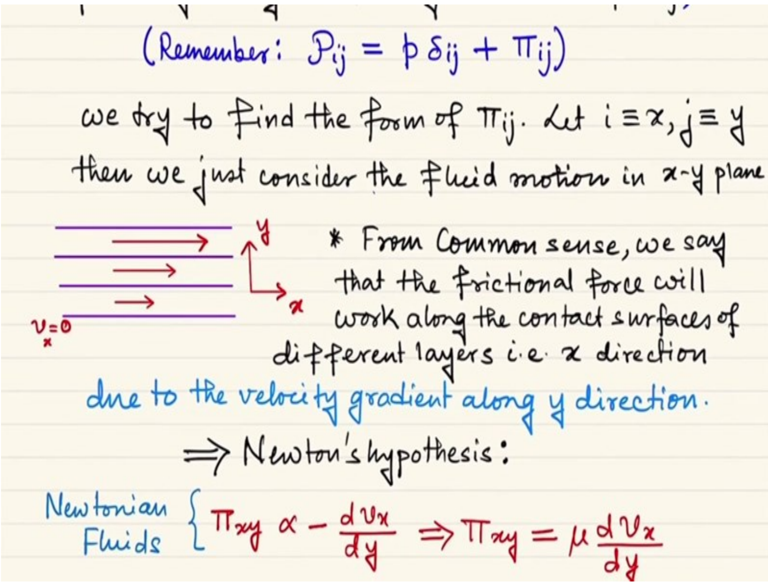

Now of course, from common sense we say that the frictional force will work along the contact surface of different layers and that is I mean what frictional force is, even this is true when we are talking about two solid surfaces, one solid surface and one liquid surface.

So, now in the in this case the flow is along x direction so, the friction will very intuitively work along negative x direction, but this friction works only because the velocity gradient along y direction. If there is no velocity gradient; that means, the arrows are all of same size then there is no frictional force.

Then from that consideration Newton posed a hypothesis and that is the macroscopic tool to get into this, I mean in business of finding $\boldsymbol{\pi}_{ij}$ is simply saying that π_{xy} should be proportional to $\frac{dv_y}{dx}$. So, if the differences of v_x along y is more and more prominent then the xy term of the $\bar{\bar{\boldsymbol{\pi}}}$ tensor will also be more and more important and there will be a minus sign because this basically tries to develop frictional entity, this tries to reduce the velocity gradient.

So, in a sense friction i.e., viscous effect is something which tries to kill the reason of its generation right and then you can say $\boldsymbol{\pi_{xy}}$ is equal to $\frac{-\mu dv_y}{dx}$, there should be a minus sign in general and for fluids where this is true, so, of course, we said that this comes from common sense and in a real fluid this is not exactly true, but sometimes this is reasonably true and for those fluids for which this type of hypothesis works is known as a Newtonian fluid.

So, most of our everyday fluids which we come across, most of them can work moderately well under this Newton's hypothesis. So, in a sense all of them are moderately well approximated as Newtonian fluids. So, Newtonian fluid simply says that the off-diagonal part of the pressure tensor that is the shear stress part is non-zero, this is called the shear stress. Why it is shear because this does not have any diagonal term.

So, shear is nothing but a combination of elongation and compression in two mutually perpendicular direction. All these off diagonal terms are giving the shear. So, this is something like a stress, you all know that pressure and stress they are the same dimension.

So, it is a force by unit area type of thing and this shear stress should be proportional to the velocity gradient and the velocity gradient should be perpendicular to the direction of the flow, that is to be understood.

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* [One can understand this from the following conside-
-ration that $d\vec{F}_{viscous} = \overline{\overline{\Pi}} \cdot \vec{dS}$

$$= \begin{pmatrix} 0 & \Pi_{xy} & \Pi_{xz} \\ \Pi_{yx} & 0 & \Pi_{yz} \\ \Pi_{zx} & \Pi_{zy} & 0 \end{pmatrix} \begin{pmatrix} dy\,dz \\ dx\,dz \\ dx\,dy \end{pmatrix}$$

So, $d(\vec{F}_{viscous})_x = \Pi_{xy}\underbrace{dx\,dz}_{dS_y} + \Pi_{xz}\underbrace{dx\,dy}_{dS_z}$

And for a given y, $d(\vec{F}_{viscous})_x = \Pi_{xy}\,dx\,dz \propto \Pi_{xy}$]

