Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

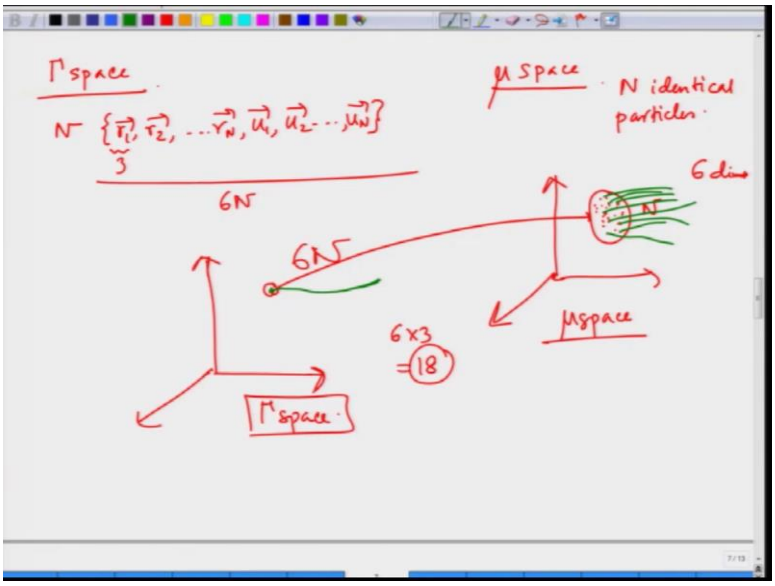
Lecture - 02
Phase space and Liouville's theorem

Let us now introduce, before even introducing let me address this question that, why I am doing all these things? When the course is all about Astrophysical Fluids, why I am describing all this passage from one level to the other level of theory. The answer is very clear, if you are interested in the research in astrophysics or you want to develop a deep insight in the astrophysics then you have to understand the scope of different levels of theory.

Now, tomorrow if you just come across a certain phenomenon and then you know the scope of different levels of theory. Then, just by verifying those conditions you can simply say, under these conditions like, this is the temperature, this is the density, so quantum description will not be there.

So, you see by that you can eliminate redundant possibilities. So, that is very essential. And also it is very good to develop a global view.

(Refer Slide Time: 01:31)



Now, let me introduce two very important concepts, one is the concept which is somehow associated with the concept of phase space, is the concept of Γ- space, another is the concept of μ- space. Now, Γ-space is, let us take a system of N identical particles, which are behaving classically.

Now, if we are just restricting ourselves in three-dimensional universe, the complete set of the system will then be given by as you can easily understand $\boldsymbol{r_1}, \boldsymbol{r_2}$ up to $\boldsymbol{r_N}$ and $\boldsymbol{u_1}, \boldsymbol{u_2}$ up to $\boldsymbol{u_N}$. So, everyone has 3 components. So, it will be having $6N$ components. So, if we now just freeze the time, that means, we take a snapshot of the system. Then, we will see that it will just give us a combination of $6N$ values.

Now, this combination is called a state of the N particle system, the classical state or the state in classical mechanics. Now, if you then consider an artificial phase space but let us suppose this is a $6N$ dimensional space. And one point is just giving you the value of or the ensemble of this $6N$ values at a given instant of time.

If this is the case, then the corresponding phase space is called a global phase space or Γ- space. So, if we have an N particle system Γ- space basically gives us the description of the global state by a single point if the time is frozen, that means one snapshot one point, one snapshot one point.

Now, let us say for example, if you have a system having 3 particles. So, you can easily understand that the number of dimensions, the dimension of Γ- space will be 6 times 3, 18.

So, every time you click and you take a snapshot you will have an ensemble of 18 possible values of $\boldsymbol{r}$ and $\boldsymbol{u}$'s right, and that will give you a single point of that corresponding Γ- space. Now, what is μ- space? μ- space is possible only when the system is composed of N identically behaving particles.

