Introduction to Astrophysical Fluids
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Lecture – 19
Hydrostatics: model of solar corona

Hi, I hope till now you have been enjoying the course of Introduction to Astrophysical Fluids. So, in the previous discussion, we were discussing of the derivation of real fluid equations. Today, we will study a very specific limit of fluid real fluid equations which is a very trivial limit actually, so it is called the hydrostatic limit. Hydrostatic we all know from our school days. So, today maybe this is a good occasion to revise all these things.

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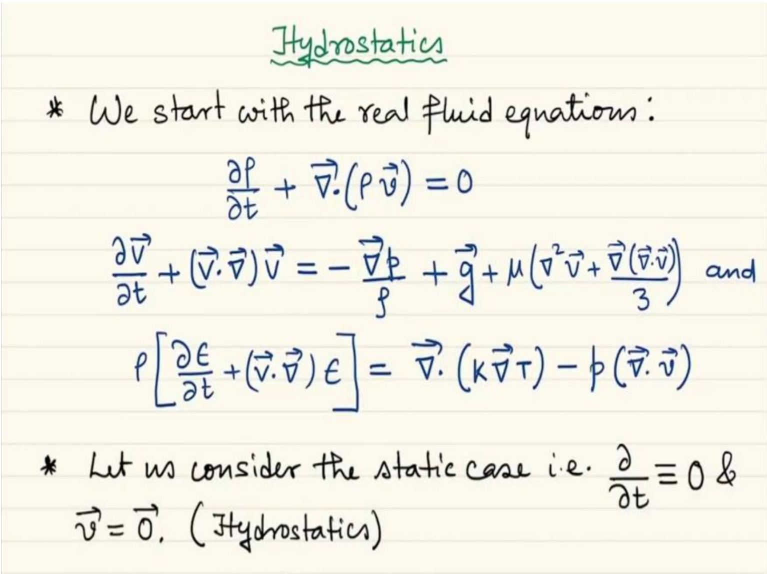

So, if you want to think a bit in a systematic way that whenever in physics we talk about a model or something, how to really verify? So, basically one very simple way to check, and this is not always easy to check whether our model is a good one or a correct one in the sense that whether the model has no inconsistencies.

So, one specific way to check that, is to check its different limits. If we succeed to reproduce our known results for that specific limit, then it is already a very good indication that there a good chance that our model does not have any serious problem, or serious inconsistencies. So, here exactly this is one of the objectives with which we will discuss a specific limit of hydrodynamics and that is the hydrostatics.

That means we consider a fluid is at rest. And for that, we will of course start with our real fluid equations, which we derived in the last discussion. So, the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$. And then you have $\frac{\partial v}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g} + \frac{\mu}{\rho}\left[\nabla^2 \boldsymbol{v} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{v})\right]$.

And finally, the equation for the evolution of the internal energy ϵ. And we can show that this is nothing but $\rho\left[\frac{\partial \epsilon}{\partial t} + (\boldsymbol{v} \cdot \nabla)\epsilon\right] - \nabla \cdot (K\nabla T) + p(\nabla \cdot \boldsymbol{v}) = 0$. So, the gradient term signifies the energy transport by conduction.

So, one simple thing is that how to reach the limit of hydrostatics. So, it simply says that first of all as the fluid is at rest, so let us say the fluid is confined in a container and it does not move. So, the fluid velocity $\boldsymbol{v}$ is equal to 0. So, the bulk velocity or the fluid velocity or the macroscopic velocity is 0. Now, that does not say that the kinetic velocity $\boldsymbol{u}$, they are 0. They are still nonzero, it is only the average velocity which is 0.

Again, another element to designate the static case, that for every variable, whether it is ρ, or $\boldsymbol{v}$ or p, there should not be any explicit time dependence. So, these two things give us the limit of hydrostatics.

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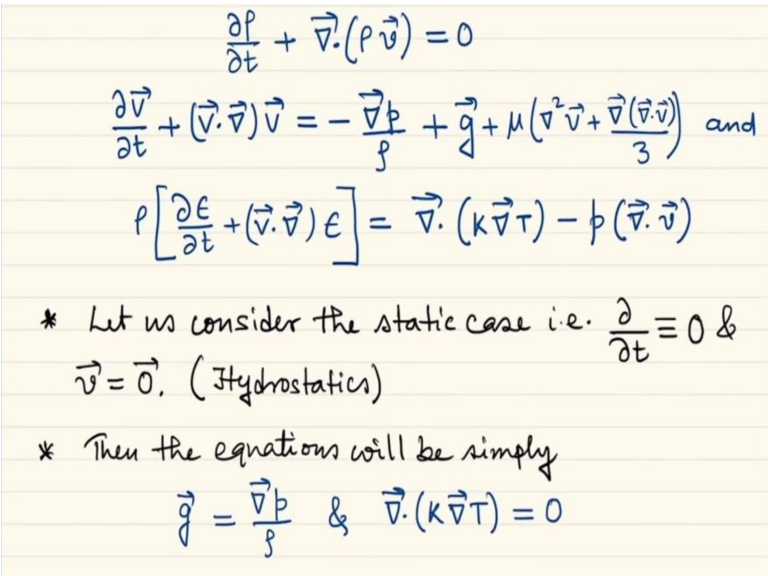

So, then you can easily see the continuity equation is trivially satisfied. So, we do not worry about this equation any longer for hydrostatics. Now, what about momentum equation? So, of course all the terms containing $\boldsymbol{v}$ are zero.