* Newton's hypothesis is experimentally found to hold only approximately. In addition it cannot give the

And one can easily understand that if we just say that this one $\frac{-\mu dv_y}{dx}$, is getting bigger and bigger simply saying or is equivalent to saying that the frictional force corresponding this term also increases and is also very easy to understand because viscous force we see is nothing but a surface type of force so $d\boldsymbol{F}_{viscous}$ is given by

$$d\boldsymbol{F}_{viscous} = \overline{\overline{\boldsymbol{\pi}}}.d\boldsymbol{s} = \begin{pmatrix} 0 & \pi_{xy} & \pi_{xz} \\ \pi_{yx} & 0 & \pi_{yz} \\ \pi_{zx} & \pi_{zy} & 0 \end{pmatrix} \begin{pmatrix} dydz \\ dxdz \\ dxdy \end{pmatrix}$$

Now if you do the dot product of this tensor $\overline{\overline{\boldsymbol{\pi}}}$ which does not have any diagonal element, when this is contracted with the area element $d\boldsymbol{s}$ you will see that the x component of this viscous force simply gives you two terms $\pi_{xy}dxdz$ plus $\pi_{xz}dxdy$. For our case, we are just concentrating at one value of y for example, at one layer so, y is constant. So, this part $\pi_{xz}dxdy$ is 0. So, we only have finally, $(dF_{viscous})x$ is nothing but $\pi_{xy}dxdz$ because dy is 0, for a given intersection between two successive layers and then you can simply say that π_{xy} is a very good proxy for $(dF_{viscous})x$.

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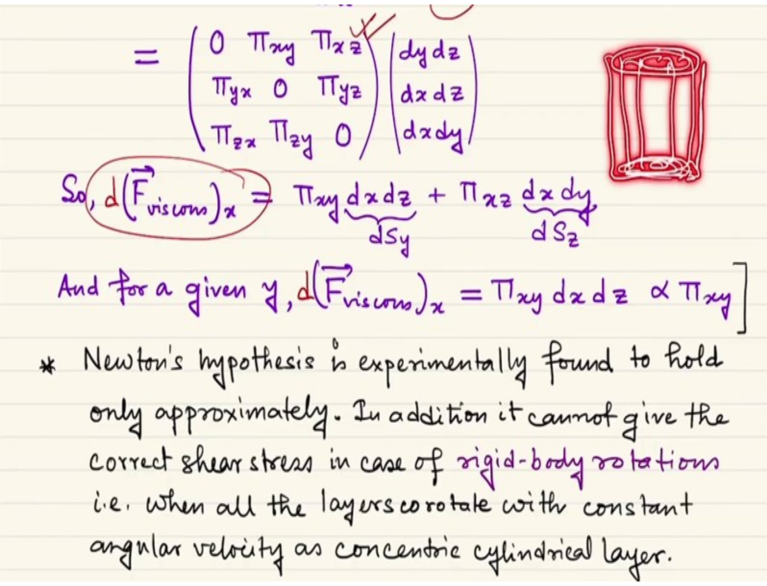

So, then that is why we say that when this π_{xy} increases then this $\frac{dv_y}{dx}$, it's modulus increases and it simply says that the x component of the viscous force also increases. So, you see that the stronger is your velocity gradient stronger will be the reply of the fluid medium to that velocity gradient in terms of internal friction or viscosity.

Now, coming to a very practical point Newton's hypothesis is experimentally found to hold only approximately, that is what I told. In addition, it cannot give the correct shear stress in case of rigid body rotations that is something very-very important. Although, I mean it is moderately valid for most of the fluids we encounter but if the fluid is rotating and actually if the fluid is rotating following at rigid body rotation; that means, now think of an imaginary concentric cylinder thing and the fluid is actually rotating in concentric cylinders with constant angular velocity; that means, although they have different linear velocities, they do not have relative angular velocity.

So, they do not have any relative angular motion with respect to the other and that basically gives you some nonzero linear velocity gradient. But the system does not need to respond in terms of shear; or in terms of its viscosity that is what I am saying.

