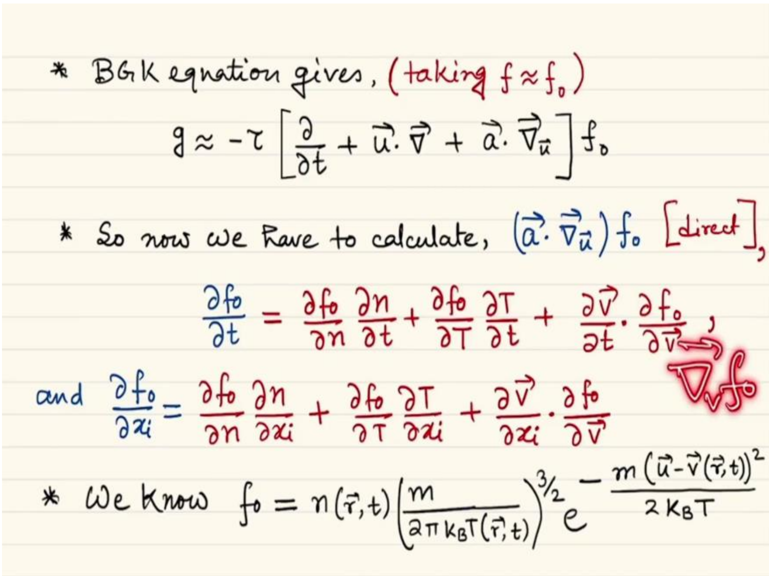
Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture – 18
Derivation of real fluid equations

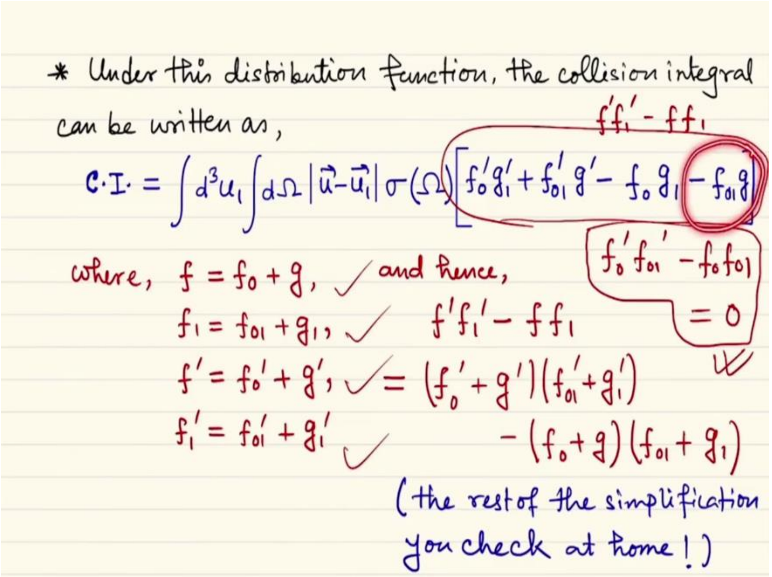
Hello. So, we continue our discussion of the Derivation of real fluid equations; starting from distribution function which has a small first order perturbation with respect to the zeroth order local Maxwellian distribution.

(Refer Slide Time: 00:35)



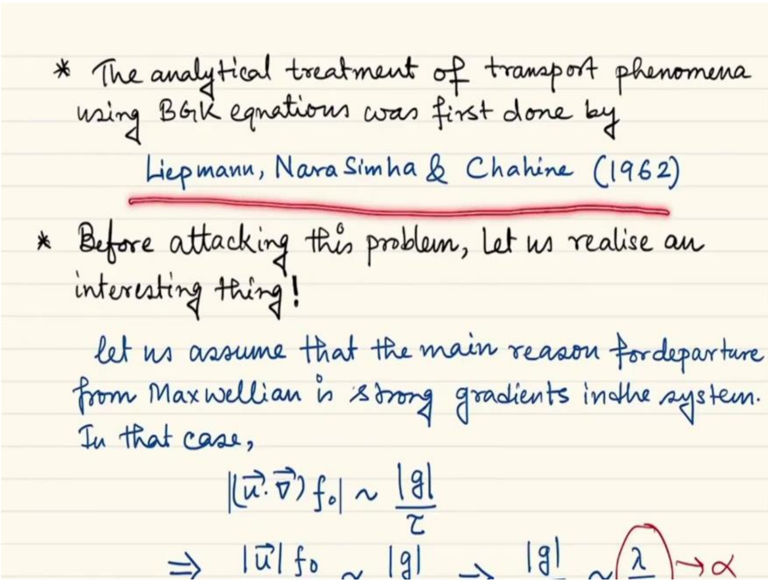
So, we just stated last time that BGK equation considerably simplifies the total formalism just by saying that only one of the term,

(Refer Slide Time: 00:57)



that is the last term of the four terms of $f_0' g_1' + f_{01}' g' - f_0 g_1 - f_{01} g$ is sufficient to adequately recover all the transport phenomena in a usual fluid.

(Refer Slide Time: 01:21)



Now, this was done by three scientists Liepmann, Narasimha and Chahine. So, we said that finally, a very reasonable approximation gives $g \approx -\tau \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla + \boldsymbol{a} \cdot \nabla_{\boldsymbol{u}} \right) f_0$. So, f_0 we know. What is f_0? f_0 is nothing but the local Maxwellian distribution.