So, if you have this type of case, then you have only the surviving terms to be

$-\frac{\nabla p}{\rho} + \boldsymbol{g} = 0$. So, this is the momentum equation for hydrostatics. So, this is the force balance equation. And now you can recover your school days message of hydrostatics that the net force acting on a fluid at rest is nothing but the force due to its pressure gradient plus the body force.

So, remember when I just was describing the real fluid equations, I said that $\boldsymbol{g}$ is the acceleration due to gravity, but this can be acceleration of any body force. So, actually when the system is at rest, the acceleration due to gravity will be exactly equal to the acceleration due to the pressure gradient force. They will counter balance each other. So, and that is the case of usual fluids confined in a container in everyday cases, because they are all always in a gravity field, the terrestrial gravity.

Now, what about the simplified form of evolution equation for internal energy? This will become, simply $\nabla \cdot (K \nabla T) = 0$. So, finally, we have these two equations to be the governing equations for hydrostatics.

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* Now in case of one dimensional case, we have

$$\frac{dp}{dz} = -\rho g \qquad [\vec{g} = -g\hat{e}_z]$$

for incompressible fluids, ρ = Constant and so,

$p = p_0$ $z = 0$

$$\underbrace{p = p_0 - \rho g z}_{\text{So the pressure of a fluid at rest increases linearly with depth}}, \text{ where } p_0 \text{ is the pressure at } z = 0$$

* Now let us take the simplest compressible case:

isothermal fluid i.e. T is constant. For this case,

Now, you see of course, we made $\frac{\partial}{\partial t} = 0$. So, nothing is evolving in time. So, the equations have changes, but they are only spatial changes, or gradients, so that is something to understand. By the definition of hydrostatics, we do not allow the explicit time variation or evolution of something because we only assume that the evolution is done only by the virtue of the flow. If the flow is not there, then there is no evolution.

Now, in the case of one-dimensional case, we have $\frac{dp}{dz} = -\rho g$, that is nothing but the one-dimensional simplification of $-\frac{\nabla p}{\rho} + \boldsymbol{g} = 0$. Why there is a minus sign? Because $\boldsymbol{g}$ is acting downwards, and we are considering our z-direction to be normally upward direction – vertically upward.

For incompressible fluid, if we now think of let us say not only any fluid at rest, but liquids, for example, at rest let us say a glass of water for incompressible fluids, ρ is constant. So, when you integrate basically this is a very simple integral. So, $dp = -\rho g dz$. And ρ, g both of them are taken to be constant, because g is the body force acceleration, so it is a gravitational acceleration which can also be reasonably assumed to be constant.

And then you just integrate, it will give you $p = p_0 - \rho g z$. And p_0 is nothing but the pressure when z is equal to 0. So, this is the best level pressure.

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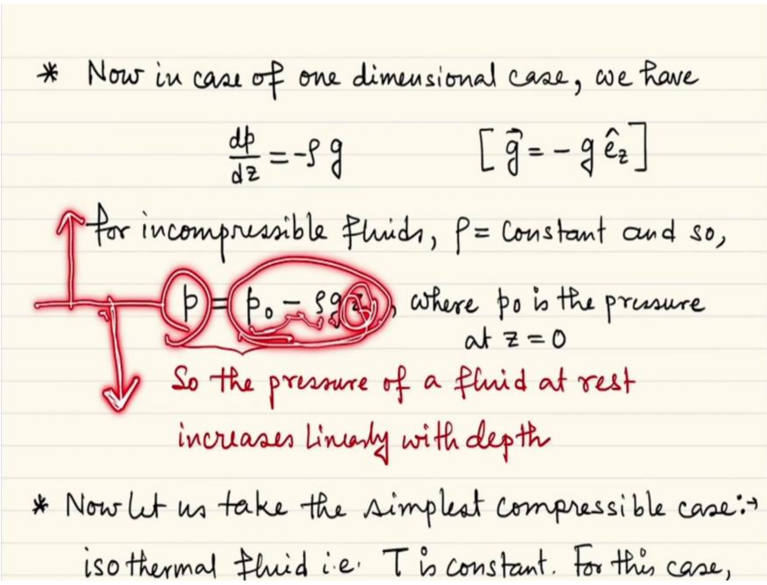

So, it simply says that when you start at level, $z = 0$, your pressure is p is equal to p_0. And when you are just changing your z either upward or downward, your pressure will be changed to p. Now, if you are changing your z in such a direction that your z's are negative, then basically we will see that $-\rho g z$ is getting bigger. So, finally, it will have a greater value of pressure.

And when you are going upward, so your z is positive from the reference level, then your pressure will be less, that is why we can again remember our school days message that the pressure of a fluid at rest increases linearly with depth, of course here the fluid should be very much incompressible.

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So the pressure of a fluid at rest increases linearly with depth

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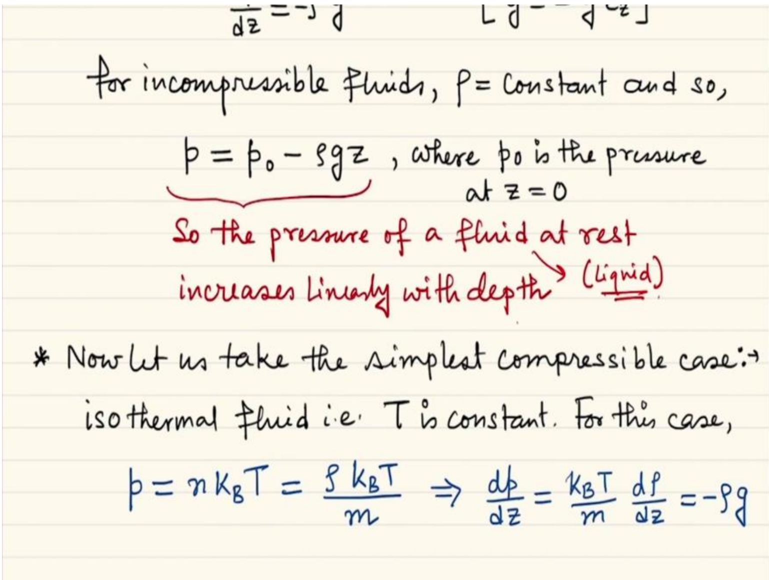

Here fluid means it is mostly liquid actually. Now, we will see what happens for the simplest compressible case.