Then, what happens? Basically, these particles can only be distinguished from one another just by their position and velocity. They have exactly identical, physical and chemical properties. They are only differing from each other by their position and velocity. So only the mechanical state is different.

If this is the case, then only you can say I can actually define another type of phase space, where every point is a single particle state. Now, say every single particle has how many numbers of variables to be identified, 3 space coordinates and 3 velocity coordinates. So, it will need 6 values right. So, then we say that in μ- space every single point is giving the description of the state of one single particle.

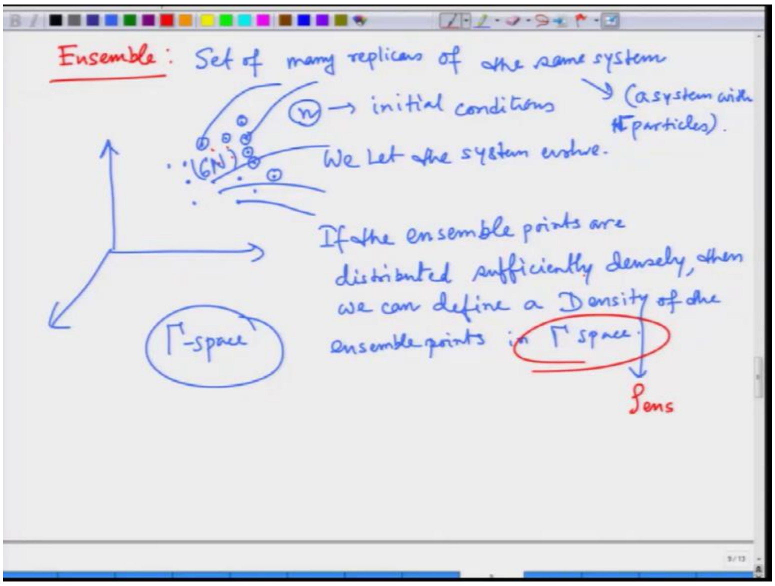
So, in this case where the Γ- space is of $6N$ dimension μ- space will simply be 6 dimensional and this is just for one single particle. So, now, if I say that what will be the corresponding mapping of one point in Γ- space to the μ- space? this will be an ensemble of N points. So, this is called the μ- space or particle specific phase space.

Now you see, when a point in Γ- space evolves in time, then that means the total system basically, is evolving along a trajectory. So, I am just taking snapshots with time and I am seeing that this point is moving. That means, that with this initial condition the system starts and then the laws of mechanics compel the system to move having all these subsequent values of $\boldsymbol{r}$ and $\boldsymbol{v}$.

Then, if this is just a line in Γ- space, a single curve, in μ- space these will not have a simple picture like that, because every one of them is now evolving on a path. You can see, so a single trajectory in Γ- space will now be an ensemble of N trajectories or a group of N trajectories in μ- space.

Now, till now I have roughly used the ensemble in order to just use the word, group. Ensemble I just said that a number of or a collection of trajectories or something. Now, there is a formal meaning of ensemble. So, from now onwards when I will use the word ensemble it will be basically used to designate a specific thing.

(Refer Slide Time: 08:14)



So, ensemble is a set of many replicas of the same system, for us the system can be a system with N particles. Then ensemble is nothing, but a collection of see if you just remember what Γ- space was, the state of the whole system of containing N particles. So, having $6N$ values in it was designated by a single point in Γ- space.

Now, let us say at an instant of time we mentally create or rather imaginarily create n number of such points, what is the meaning of that basically? We are creating n number of collections of $6N$ values. That means, we are creating n number of initial conditions. And we let the system evolve. This is we are talking in terms of Γ- space.

So, every member of the ensemble will then take a path of evolution. Now, if thus, number of ensembles, the number of rather ensemble points are distributed sufficiently densely, then we can define a density of the ensemble points Γ- space. And that density is designated by ρ_{ens}.