So, now we have to calculate the three term. Now, we know that f_0 is the local Maxwellian distribution. So, calculating $(\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}}) f_0$ is reasonably direct because in f_0, if you see the form of f_0 the dependence on $\boldsymbol{u}$ is explicit dependence and that is why this is direct to calculate. So, please calculate at home.

Once again in this course it will be not possible to do all the steps of intermediate algebra, but you are supposed to check all the steps of algebra at home; if in case you are blocked you can ask me or you can just refer to one of the books which I suggested in your course handout.

Then we have to calculate also $\frac{\partial f_0}{\partial t}$ and $\frac{\partial f_0}{\partial x_i}$, which corresponds to first two terms in the expression of g. Now, for that we have to realize the fact that in f_0, which is the local Maxwellian, the explicit space and time dependent comes through three variables n, T and $\boldsymbol{v}$.

So, when we will try to calculate $\frac{\partial f_0}{\partial t}$ the total expression will be simply

$$\frac{\partial f_0}{\partial t} = \frac{\partial f_0}{\partial n}\frac{\partial n}{\partial t} + \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial f_0}{\partial \boldsymbol{v}}\frac{\partial \boldsymbol{v}}{\partial t}.$$

Where, $\frac{\partial f_0}{\partial \boldsymbol{v}} = \nabla_{\boldsymbol{v}} f_0$, so in the similar way you can also write $\frac{\partial f_0}{\partial x_i}$ as a summation of three terms through their implicit dependence of n, T and $\boldsymbol{v}$ (see below).

(Refer Slide Time: 04:31)

$$g \approx -\tau \left[\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \vec{a}\cdot\vec{\nabla}_{\vec{u}}\right] f_0$$

* So now we have to calculate, $(\vec{a}\cdot\vec{\nabla}_{\vec{u}}) f_0$ [direct],

$$\frac{\partial f_0}{\partial t} = \frac{\partial f_0}{\partial n}\frac{\partial n}{\partial t} + \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial \vec{v}}{\partial t}\cdot\frac{\partial f_0}{\partial \vec{v}},$$

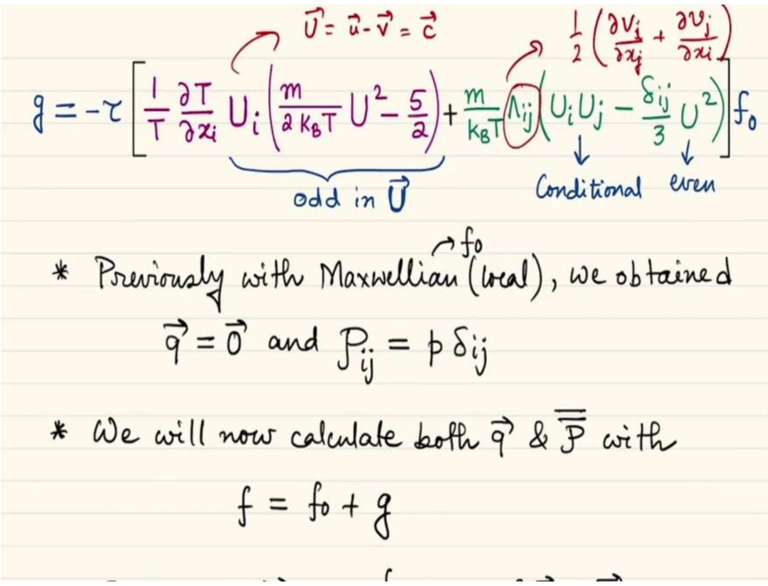
and $$\frac{\partial f_0}{\partial x_i} = \frac{\partial f_0}{\partial n}\frac{\partial n}{\partial x_i} + \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial x_i} + \frac{\partial \vec{v}}{\partial x_i}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

* We know $$f_0 = n(\vec{r},t)\left(\frac{m}{2\pi k_B T(\vec{r},t)}\right)^{3/2} e^{-\frac{m(\vec{u}-\vec{v}(\vec{r},t))^2}{2k_BT}}$$

* Now a few steps of straight forward but careful Algebra (Please Check) finally give

So, now you have to do step by step some straight forward but careful algebra. If you make some mistake over here then you will be lost. So, do slowly even at home when you will practice, do slowly but step by step do not do any step jump.

(Refer Slide Time: 04:53)



And this will finally give you g to be equal to the whole thing (see above). So, it may look like a bit I mean lengthy or a bulky, but believe me if you try to understand segment wise this is quite interesting.

So, inside the bracket there are several things. So, $\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla + \boldsymbol{a}\cdot\nabla_{\boldsymbol{u}}\right)f_0$ all these things give us finally, a factor times f_0. So, finally, again this whole thing is proportional to f_0 as well this is also proportional to τ.

That is very very interesting that it directly says that when your system is strongly collisional, τ is very small-your g is small; when τ is big and your systems collisionality is very weak then you have basically a considerable g.

So, g is consisting of globally two type of contributions one comes from $\frac{1}{T}\frac{\partial T}{\partial x_i}U_i(\frac{m}{2K_BT}U^2 - \frac{5}{2})$ and this one is proportional to $\frac{\partial T}{\partial x_i}$, another one proportional to Λ_{ij}; if you remember what Λ_{ij} was! So Λ_{ij} was just the velocity gradient tensor like $\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, this is a symmetric tensor.

So, one term should be proportional to the temperature gradient and you can see that term has an odd dependence on the components of $\boldsymbol{U}$ because there is always one U_i which is unpaired, which makes component wise the thing odd.