So, once again we are always in the hydrostatic regime and we are talking about compressible case with a very simple closure, which is call the isothermal closure, that means T is constant. If T is constant, then you know from kinetic definition $p = nk_BT$; and n can be written as $\frac{\rho}{m}$. So, finally you will have $p = \frac{\rho k_B T}{m}$.

So, now if you just remember $\frac{dp}{dz} = -\rho g$, then instead of p you can write $\frac{\rho k_B T}{m}$. And $\frac{k_B T}{m}$ is constant because T is constant, comes out. And this is nothing but $\frac{k_B T}{m}\frac{d\rho}{dz} = -\rho g$.

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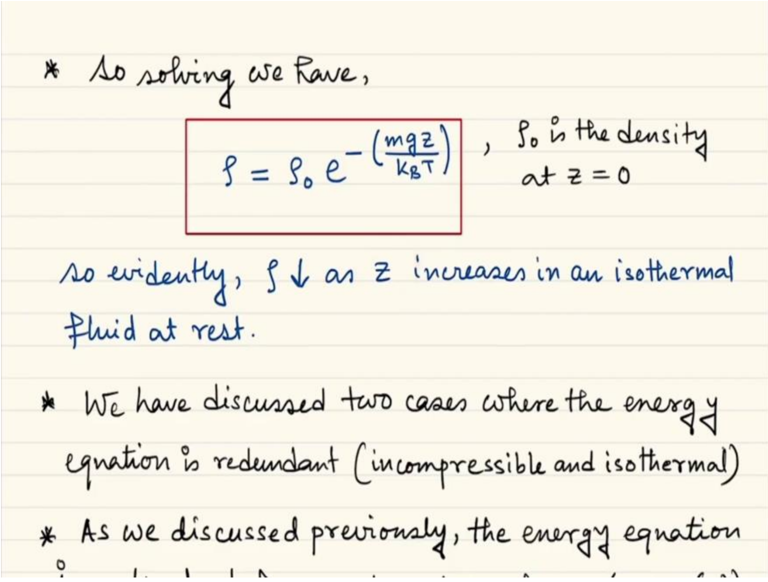

So, if we solve that you know this will be nothing but a exponential law for the density as a function of z. We are always in one dimension for simplicity and this is simply

$\rho = \rho_0 e^{-\frac{mgz}{k_B T}}$. So, where ρ_0 is nothing but the density at z is equal to 0, at reference level. So, evidently here we can see pressure density also decreases, but in a different way.

So, pressure was decreasing in a linear way for incompressible fluid; but for isothermal fluids density is decreasing, but exponentially as z increases ok. So, these things you also may have learnt, all these things are very known results.

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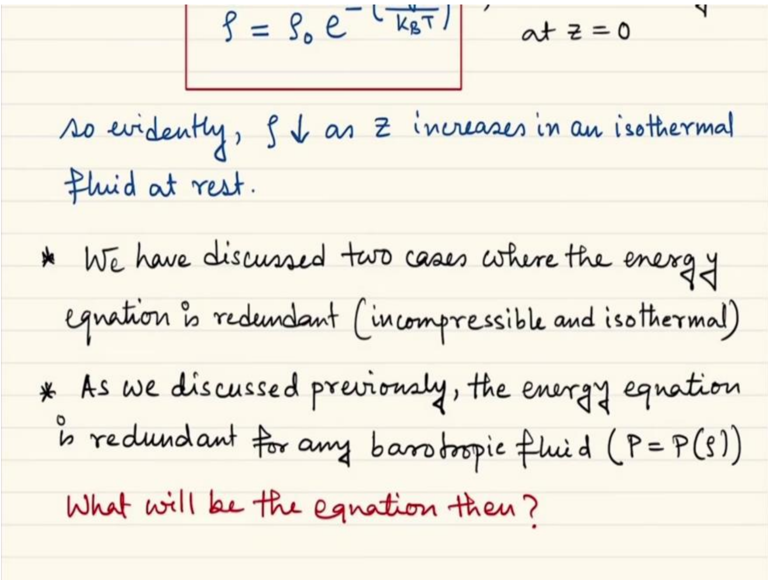

So till now we have discussed one case of incompressible fluids that is the liquids, and one case for isothermal fluids. For two cases, we know that the only useful or necessary equations are the equations of continuity and equation of momentum. But in our current hydrostatic case, equation of continuity is trivially, satisfied always. So, equation of momentum is the only equation to be read.

So, energy equation to be very honest is redundant to constitute the dynamical theory for incompressible and isothermal fluids. There is another class of fluid which you know very well, barotropic fluids, where p can be written as a function of the density only under that condition the fluid is known as barotropic fluid. We should be only worried about or concerned about the momentum equation and not the energy equation. But then what will be the equation?