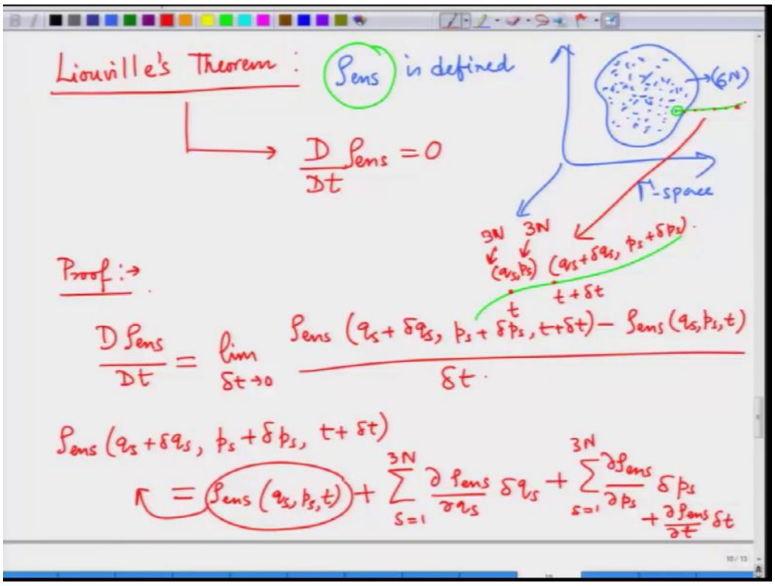
So, once again every point is designating one single state, one global state of the system containing N particles and now, I am saying that at an instant of time we are imaginarily making n numbers of such collection of $6N$ values. So, every single entity of the set contains $6N$ values. So, finally, we have n number of this $6N$ values, n number of this type of collection each of which contains again $6N$ values.

So, this cluster or this total collection of n points is known as ensemble and a single point of ensemble is known as an ensemble point. And ensemble point is nothing but a possible initial condition. And then after that we basically let the system evolve according to the evolution trajectory.

Now, what we claim is that, if the ensemble points are distributed sufficiently densely, that means, we are creating such an ensemble where we have artificially or imaginarily created a very large number of possible collections of $6N$ values, meaning collection of points each of which contains $6N$ values of $\boldsymbol{r}$ and $\boldsymbol{v}$.

Then we can define that in Γ-space, the ensemble points are very very dense and then we can define a density of the ensemble points. Look, ensemble point density has nothing to do with the real point density. Ensemble points density is a density which is defined on phase space. When we say that the ensemble points are distributed sufficiently densely, that simply means that, we have created a sufficiently large number of collection of points where each points are giving a possible state of $\boldsymbol{r}$ and $\boldsymbol{v}$'s, that is it. Now, this density of this ensemble points can be designated as ρ_{ens}.

(Refer Slide Time: 14:33)



Let us now talk about a very fundamental theorem which is known as Liouville's theorem. What is the statement of Liouville's theorem? Liouville's theorem says, let us say we are in such a Γ-space where ρ_{ens} is defined. That means, the ensemble points are very densely spaced in the Γ-space.

Now, we concentrate on one single member of the ensemble. That means in Γ-space we have sufficient number of points, each point remember again contains $6N$ values. Now, these points are very very densely spaced, now we concentrate on one such point and we trace its evolution with time.

If we just trace its trajectory in phase space, then we know what ρ_{ens} is or we can calculate what ρ_{ens} is at every point along the trajectory of this particle. So, basically the strategy is that first we have to create a dense ensemble, then we define the density of the ensemble ρ_{ens}, then we just concentrate on one single member of the ensemble and we trace its evolution. Now, we calculate again the ρ_{ens} at every point of the trajectory of the evolution.