Now, another part which is depending on the velocity gradient has two parts – one is of course, clearly even, the another is conditionally even or odd. So, let us say if you multiplied with $U_iU_j - \frac{\delta_{ij}}{3}U^2$, something where both U_i and U_j can pair, like if you just multiply U_i and U_j, with all the thing, then this part will become behave as a even combination.

On the other hand, if you just multiply with a single U_i , then U_i will pair, but U_j will remain unpaired or uncoupled. So, that will give you an odd contribution. So, this is the conditional part, $\delta_{ij}U^2$ is even always because it is U^2.

So, you see after all these simplifications finally, we have a form of g which looks like a bit frightening, but believe me finally this is a very handy form. So, you can decouple the total contribution as a sum of a contribution which is proportional to the linear temperature gradient another is a linear velocity gradient.

So, if you remember now, that from your basic knowledge or previous knowledge of fluid dynamics or something, it is somehow expected that the heat transfer should be related to temperature gradient term and momentum transfer or the viscosity basically you know that viscosity is nothing but a transfer of momentum or transport of momentum should be related to Λ_{ij}, and one of the very simplest models we know is the Newtonian fluid model that I will come later.

But, roughly speaking you can easily understand that maybe the temperature gradient part is related to the thermal conduction or thermal transport part and Λ_{ij} it is related to the momentum transport part. And, we will see that there is no exception to that, what we have guessed is actually correct.

So, previously with Maxwellian distribution we had simply, $\boldsymbol{q}$ is equal to 0, and the pressure tensor was trivially diagonal with a single value which says that all the diagonal elements were equal. So, it was simply a scalar. So, we will now calculate again $\boldsymbol{q}$ and $\bar{\bar{P}}$

with this new distribution function and as we can expect that, $\boldsymbol{q}$ will no longer be 0, P_{ij} will no longer be purely diagonal right.

(Refer Slide Time: 11:05)

odd in $\vec{U}$ Conditional even

* Previously with Maxwellian (local) $\overset{\curvearrowright f_0}{}$, we obtained

$$\vec{q} = \vec{0} \text{ and } P_{ij} = p\,\delta_{ij}$$

* We will now calculate both $\vec{q}$ & $\overline{\overline{P}}$ with

$$f = f_0 + g$$

* Evidently, $\vec{q} = \frac{m}{2}\int (f_0 + g)\, U^2 \vec{U}\, d^3\vec{U}$

$$= \frac{m}{2}\int g\, U^2 \vec{U} d^3\vec{U}$$

So, for $\boldsymbol{q}$ we will have by definition $\frac{m}{2}\int (f_0 + g)U^2\boldsymbol{U}d^3\boldsymbol{U}$. So, $\frac{m}{2}\int f_0 U^2\boldsymbol{U}d^3\boldsymbol{U} = \boldsymbol{0}$. So, finally, the term which remains is $\frac{m}{2}\int gU^2\boldsymbol{U}d^3\boldsymbol{U}$. So, non-zero $\boldsymbol{q}$ basically is an outcome of non-zero g.

(Refer Slide Time: 11:37)

* A careful observation clearly indicates that the part proportional to $\frac{\partial T}{\partial x_i}$ will contribute to the non-vanishing expression for $\vec{q}$.

* Again some steps of Algebra give,

$$\vec{q} = -K\vec{\nabla}T \quad \text{where}$$

$$K = \frac{\tau m}{6T}\int d^3U\, U^4 \left[\frac{m}{2k_B T} U^2 - \frac{5}{2}\right] f_0$$

$$= \frac{5}{2}\tau \frac{n k_B^2 T}{m} \rightarrow K \text{ is thermal conductivity}$$

So, a careful observation now here clearly indicates that the part of g which is proportional to $\frac{\partial T}{\partial x_i}$ will contribute to the non-vanishing expression of $\boldsymbol{q}$, why is that? That is something very interesting and we have to learn how to check that.

So, here you can see component wise $\frac{m}{2}\int gU^2\boldsymbol{U}d^3\boldsymbol{U}$, there is an odd functionality, because if you just write it in components you will have $\left(U_x + U_y + U_z\right)U^2$ and from this one if you just do the multiplication, then some of the component of $\Lambda_{ij}(U_iU_j - \frac{\delta_{ij}}{3}U^2)$ will pair with U_iU_j part, some component may or may not pair, but finally, the other component will be remained uncoupled.

So, if in case some of them will pair with U_i, then one will remain uncoupled; if none of them pair with that then simply both of them will be uncoupled. So, in any case this will vanish. Same thing for U^2, this is an even. So, even times odd gives you something odd with respect to each component of the velocity.

Now, here you have an odd contribution component wise for the velocity. Once again when I say $\boldsymbol{U}$ just remember $\boldsymbol{U}$ is nothing but the fluctuation velocity which sometimes we call $\boldsymbol{c}$ ok. So, just for recapitulation $\boldsymbol{U} = \boldsymbol{u} - \boldsymbol{v} = \boldsymbol{c}.$

Now, only the part with $\frac{\partial T}{\partial x_i}$ has an even contribution and that is why you can easily see that the only possibility that the $\boldsymbol{q}$ has a nonzero contribution may only come from temperature

Now, again here you have some few steps of cumbersome algebra I agree, but systematic algebra, so for this type of thing rather than doing this algebra it is rather interesting to understand the physics. So, some steps of algebra can give you $\boldsymbol{q} = -K\nabla T$.