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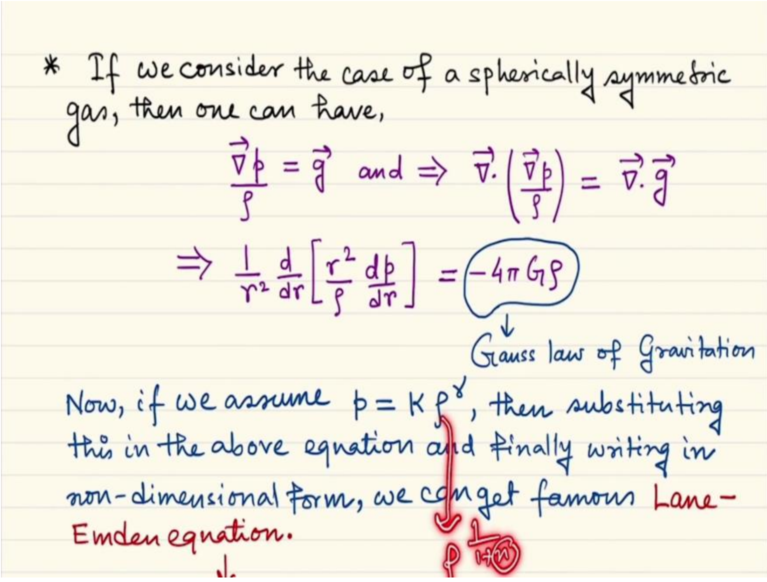

Well, the basic vectorial equation should be the same $\frac{\nabla p}{\rho} = \boldsymbol{g}$. Then, now we will do something in interesting, and that people in general do when they are working on stellar structures. So, what they do? They use this type of equation to determine the stellar structure of a polytropic gas which is self gravitating, to study its equilibrium.

So, the traditional way is to take the divergence of both sides. So, you will have $\nabla \cdot \left(\frac{\nabla p}{\rho}\right) = \nabla \cdot \boldsymbol{g}$. What is the advantage of taking divergence? That finally, you can use Gauss's law of gravitation to find $\nabla \cdot \boldsymbol{g} = -4\pi G\rho$, ρ is the density. So, on the LHS you have p which is the function of density and on RHS you have ρ again.

So, finally what happens that you do not have to think of the body force field again. So, you just have everything in terms of the fluid density. And the pressure, but pressure is also in terms of the fluid density because this is a barotropic fluid.

So, now, if we assume the closure to be $p = K\rho^{\gamma}$, although in some literatures you can see another different type of writing. So, they sometimes write in astrophysics $\rho^{\frac{1}{1+n}}$, and they call this n as polytropic index, but finally this $\frac{1}{1+n}$ is nothing but γ.

So, if we now assume p is equal to $K\rho^{\gamma}$, where γ is our polytropic index in fluid dynamics in general, then substituting this in this above equation and finally you just write, assuming spherical symmetry, now thinking of a polytropic self gravitating gas, the most natural thing to assume is the spherical symmetry. So, the divergence in spherical symmetric case is given by $\frac{1}{r^2}\frac{d}{dr}$, so that is the divergence operator. And what is inside? $\frac{dp}{dr}$, which is nothing but gradient of p. And $\frac{\nabla p}{\rho}$ is nothing but $\frac{1}{\rho}\frac{dp}{dr}$. So, when you write the whole thing in case of spherical symmetry, you will have $\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dp}{dr}\right] = -4\pi G\rho$. And when this is written again I said in advance that replacing p by $K\rho^{\gamma}$, and then finally, writing everything in terms of the non-dimensional quantities ,

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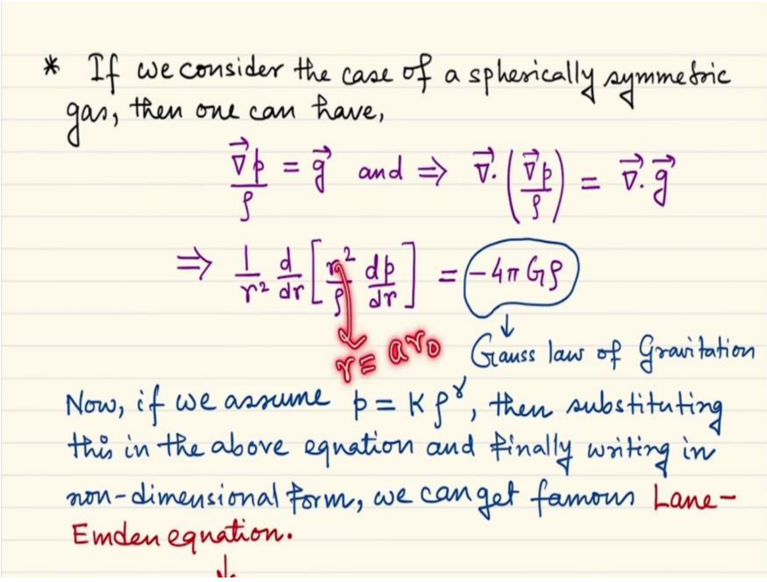

which you can do by saying that r is equal to ar_0, these types of things.

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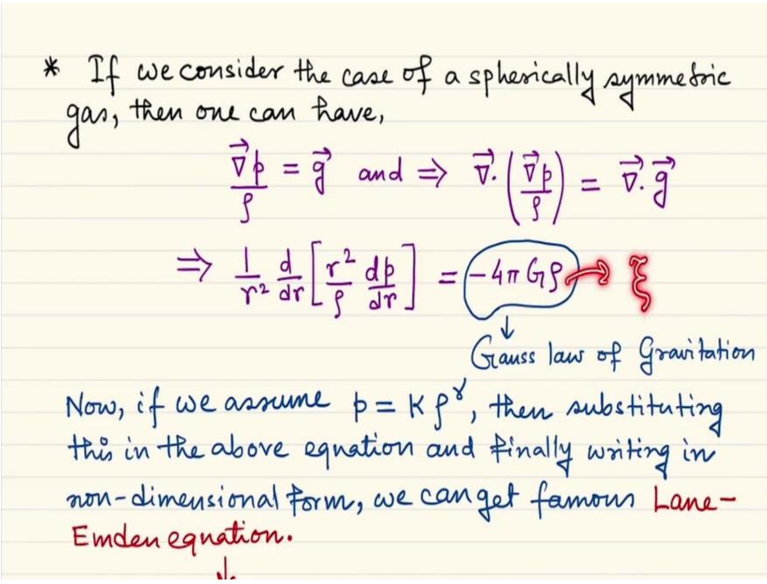

If you do that and you can also define some non-dimensional quantities for ρ like ξ. If you do that finally, you will be given an equation which is known as the famous Lane-Emden equation.