So, if we calculate all these things basically finally, Liouville's theorem says that $\frac{D\rho_{ens}}{Dt}$ will be equal to 0, where $\frac{D}{Dt}$ is the time derivative of ρ_{ens} along the trajectory. What is the proof of that theorem? This is also very interesting. Let us do this in an elaborate way, we take two points on the trajectory. Let us say they are infinitesimally placed. So, one point is at time instant t and the other point at time instant $t + \delta t$ (see picture above). And their states at t are given simply by (q_s, p_s), I am just writing in terms of the generalized coordinates and generalized momentum. But for your convenience just let me tell you that this is nothing but the position and the velocity for example. And states at $t + \delta t$ will be $(q_s + \delta q_s, p_s + \delta p_s)$.

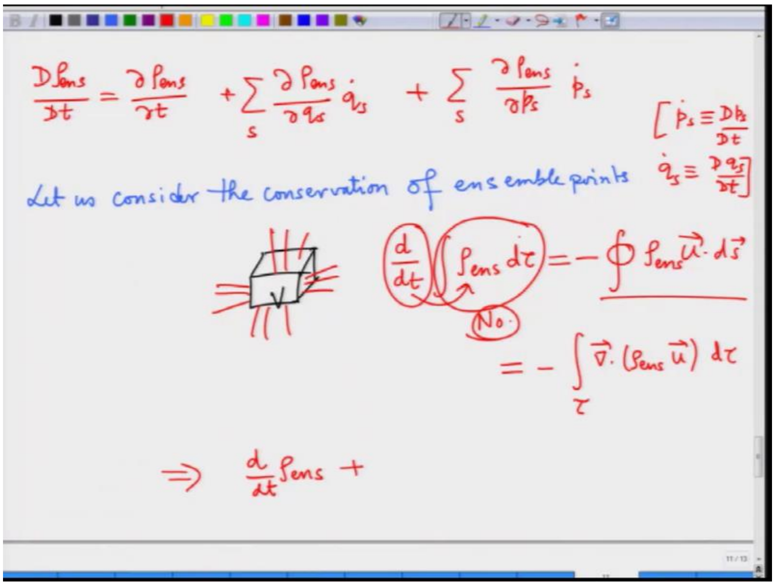
Now you see that here, although I just wrote (q_s, p_s) basically, every q says there are $3N$ number of values for that corresponding to that q_s and for p_s is the same. So, $3N$ plus $3N$ again $6N$ values, must be so. Now, we want to calculate $\frac{D\rho_{ens}}{Dt}$ and we simply say that this is nothing but, $\lim_{\delta t \to 0} \frac{\rho_{ens}(q_s+\delta q_s, p_s+\delta p_s, t+\delta t) - \rho_{ens}(q_s, p_s, t)}{\delta t}$.

Once again, when I am saying all these things, $q_s + \delta q_s$'s are also $3N$ quantities. Then, you can again write by doing some Taylor type of expansion you can write,

$$\rho_{ens}(q_s + \delta q_s, p_s + \delta p_s, t + \delta t) = \rho_{ens}(q_s, p_s, t) + \sum_{s=1}^{3N} \left[\frac{\partial \rho_{ens}}{\partial q_s} \delta q_s + \frac{\partial \rho_{ens}}{\partial p_s} \delta p_s\right] + \frac{\partial \rho_{ens}}{\partial t} \delta t.$$

This is just we can get from Taylor's expansion. So, if you just take $\rho_{ens}(q_s, p_s, t)$ to the left hand side and divide the both sides by δt and taking the limit $\delta t \to 0$.

(Refer Slide Time: 21:16)



Then, basically you can see you will be simply given

$$\frac{D\rho_{ens}}{Dt} = \frac{\partial\rho_{ens}}{\partial t} + \sum_s \left[\frac{\partial\rho_{ens}}{\partial q_s}\dot{q}_s + \frac{\partial\rho_{ens}}{\partial p_s}\dot{p}_s\right]$$

where dots are nothing, but the $\frac{D}{Dt}$'s. So, $\dot{p}_s$ is equivalent $\frac{Dp_s}{Dt}$ and $\dot{q}_s$ is $\frac{Dq_s}{Dt}$. Now, here you see that here you have three contributions.