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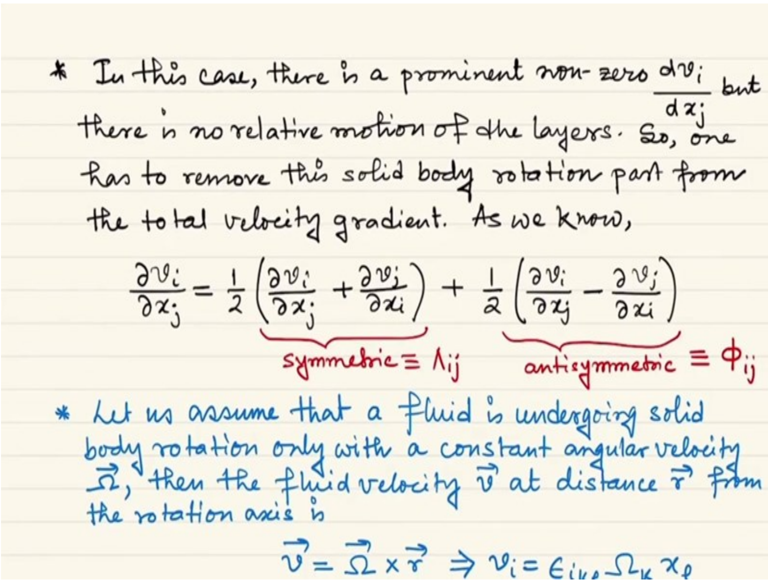

Because all the layers are moving are corotating. Solid body rotation means $\boldsymbol{\Omega}$ is constant, the angular velocity is constant. Then there is of course, a prominent non zero $\frac{dv_i}{dx_j}$ because your angular velocity $\boldsymbol{\Omega}$ is constant.

So $\boldsymbol{\Omega} \times \mathbf{r}$ which is nothing but the linear velocity $\boldsymbol{v}$ that is of course, a function of $\boldsymbol{r}$. So, that will be different at different $\boldsymbol{r}'s$, but finally, since the system is rotating in such a way that some global rotation is followed by every point of the system so, there is relative motion of the layers of fluid. Of course, here we are just thinking that the whole system is only following a global rotation there is no other flow.

Then what happens that one has to remove the solid body rotation part from the total velocity gradient. Now maybe I should explain once again that in case of solid body rotation what happens that we have a non-zero velocity gradient, but the viscous force is 0 because there is no relative motion.

So, it is actually not a very correct statement if we simply say that the viscous force or the shear stress will be proportional always to the $\frac{dv_i}{dx_j}$ but rather we should say that the shear stress should be proportional to the part of the velocity gradient which does not include the solid body rotation.

How to do that? We can do a very simple thing, we can write the velocity gradient component as a sum of a symmetric part and an antisymmetric part, symmetric part we call capital $\Lambda_{ij} = \frac{dv_i}{dx_j} + \frac{dv_j}{dx_i}$ and the antisymmetric part we called $\Phi_{ij} = \frac{dv_i}{dx_j} - \frac{dv_j}{dx_i}$; just π_{ij} for example. So, this is just I mean a plus b plus a minus b type of thing okay.

Now, let us assume that the fluid is undergoing solid body rotation; that means, $\boldsymbol{\Omega}$ is constant. So, $\boldsymbol{v}$ is equal to some constant vector cross $\boldsymbol{r}$ .So, then v_i the ith component of the velocity vector is nothing but equal to $\epsilon_{ikl}\Omega_k r_l$, ϵ you all know this is a Levi-Civita symbol okay.

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has to remove this solid body rotation part from the total velocity gradient. As we know,

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{symmetric} \equiv \Lambda_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{antisymmetric} \equiv \Phi_{ij}}$$

* Let us assume that a fluid is undergoing solid body rotation only with a constant angular velocity $\vec{\Omega}$, then the fluid velocity $\vec{v}$ at distance $\vec{r}$ from the rotation axis is

$$\vec{v} = \vec{\Omega} \times \vec{r} \Rightarrow v_i = \epsilon_{ikl}\Omega_k x_l$$

$$\Rightarrow \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\epsilon_{jkl}\Omega_k x_l\right) = \epsilon_{jkl}\Omega_k \delta_{li} = \epsilon_{jki}\Omega_k$$

and similarly, $\frac{\partial v_i}{\partial x_j} = \epsilon_{ikj}\Omega_k \Rightarrow \Lambda_{ij} = 0$