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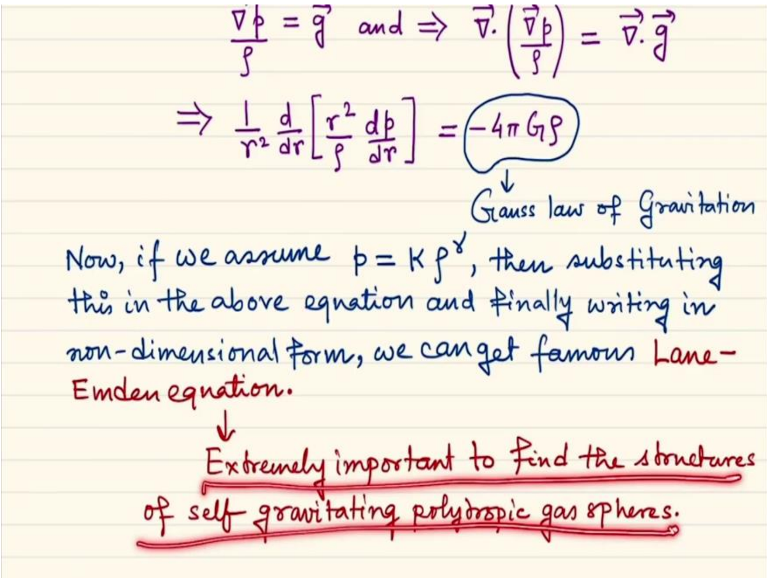

So, Lane-Emden equation finally gives you important information about the structures of the self gravitating polytropic gas spheres, so that is the advantage of doing that. But here in the scope of this course, we are not really studying stellar structures, but I am just telling that hydrostatic which is a simple limit of hydrodynamics is also very very important in certain aspects of astrophysics.

So, this type of thing is very interesting for studying stellar structures, stellar evolution, because this type of equilibrium can be thought to be one of the best phase of star forming gas.

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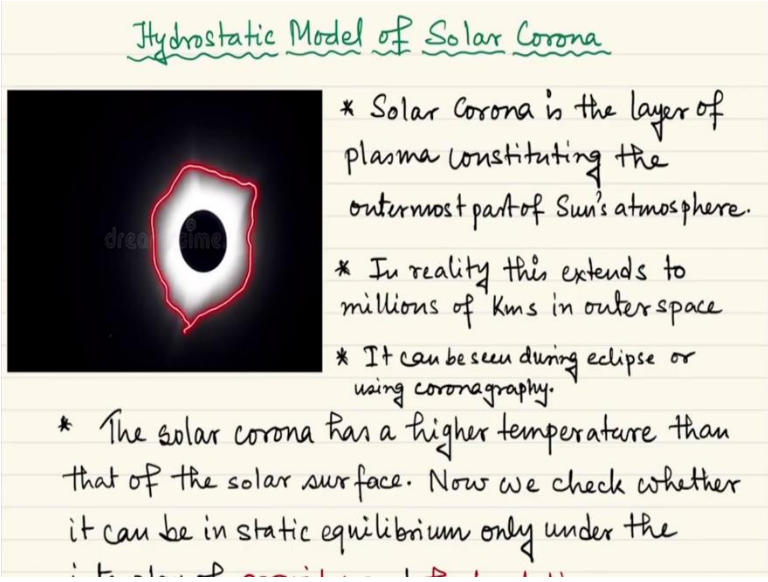

So, now we reproduced finally some hydrostatic results which we already knew. And we also said how hydrostatics, how the treatment of hydrostatics can be interesting in the domain of astrophysics, rather stellar structure.

But now we will do a problem which is of utter interest in for our course, and we will see that basically when we model, in some of the cases it may happen that the model is done assuming something and finally, it comes out negating or contradicting the basic assumption. So there is nothing to get disappointed for that. It simply says that we have to finally search for the reason for that inconsistency, that is how one should do research.

So, people try to fit hydrostatic model to Solar Corona. Now, to know that we have to know what solar Corona is. So, solar Corona is nothing but the layer of plasma constituting the outermost part of the Sun's atmosphere. So, you all know that in the Sun there is a photosphere and just above photosphere, you can have a layer of plasma. You can see this in the total eclipse, the black thing basically covers up to photosphere. And after that this thing which has a burning type of structure, is the solar Corona (see figure above). So, we know now that in reality the plasma is actually nothing but a flow of plasma.

So, the Corona is not static at all. So, the Corona actually it extends to millions of kilometers in outer space that we know. But when we did not know about it, then people started different attempts to understand the several features of Corona and they started by fitting the simplest hydrostatic model to the solar Corona.

So, just one information solar Corona in general can be seen only during eclipse, total eclipse, this is very fantastic view actually. And otherwise it can be seen using coronagraphy.