So, we already showed that this thing will be so, the part of g which is proportional to $\frac{\partial T}{\partial x_i}$ will contribute only. So, $\boldsymbol{q}$ should be expected to be proportional to gradient of temperature and if you say that I now suppose that $\boldsymbol{q}$ is equal to some constant, of course, I am writing minus for the conventional purpose, but minus K times gradient of temperature then one can simply calculate just by comparing and by replacing g

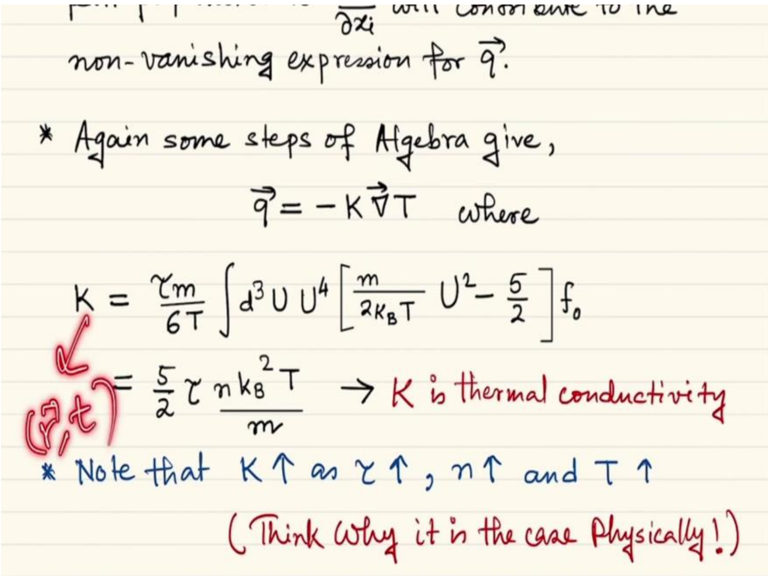
$$K = \frac{\tau m}{6T}\int d^3\boldsymbol{U}U^4\left[\frac{m}{2k_BT}U^2 - \frac{5}{2}\right]f_0.$$

So, you see that finally, again this integration is all over all the components of the $\boldsymbol{U}$. So, it is integrated over $\boldsymbol{U}$.

So, if you do again the integration from minus infinity to plus infinity that is the I mean domain of integration for classical cases you can show that $K = \frac{5}{2}\tau\frac{nk_B^2T}{m}$.

So, if you can do that finally, try to understand what this is. So, this is some quantity which is the proportionality constant of $\boldsymbol{q}$, it may or may not be constant. But at least if you just see the expression, it should not depend on $\boldsymbol{U}$. So, of course, there is T and n.

(Refer Slide Time: 17:21)



So, it will be a function of in general $\boldsymbol{r}$ and t. So, this is not an absolute constant, but this is not a function of kinetic velocity. So, this is another macroscopic quantity if you want. This K, when you match your previous results of real fluids you can see that this is nothing but the thermal wave conductivity.

Now, the physics comes here, you have to note that K is proportional to τ, K is proportional to n, K is proportional to T. Of course, when I am saying proportional to τ, that means, n and T are kept constant. So, by the rule of joint variation you can say that K is proportional to τnT the product, when all of them vary and individually saying that K actually increases.

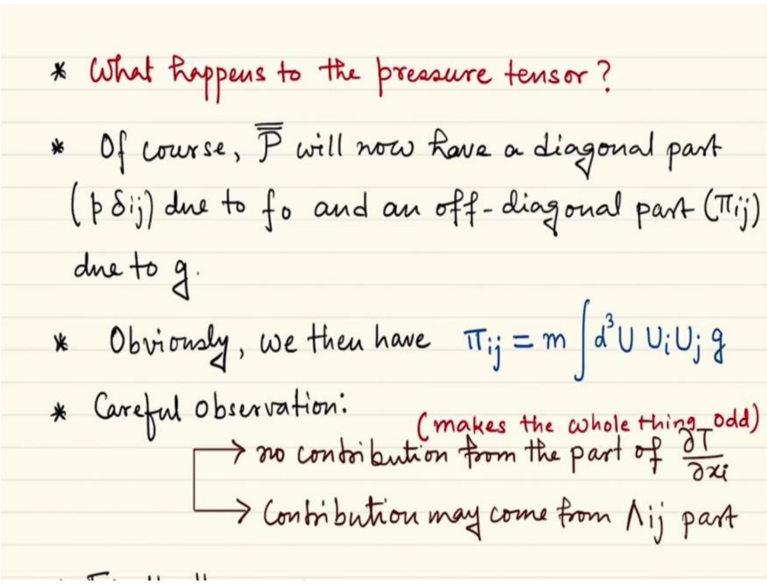
Let us say that n is constant, T is constant, then K increases with τ. So, if in a system for example, we just do not change the temperature and we do not change the density, but we just simply say that the collisionality is now becoming weaker and weaker then strangely the thermal conductivity increases. So, that the system is a fluid. So, that is the problem. So, what is the the problem here? because n is constant I understand, but then when τ is larger, that simply says that the mean collision time is larger, how can that correspond to a greater K? That is simply because in order that the system still behaves like a fluid it should have a greater transportability.

So, this is the this is the story of the thermal transport. So, K is the responsible for the thermal transport. So, at least the duty of K is to do the thermal transport efficiently when τ increases; that means, the collision basically decreases, so, mean collision time increases.