Now finally, we have to prove this is 0, right. How to do that? Ok now, we start by doing a simple thing that is we will try to discuss rather we will try to calculate the dynamics of the ensemble points in the Γ-space. What will be this? Now, till now we have said that the ensemble points are so densely spaced that you can define a density for the ensemble points. Now let us consider the conservation of ensemble points of course, in Γ-space.

What is the meaning of that? If you just take any arbitrary volume element, then the rate of change of the number of ensemble points in a test volume V, will be equal to the flux rate of the particles through the boundary surfaces enclosing the volume.

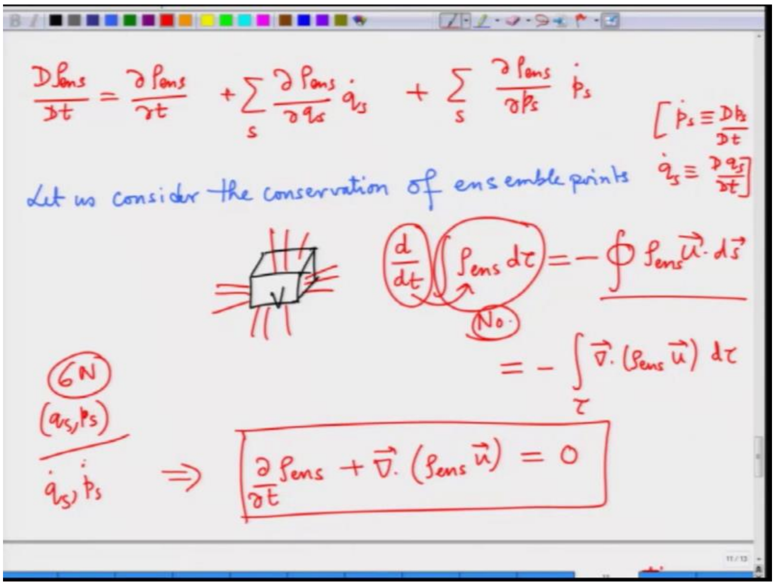
That something you all know from the basic hydrodynamics, that when some fluid is not having any source or sink inside this volume, that means the rate of change of the mass of the fluid inside the volume will be exactly equal to the flux rate of the particles or flux rate of the fluid through the surfaces.

So, how to write that? The formal way of writing this, is $\frac{d}{dt}\int \rho_{ens} d\tau$, this is the total time derivative of the mass of the ensemble points. Now, again once again they are not real particles. So, when I talk about mass this is a representative mass this is nothing but the density times the volume.

So, this is the I mean the rate of change of $\int \rho_{ens} d\tau$. So, basically what is this? This is the total number. So, there is a change of the number of the particles from an arbitrary volume in Γ-space that will be equal to $-\oint \rho_{ens}\boldsymbol{u}\cdot \boldsymbol{dS}$. That is just to say that, if the change of mass is positive, it should be coming inside and if the change of the mass is negative then it should come out of the surface.

So that, basically says that these two will be related via negative sign. Now, what is the surface integral? By Gauss's divergence theorem you all know this is nothing but $-\int_{\tau} \nabla \cdot (\rho_{ens}\boldsymbol{u}) d\tau$.

(Refer Slide Time: 26:40)