This is a question for you to think, do you know why we have to wait for a total eclipse or we have to use coronagraphy? Why one cannot see corona in naked eye or in normal photography? So, please think about that, and now actually you can search. And if you find the answer it is okay. If you want to share your answer and try to verify cross check, you can write in the forum.

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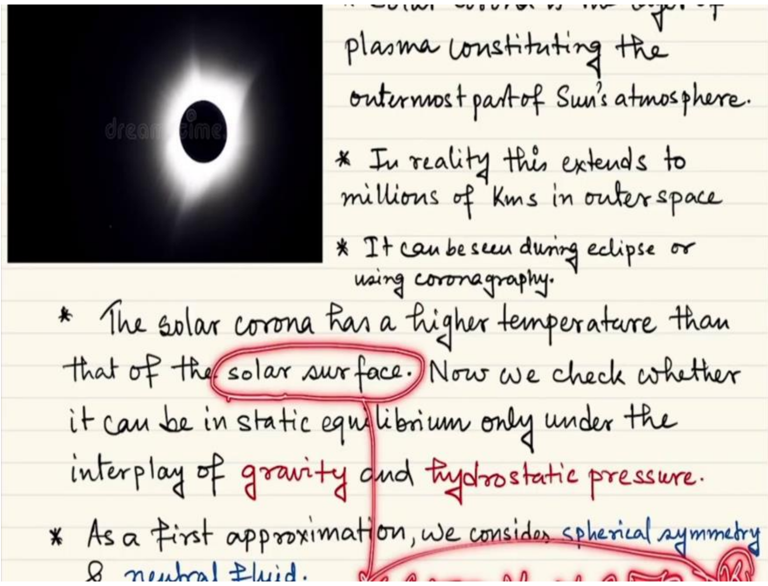

Now, coming to the basic point, the point is that the Solar Corona has a higher temperature and it is not like one or two times higher. This is actually roughly I mean 150 to 400 times higher temperature than that of the solar surface.

You know maybe the normal temperature of the solar surface this is nearly 6000 Kelvin. So, if it is like 250 times of that, so maybe you can think what can be the temperature of solar Corona if you calculate roughly.

So, now why this has a higher temperature that we will discuss later in some other aspects if that arises. But here we now check that whether this Corona or rather Coronal plasma here although we are talking about plasma, we will just use normal fluid model, this is a very simplistic model.

Now, we will check whether this fluid can be a kept in a static equilibrium only under the interplay of gravity and hydrostatic pressure, that is the two things I was telling you. The force balance for a normal fluid at rest is the interplay between the gravity and the hydrostatic pressure.

So, for a star forming gas basically the self gravitating force acts radially inwards, and it tries to make the system denser, whereas the counter play is done by the hydrostatic pressure which is radially outward. And thereby trying to reach an equilibrium. Now, whether always the equilibrium is reached or not that is another story, that is the story of stellar evolution, that is also very interesting.

So, now here as a first approximation, we will do two simplifications one is that we consider spherical symmetry, although we can see that this is not quite spherical, but roughly. And then also we consider this as a neutral fluid, although this is a plasma.

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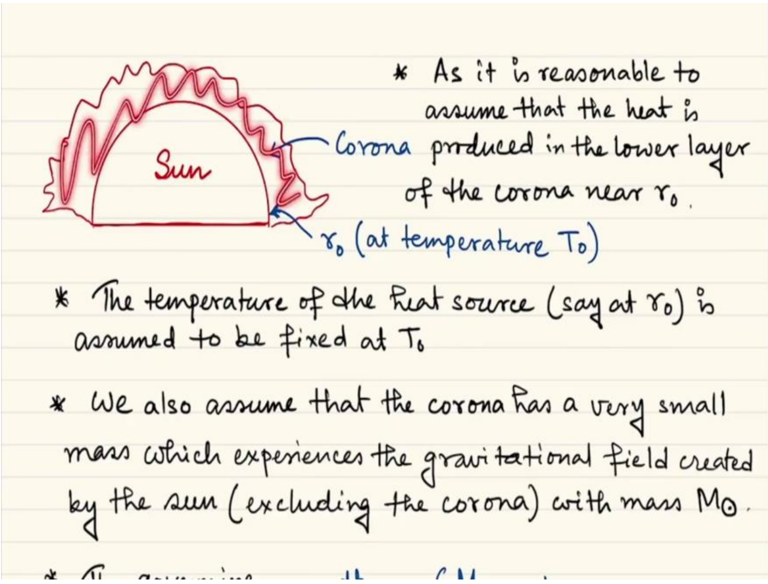

We will also see that at one point, we will use some property of plasma that is also important. So, here basically we are trying to do some model. So, according to the necessity of the model, we will do several things. So, this is a schematic figure of the Sun and the Corona (see above). The outer most part of the solar atmosphere is the Corona.

And this is very reasonable to assume that the heat is produced in the lower layer of the corona near r_0. So, we are assuming that basically as the Corona is hotter than the photosphere, then it is very reasonable to assume that somewhere between the outer part of the photosphere and the base part of the Corona, some energy is produced. And that is responsible for that higher temperature of the factor 250 or something.

Let us say this is produced at a radius r_0, this is the base of the Corona. So, to be very very honest, this is not exactly the photosphere, but roughly this is close to the photosphere. Now, you can also check the detailed structure of a Sun, there are several regions starting from its center. So, radiation zone, convective zone, again radiation zone, so you can just see all these things. So, this part should be clear when you are doing this type of studies.

So, then we will just think that this plasma or this fluid is heated due to some heating mechanism which is at radius r_0. So, at this radius we can assume that there is a furnace or a heater a burner. And that burner has a fixed temperature T_0 which is a very high temperature, and this is giving energy. So, the burner gives energy to this plasma.