So, if this is the case then you can see that the systems conductivity should be strong enough, so that the system can efficiently transport energy from one part to the other otherwise the system is no longer fluid. But here we have supposed that already we are representing this as a macroscopic thing.

So, being a macroscopic entity where the collective effects are important the necessity that τ is higher is that K must be higher. The same thing if your n is higher, then K is also higher; if your temperature is higher, temperature gradient can also be expected to be higher and then the energy transport should be efficient as well. So, K should be higher. So, these things are in consistency with each other.

(Refer Slide Time: 20:59)

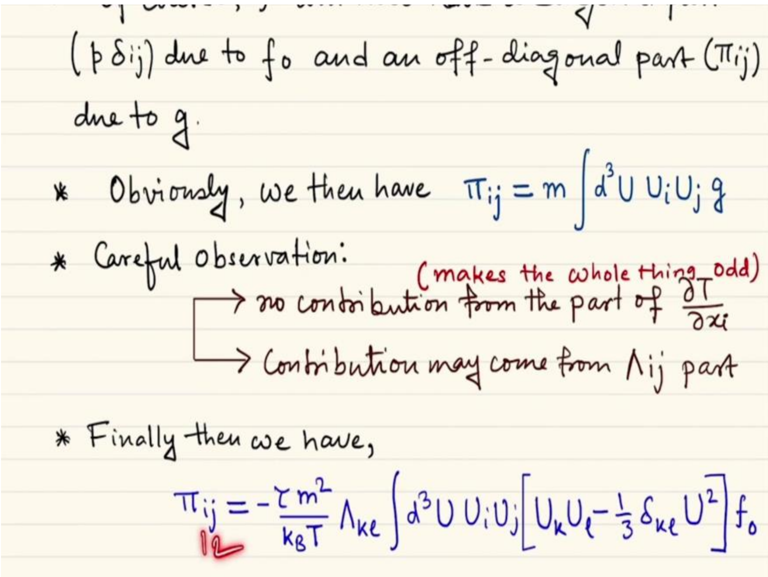


Now, the question is that was the story of the heat flux tensor. Now, what is the destiny of the pressure tensor, what happens to it? Of course, $\bar{\bar{P}}$ will now have a diagonal part $p\delta_{ij}$ due to f_0 plus an off-diagonal part π_{ij} due to the perturbation g. So, obviously, we now can write that $\pi_{ij} = m\int d^3\boldsymbol{U} U_i U_j g$.

Now, again careful observation simply says, if you just take the part which is proportional to $\frac{\partial T}{\partial x_i}$, it will remain odd component wise and making this 0. So, there is no chance of having a contribution from temperature gradient part.

Contribution now may come from Λ_{ij} term. Now, of course in Λ_{ij} when you have this i is equal to j and this part can also contribute when i not equal to j, but ij is exactly equal to this ij. So, that $U_i U_j$ is now paired with $U_i U_j$ gives you $U_i^2\ U_j^2$ square.

(Refer Slide Time: 23:17)

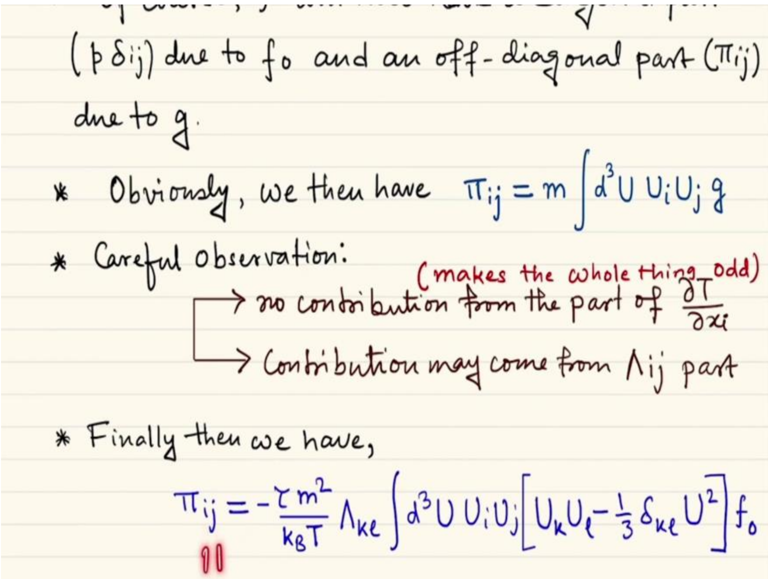


So, even multiplied with conditional even makes it globally even. So, if you finally, write the expression explicitly you will see this is nothing but

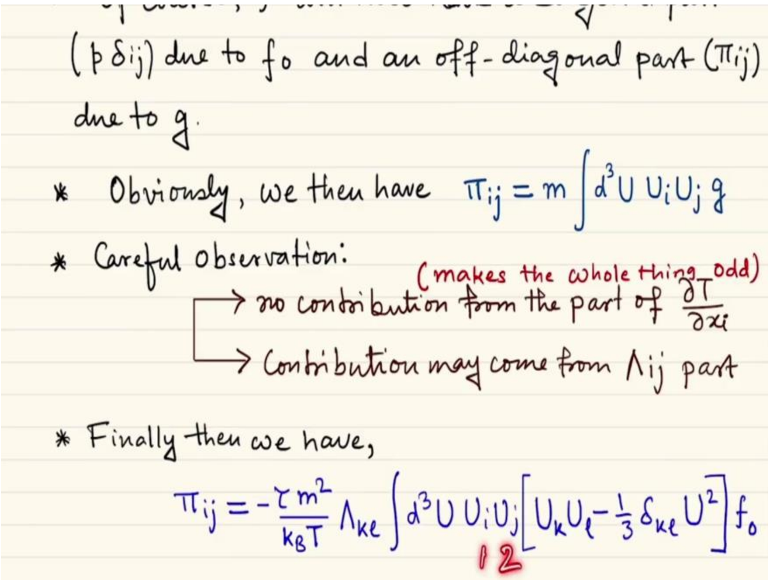
$$\pi_{ij} = -\frac{\tau m^2}{k_B T}\Lambda_{kl}\int d^3\boldsymbol{U} U_i U_j \left[U_k U_l - \frac{1}{3}\delta_{kl}U^2\right] f_0.$$