And so, this is the antisymmetric tensor of rank 3 you know and there is a sum on k and l okay. Now, if you do $\frac{dv_j}{dx_i}$. So, then $\frac{d}{dx_i}$ of this whole expression $\epsilon_{jkl}\Omega_k r_l$ will go there just here in case it is v_j. So, I have replaced the i by j over here. So, now, if you see this is nothing but equal to $\epsilon_{jkl}\Omega_k \delta_{li}$. Now Ω_k is constant so, it comes out and finally, you have $\frac{dx_l}{dx_i}$ and it is nothing but δ_{li}. So, if l is equal to i then it is 1 otherwise it is 0 and if we have this type of thing then this l will be equal to i. So, finally, this is nothing, but $\epsilon_{jki}\Omega_k$.

So, we now have only one sum on k ok and similarly we have $\frac{dv_i}{dx_j}$ which is equal to $\epsilon_{ikj}\Omega_k$, you can show that. Please check that at home this type of exercises is super important for your own development. Of course, you can see that k is a dummy index. So, whether I can write k or m does not matter, but then when I try to write the sum again to show that what is the value of Λ_{ij} then again, I can take k for both the terms. So, that I can group $\frac{dv_j}{dx_i}$ and $\frac{dv_i}{dx_j}$. if I add them, we will see that there will be term wise cancellation of every term because ϵ_{jki} is minus ϵ_{ikj}, that is simply because of the antisymmetric nature of Levi-Civita symbol and this finally, leads to the vanishing of Λ_{ij} tensor $\Lambda_{ij} = 0$.

So, it simply says that when this body has a solid body rotation, the symmetric part of this velocity gradient actually vanishes identically. So, the if we wanted assign the property of the solid body rotation to some part of this velocity gradient that will be this part Φ_{ij}.

So, actually this part Φ_{ij} carries the signature of rotation, the antisymmetric part and the symmetric part is free of solid body rotation effect because this vanishes for a solid body rotation.

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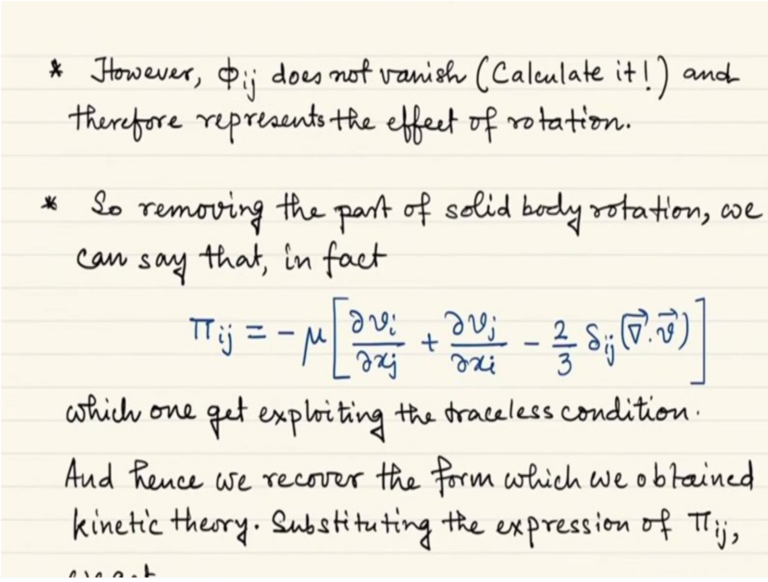

However, actually one can show explicitly that Φ_{ij} does not vanish, but how much it is? Once again calculate, this is super easy because I already showed some steps how to manage that previously. I calculated capital Λ_{ij} and therefore, Φ_{ij} it represents the effect of rotation okay.

So, finally, removing the part of the solid body rotation, we can actually say that the correct statement will be the shear stress should be proportional to the symmetric part of the velocity gradient tensor $\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i}$. But now we do should not forget that there is another condition that π_{ij} should be a traceless tensor.

So, π_{ii} summation over i should be 0 and if we consider that we can actually show by some calculations that you have already seen when we were talking, I mean we were deriving the expression for π_{ij} in case of kinetic theory starting from kinetic theory. Then you know that how to do that, π_{ij} is nothing but equal to

$$\pi_{ij} = -\mu\left(\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i} - \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}.\boldsymbol{v}\right)$$

Now calculate π_{ii}. So, $\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i}$ will be $2\boldsymbol{\nabla}.\boldsymbol{v}$ and δ_{ii} will be 3, so $\pi_{ii} = 0$, i.e., π_{ij} is traceless.