Now, we also assume that the Corona has a very small mass which experiences the gravitational field created by the Sun. Now, that is quite reasonable because if you just see that Sun is highly dense, and the mass is super high with respect to the plasma mass.

So, this is a good approximation that the mass of the plasma basically experiences the gravitational pull by the Sun, excluding the Corona. Actually Corona does not add much to the mass of the Sun.

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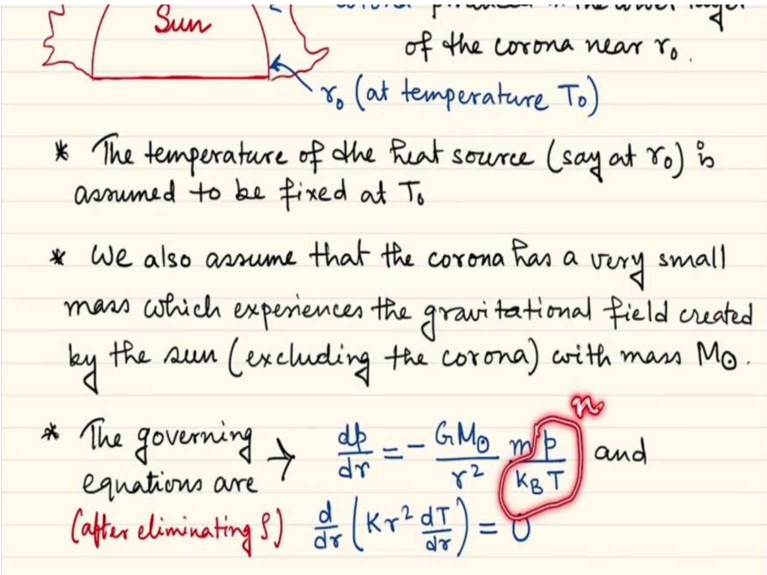

So, how much it can add? That is also a good question. Again search in the internet for some estimates. So, a very good way to learn astrophysics is to have estimates. If you know several types of estimates in order of magnitude, then you have learnt actually 50 percent of the essentials of astrophysics.

Now, so with this thing we can write the governing equations and which should look like now, we are just eliminating ρ using the expression of a relation of p and ρ using ideal gas type equation which comes from kinetic theory.

So, you can simply see that $\frac{dp}{dr} = -\frac{GM_\odot}{r^2}\frac{mp}{k_B T}$. Where $M_\odot$ is the Solar mass and m is the mass of one particle of the Corona plasma.

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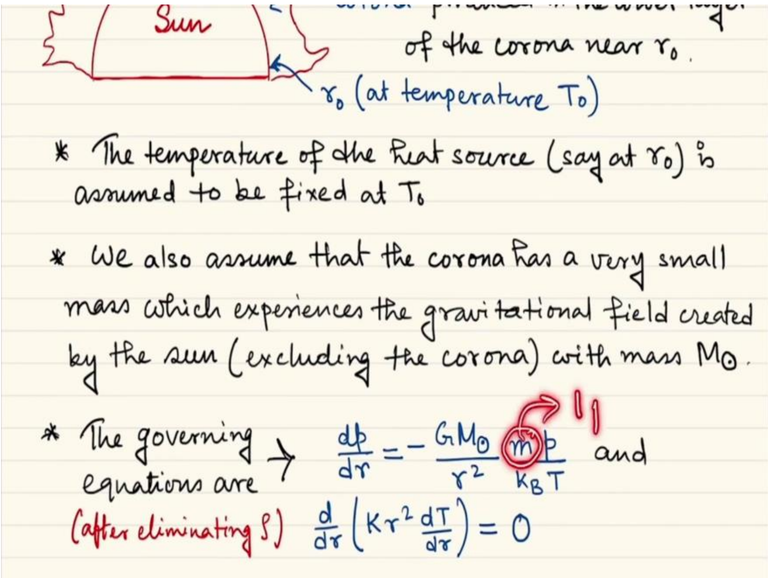

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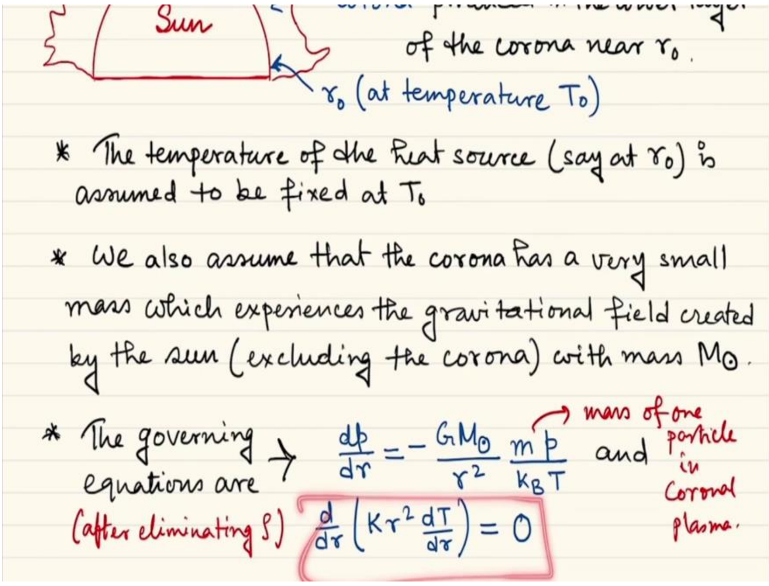

And $\frac{p}{k_B T}$ is the number density of the particles. And m times the number density gives us the mass density of the Coronal fluid or the Coronal plasma. The energy equation is $\frac{d}{dr}\left(Kr^2\frac{dT}{dr}\right) = 0$.