So, that is the most general way you have to write. So, of course, here you see this i and j they are the free index over there. And kl's are dummy indices; that means, there is a sum over these indices.

(Refer Slide Time: 24:21)

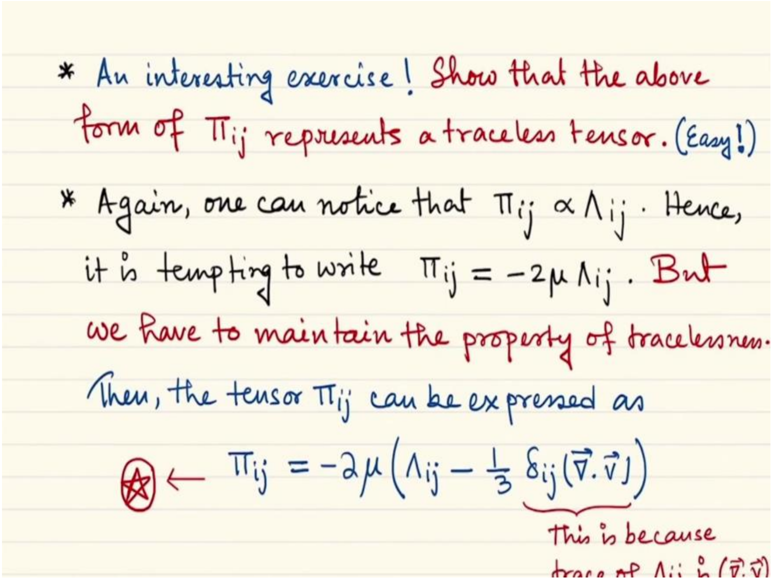


(Refer Slide Time: 24:25)



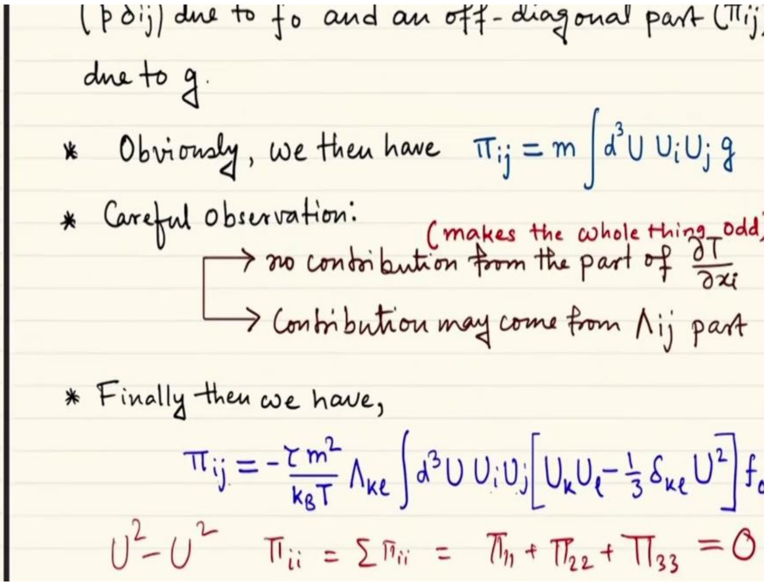
For example, and then when you have fixed ij as 1 and 2 respectively for example, then you make a sum over all the values of kl, because repetitive or dummy indices they are basically designating sums.

(Refer Slide Time: 24:59)



The above form of π_{ij} represents a traceless tensor. Why? what is the meaning of traceless tensor?

(Refer Slide Time: 25:15)



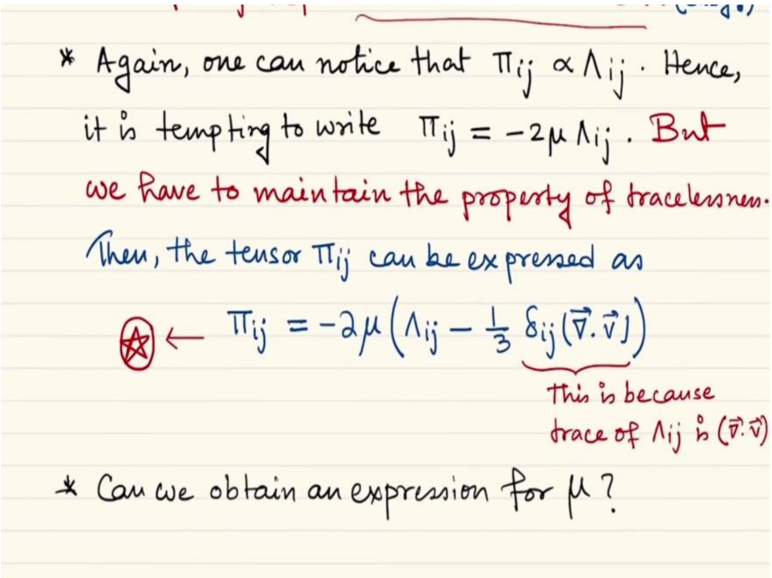
That means, $\pi_{ii} = \sum_i \pi_{ii} = \pi_{11} + \pi_{22} + \pi_{33} = 0$.