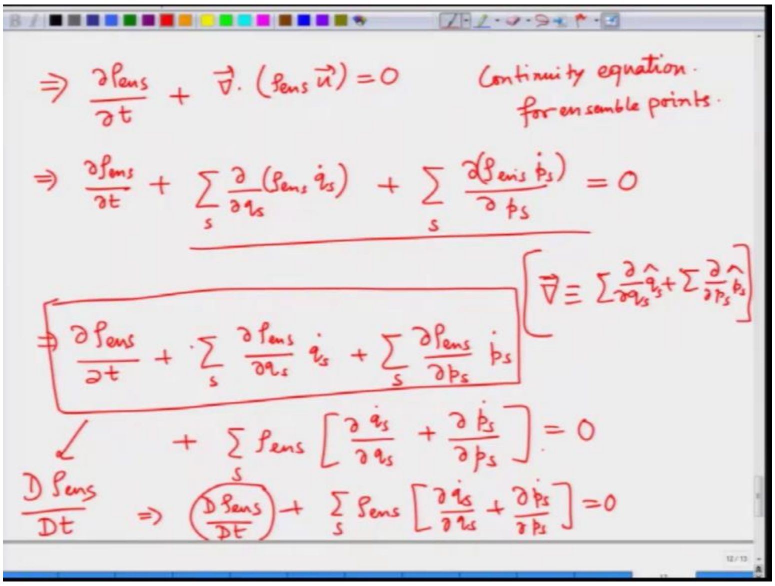


Since, this is a volume element, and space and time they are not seeing each other. So, basically by the Leibniz rule, the differentiation w.r.t time actually can enter inside the integration as a partial differentiation. So,

$$\frac{\partial \rho_{ens}}{\partial t} + \nabla \cdot (\rho_{ens}\boldsymbol{u}) = 0.$$

This is the equation of continuity for the ensemble points. Now, what is the advantage I get from this equation?

(Refer Slide Time: 27:39)



If you now see there is a lot of advantage because, this divergence term you can now write like

$$\nabla \cdot (\rho_{ens}\boldsymbol{u}) = \sum_s \left[\frac{\partial(\rho_{ens}\dot{q}_s)}{\partial q_s} + \frac{\partial(\rho_{ens}\dot{p}_s)}{\partial p_s}\right]$$

Now why is this, can you understand this point? That is because $\boldsymbol{u}$ is nothing but the velocity of the ensemble points during its evolution. So, $\boldsymbol{u}$ is nothing but the Γ-space velocity if the Γ-space is of $6N$ dimension and the point position coordinates are (q_s, p_s) .

So, remember p_s is also inside the position coordinate, all of them together gives you the $6N$ dimensions. Then the corresponding velocity in Γ-space will be just $\dot{q}_s$ and $\dot{p}_s$, dots are nothing but the time derivatives of them, the total time derivative. Now, if you have this, then $\boldsymbol{u}$ can be written as $\dot{q}_s$ and $\dot{p}_s$'s, again $\dot{q}_s$'s has $3N$ values and $\dot{p}_s$'s has $3N$ values.

So, $\boldsymbol{u}$ also has $6N$ values, if the time is frozen over there. Now, and why the other things? because divergence is nothing but the partial differentiation with respect to the position coordinates. Here the position coordinates have $3N$ q's and $3N$ p's so, that will be $\nabla \equiv \sum \left[\frac{\partial}{\partial q_s}\widehat{q_s} + \frac{\partial}{\partial p_s}\widehat{p_s}\right]$. That is our divergence here of course, with the unit vectors.

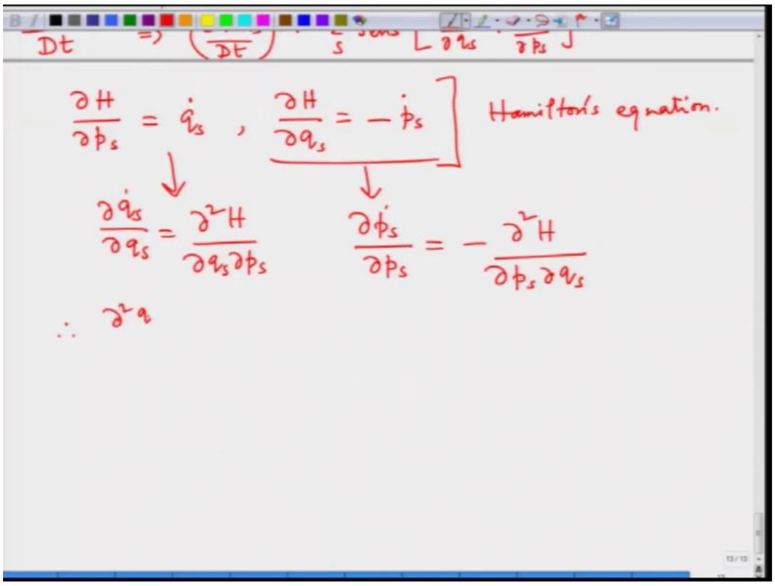
Now finally, expanding the divergence, where each derivative contains two terms, we can write