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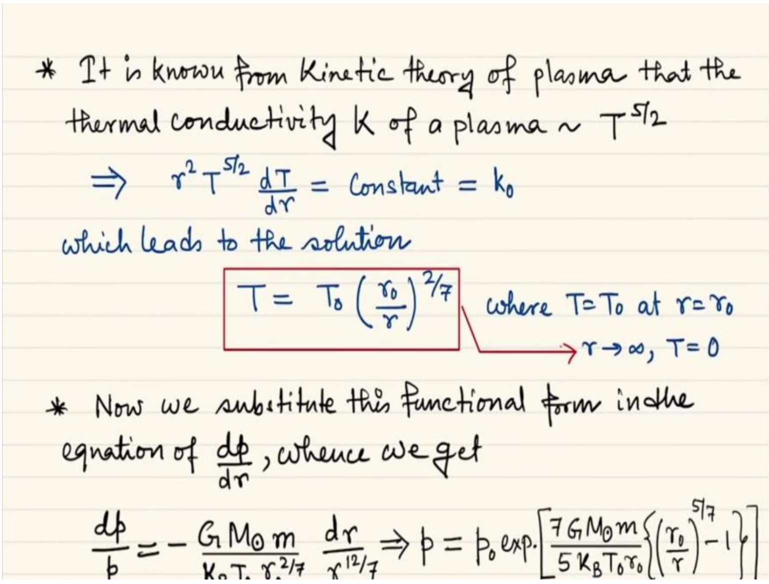

Now, it is known from kinetic theory of plasma, now you see we use some properties of plasma over here that the thermal conductivity K of a plasma has a functional dependence of temperature as $T^{\frac{5}{2}}$. So, now remember when we were discussing about the construction of the dynamical theory for real fluids, at one point I said that the K can sometimes have some explicit dependence on space, and actually it may happen that K sometimes has an explicit dependence on temperature, and that also finally solves the problem. That K and T, they should not be considered as two different variables, K in most of the cases can be an explicit function of temperature as well. For example here, so finally then using 0 divergence condition, you can have, $r^2 T^{\frac{5}{2}} \frac{dT}{dr} = constant = k_0$.

And if you now integrate this equation, you will have $T = T_0 \left(\frac{r_0}{r}\right)^{\frac{2}{7}}$, where T_0 is nothing but the temperature at r_0. Now, from this equation, if I ask you to tell me the temperature at r is equal to infinity or $r \to \infty$, then you can easily say that at infinity the temperature is 0. Well, that is ok, that is quite reasonable. This is a hydrostatic model, then after a finite distance the plasma is no longer there. So, the energy cannot go there by conduction process, because the energy has to go there only by conduction, it is not radiation, this is a conductive process. So, T is equal to 0 simply says that the plasma cannot go up to infinity, and that is completely true, that is completely consistent with hydrostatic model. So, this is confined in a finite space.

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$$\Rightarrow \quad r^2 T^{5/2} \frac{dT}{dr} = \text{Constant} = k_0$$

which leads to the solution

$$\boxed{T = T_0 \left(\frac{r_0}{r}\right)^{2/7}} \quad \text{where } T = T_0 \text{ at } r = r_0$$

$\rightarrow r \to \infty,\ T = 0$

* Now we substitute this functional form in the equation of $\frac{dp}{dr}$, whence we get

$$\frac{dp}{p} = -\frac{G M_{\odot} m}{K_B T_0 r_0^{2/7}} \frac{dr}{r^{12/7}} \Rightarrow p = p_0 \exp.\left[\frac{7 G M_{\odot} m}{5 K_B T_0 r_0}\left\{\left(\frac{r_0}{r}\right)^{5/7} - 1\right\}\right]$$

But p does not vanish as $r \to \infty$! The source of the problem is the static model $\Rightarrow$ Corona is expanding!

Now, we substitute this functional form in the equation of $\frac{dp}{dr}$ and we will see what happens. Now, we will see to our utter surprise that $\frac{dp}{p} = -\frac{GM_{\odot}m}{k_B T_0 r_0^{\frac{2}{7}}} \frac{dr}{r^{\frac{12}{7}}}$, and if you integrate that completely just by using that at $r = r_0$, $p = p_0$, then finally you will have the expression for the pressure to be $p = p_0 \exp\left[\frac{7GM_{\odot}m}{5k_BT_0r_0}\left\{\left(\frac{r_0}{r}\right)^{\frac{5}{7}} - 1\right\}\right]$. And now if I ask you what will be the pressure as $r \to \infty$?

Well, as r tends to infinity $\frac{r_0}{r}$ will be 0. So, the pressure will $p = p_0 \exp\left[-\frac{7GM_{\odot}m}{5k_BT_0r_0}\right]$. So, you will have a nonzero pressure. So, pressure does not vanish at $r \to \infty$, so that means, this is inconsistent with the static model. So, what is the meaning of that? That means, the static model, although it was good for temperature, but for pressure it creates an inconsistency.

And that is why Parker then actually correctly predicted that the Corona is expanding. And not only the corona is expanding, this is expanding from all the surfaces, we will see this in the next lecture, so that is the origin of different type of stellar winds, for example, solar wind.

So, you see that an inconsistency in a simplified model could actually lead to a very deep conclusion about the existence of solar wind, because it is true that nowadays we all know that there is a plasma flow radially outward from the Sun. But beforehand people did not know and that is why they applied the hydrostatic model. They applied it because for them it was the most reasonable one, but when they failed, then they started to think otherwise. So, in the next lecture we will discuss about solar wind.

Thank you very much.