And, how is that possible? Let say we just take π_{11} so, $i = 1, j = 1$. So, every time you have $U^2 - \frac{1}{3}U^2$. So, after when you sum all these three things you will have finally $U^2 - U^2$ because $\frac{1}{3}U^2$ will then contribute thrice to make it a U^2.

And it will give you something a traceless tensor. So, this is the necessary condition for a traceless tensor basically.

So, again you can notice here that π_{ij} when we just defined that from its basic definition it is proportional to Λ_{ij}. So, if it is proportional to Λ_{ij}, that is also our consideration that this part will now contribute for π_{ij}.

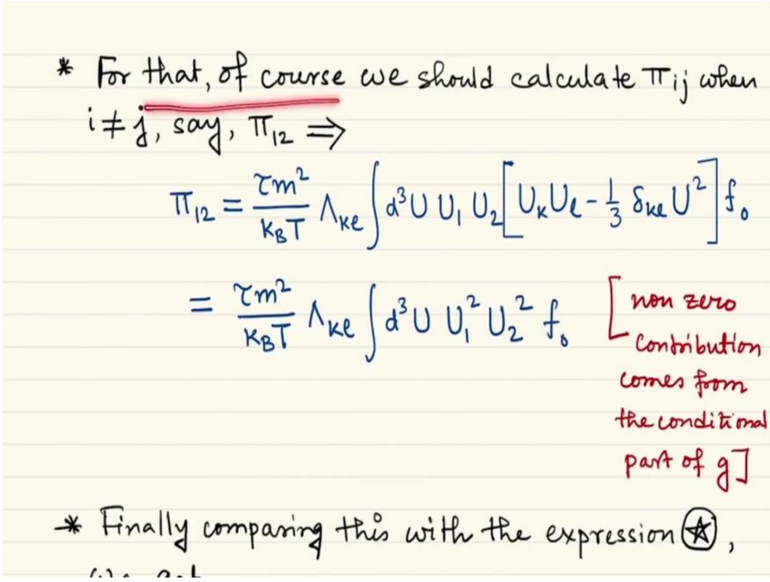
(Refer Slide Time: 28:21)



So, we can simply say if this is proportional to Λ_{ij} then of course, it is tempting to write that this π_{ij} is equal to some constant times Λ_{ij}, but we have to remember always that the property of tracelessness should be satisfied. And, for that the correct way of writing mathematically $\pi_{ij}$is not simply $-2\mu\Lambda_{ij}$, but $\pi_{ij} = -2\mu(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\nabla \cdot \boldsymbol{v})$.

And, we subtracted the divergence part because we know that the trace of Λ_{ij} is divergence of $\boldsymbol{v}$. So, again starting from this expression this is another small exercise for you to check that $\sum_i \pi_{ii} = 0$. Now we have introduced μ as a proportionality constant but can we obtain an expression for μ?

(Refer Slide Time: 29:47)



For that of course, we should calculate π_{ij} when i is not equal to j. Why?

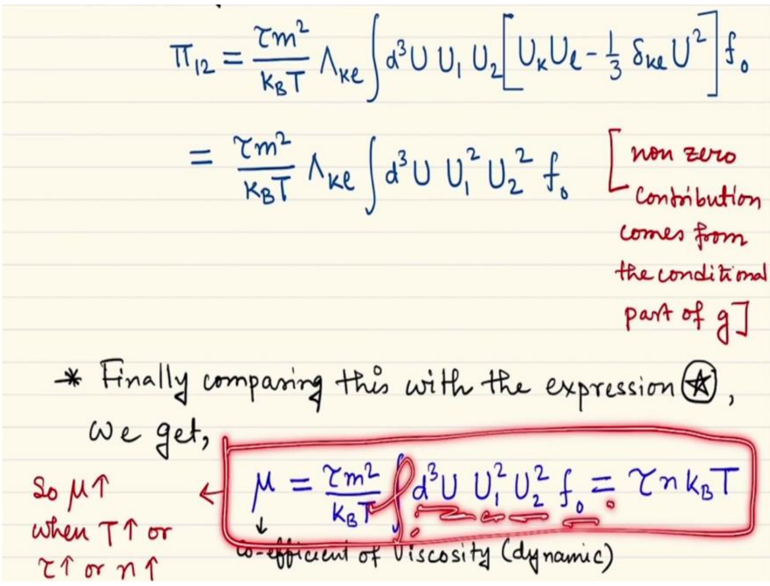
So, it is a very small brain twister can you immediately tell me from the expressions of π_{ij}, why we have to calculate π_{ij} with i not equal to j to calculate μ? Think about it!

And, this expression finally comes out to be

$$\pi_{12} = \frac{\tau m^2}{k_B T} \Lambda_{kl} \int d^3U U U_1 U_2 \left[U_k U_l - \frac{1}{3}\delta_{kl} U^2\right] f_0,$$

because the nonzero contribution only comes from the conditional part of g. Finally, we compare this with the full expression for π_{ij}.