$$\frac{\partial \rho_{ens}}{\partial t} + \sum_s \left[\frac{\partial \rho_{ens}}{\partial q_s}\dot{q}_s + \frac{\partial \rho_{ens}}{\partial p_s}\dot{p}_s\right] + \sum_s \rho_{ens}\left[\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s}\right] = 0.$$

Now, first three terms on the LHS are the definition of your $\frac{D\rho_{ens}}{Dt}$. If you just go through the this definition you will see this is nothing but this terms. So, then finally, we can write from here $\frac{D\rho_{ens}}{Dt} + \sum_s \rho_{ens} \left[\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s}\right] = 0$. Now, we know that we have to show the term in the parenthesis is equal to 0.

So, now, remember there is an equation called Hamilton's equation and as the microscopic systems are all conservative in nature, then basically for every position and for every momentum, you can define a Hamiltonian of a conservative system.

(Refer Slide Time: 33:27)



And then finally, you can write that for the total Hamiltonian you can simply write

$$\frac{\partial H}{\partial p_s} = \dot{q}_s, \qquad \frac{\partial H}{\partial q_s} = -\dot{p}_s$$

Now, this Hamiltonian is a Hamiltonian for the real N particles, who is having q_s and p_s as their coordinates and momenta. Or, now you do not have to go to momenta you just say it is something like velocity and they just admit this Hamilton's equation.

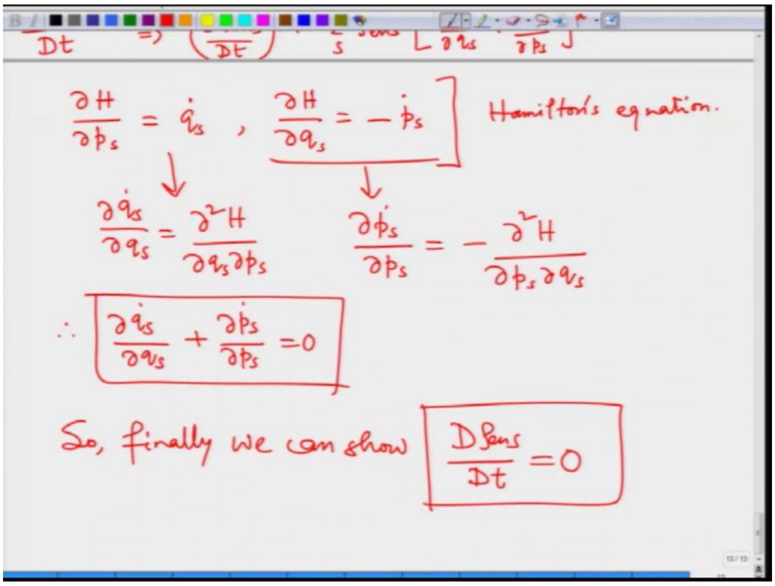
So, once again what I said that here basically, that these total N coordinates that every states were given by $\boldsymbol{r}$ and $\boldsymbol{v}$. So, it is more properly saying that it is not $\boldsymbol{r}$ and $\boldsymbol{v}$, but $\boldsymbol{r}$ and $\boldsymbol{p}$, $\boldsymbol{p}$ is the momentum. And then basically using Hamilton's equation will be much more easier so, q_s will be generalized coordinate and p_s is the generalized momentum.