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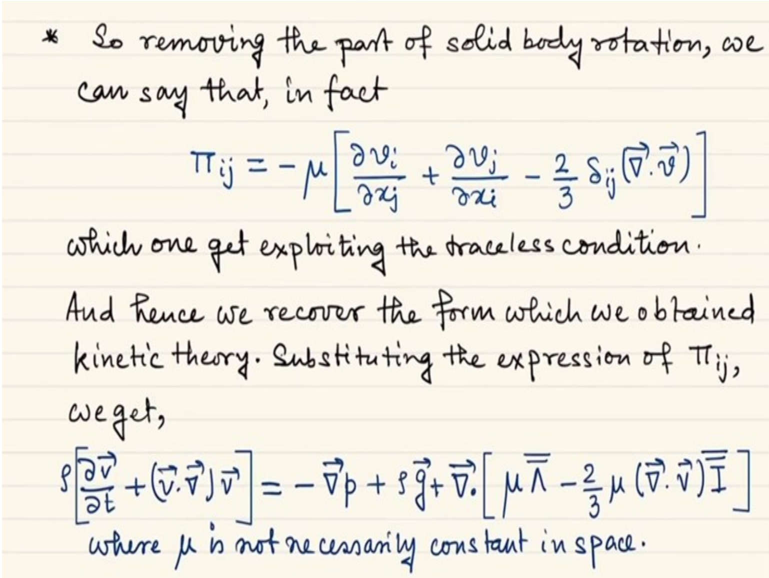

Finally, we recover macroscopically what we found microscopically but for that we needed a hypothesis and hence we see the something what we expected and substituting now the value of π_{ij} in the so-called momentum evolution equation which we call Navier-Stokes equation as well

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v}\right] = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \boldsymbol{\nabla}.\left[\mu\,\bar{\bar{\boldsymbol{\Lambda}}} - \frac{2}{3}\mu(\boldsymbol{\nabla}.\boldsymbol{v})\,\bar{\bar{\boldsymbol{I}}}\ \right]$$

So, p is the diagonal element of the pressure tensor plus ρ times $\boldsymbol{g}$ this is the body force density plus divergence of π_{ij}. Of course, here at this point we have not considered μ is necessarily constant in space, that is very general.

And we will use this expression when we will talk of the dynamics of accretion disks. Because in case of accretion disks this is most reasonable, I mean it is actually found by different studies that the coefficient of viscosity is not constant in space. So, this equation is perfect for studying the momentum evolution for the accretion disks.

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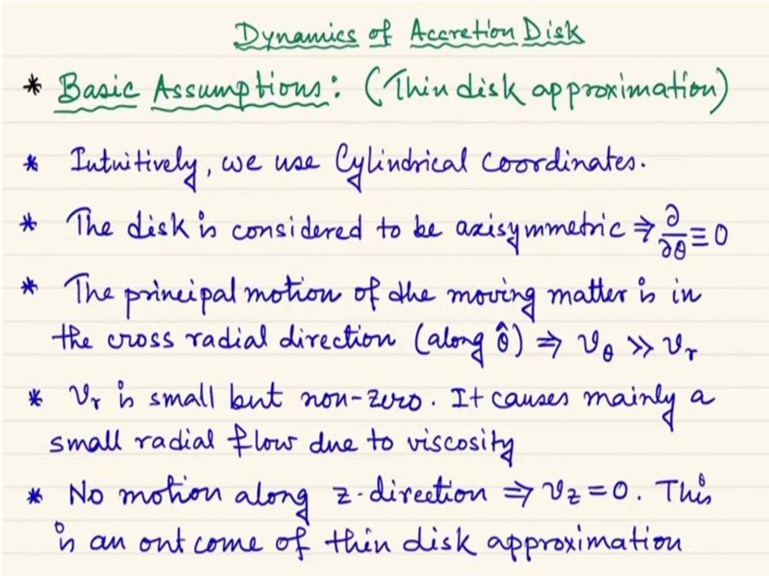

So, after that we will again just come to the basic assumptions of accretion disk for recapitulation and then I will address the dynamics of the accretion disk, that I will do in the next lecture ok.

Thank you very much.