So, by Hamilton's equation we have

$$\frac{\partial \dot{q_s}}{\partial q_s} = \frac{\partial^2 H}{\partial q_s \partial p_s}, \qquad \frac{\partial \dot{p_s}}{\partial p_s} = -\frac{\partial^2 H}{\partial p_s \partial q_s}$$

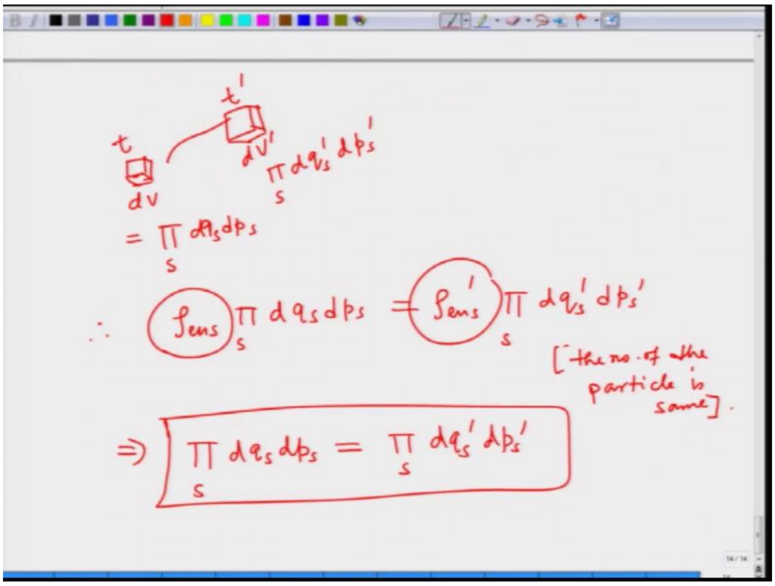
As you know that, ∂p_s and ∂q_s for normal state functions are always interchangeable, and Hamiltonian is a state function.

(Refer Slide Time: 36:07)



So, $\frac{\partial \dot{q_s}}{\partial q_s} + \frac{\partial \dot{p_s}}{\partial p_s} = 0$. So, finally we can show $\frac{D\rho_{ens}}{Dt} = 0$. So, this is the proof of Liouville's theorem.

(Refer Slide Time: 37:01)



Now, there is an interesting corollary of the Liouville's theorem.

Let us say, that some of the ensemble points which were initially contained in an elementary volume dV, dV which you can write now as a product of $dq_s\, dp_s$ at time t. Now after some finite instance, the element at t' will be dV'.

So, some of the ensemble points which were initially contained in an elementary volume dV, now they are contained in dV' . So, the number of the particles are same, that means the mass is the same.

So, we can say $\rho_{ens} \prod_s dq_s dp_s = \rho'_{ens} \prod_s dq'_s dp'_s$. Because, the number of the particles are same.

Now, Liouville's theorem tells us that, if it is measured along the trajectory of one of the ensemble points, not arbitrarily, but along the trajectory of one of the ensemble points, then basically, on this trajectory we will have $\prod_s dq_s dp_s = \prod_s dq'_s dp'_s$.

So, you see that if you take some elementary volume in the phase space where this volume is containing initially some of the (q_s, p_s), some of the ensemble points of course. And then you let them evolve and you trace them, let us say you first take some ensemble points you color them so that you always can recognize them.

Then, you trace them and after some time at some later instant t', this packet of particles are now having another volume, before it was dV now it is dV'. But we have always given the restriction that we are always tracing the number of the particles or the the collection of the particles, particles here means the ensemble points not the real particles which are tracked along the trajectory.

Then, what happens? Then you will see that the elementary volume which they occupied before and which they occupy now they are exactly the same. So, that means, that in case of Liouville's theorem we can say that along the trajectory of a real evolution basically, the phase space volume does not change. So, the phase space volume behaves like an incompressible fluid element along the trajectory of an ensemble point.

So, in the next lecture, we will be trying to discuss about the classic equation or the basic equation of the kinetic theory which is the equation of Boltzmann.

Thank you.