(Refer Slide Time: 33:07)



And, if you do that you will finally, see that

$$\mu = \frac{\tau m^2}{k_B T} \int d^3 U U_1^2 U_2^2 f_0$$

that is equal to, if you correctly do the integration, finally you will see this is nothing but $\tau n k_B T$.

Of course, you can calculate mu just by taking let us say i is equal to 2, j is equal to 3 or i is equal to 2, j is equal to 1 or i is equal to 1, j is equal to 3 anything, but i is not equal to j. You should find the same result to our utter surprise that just like K, μ is also proportional to τ, μ is also proportional to n and T. So, μ increases when T increases, τ increases or n increases. Why?

Now, I discussed about these dependencies for K, now for μ you have to think. This μ from our previous knowledge of fluid equations, the fluid dynamists can match and call that as coefficient of viscosity rather dynamic viscosity coefficient of dynamic viscosity. When this is divided by the ρ, by the density mass density, it is called the coefficient of kinematic viscosity, but in general this is called roughly coefficient of viscosity.

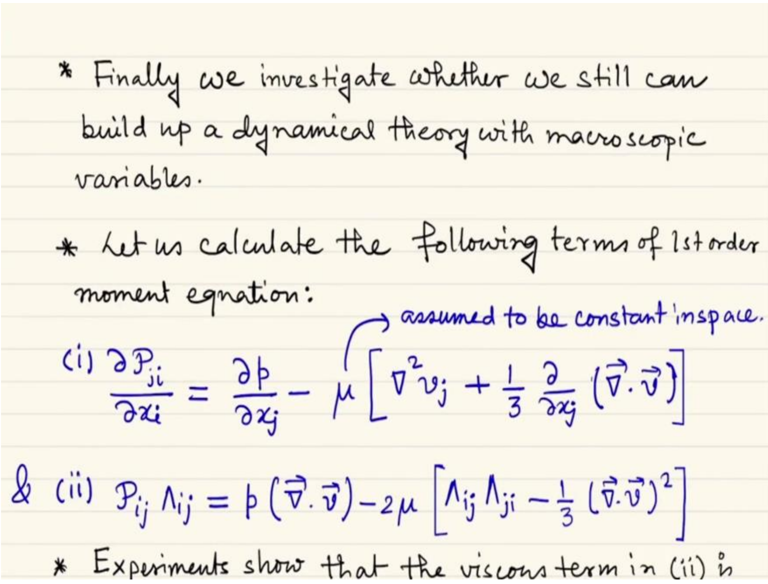
So, one thing is true that when temperature increases coefficient of viscosity should increase. Now, tell me one thing, it is true, I know for gas, when you increase the temperature μ increases, but what happens for oil? When you heat oil in a frying pan and you increase the temperature of it, you will see the oil is moving around much more easily on the pan. So, can you reconcile that?

So, this is a question not exactly related to the course, but this is a general physics question. So, what happens? So, why the viscosity of a gas and the viscosity of a liquid have different type of temperature dependencies? So, just think.

Here we are talking about mostly about fluid which is gas because here we are just trying to make a macroscopic theory from that of kinetic theory and kinetic theory is only valid for classical gases.

For classical liquid, you cannot make kinetic theory because of the simple fact that in a liquid you have complicated interactions, long range interactions, other than the instantaneous interactions due to binary elastic collisions.

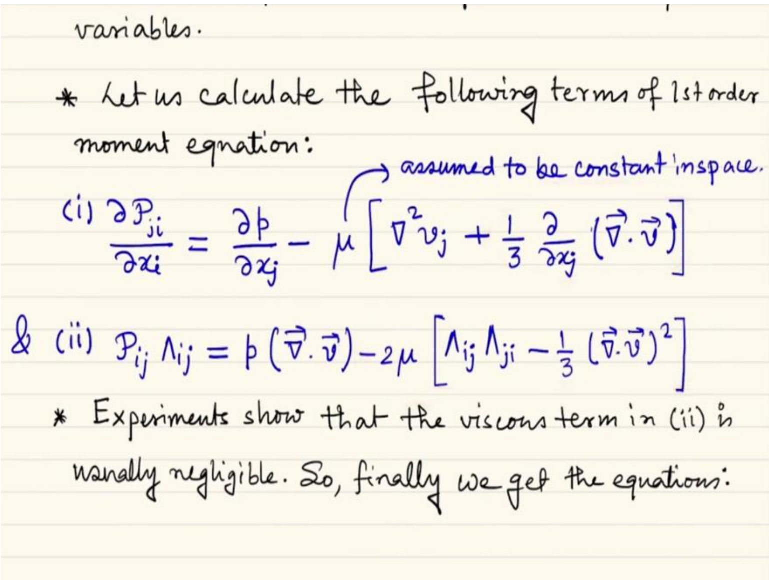
(Refer Slide Time: 36:29)



So, finally our objective was to search whether we can still build up a dynamical theory for macroscopic variables for real fluids or not. Now, for that let us calculate the following terms of first order moment equation the $\frac{\partial P_{ji}}{\partial x_i}$. So, for diagonal P we had only $\frac{\partial p}{\partial x_j}$, for local Maxwellian, now we have the additional term like $-\mu\left[\nabla^2 v_j + \frac{1}{3}\frac{\partial}{\partial x_j}(\nabla\cdot\boldsymbol{v})\right]$ where μ is assumed to be constant in space and which is very reasonable for most of the fluids.

So, there is a case, when we will talk about the physics of accretion disc, the μ is not constant in space and that actually adds a separate physical flavor to the problem, but here most of the cases we have this.

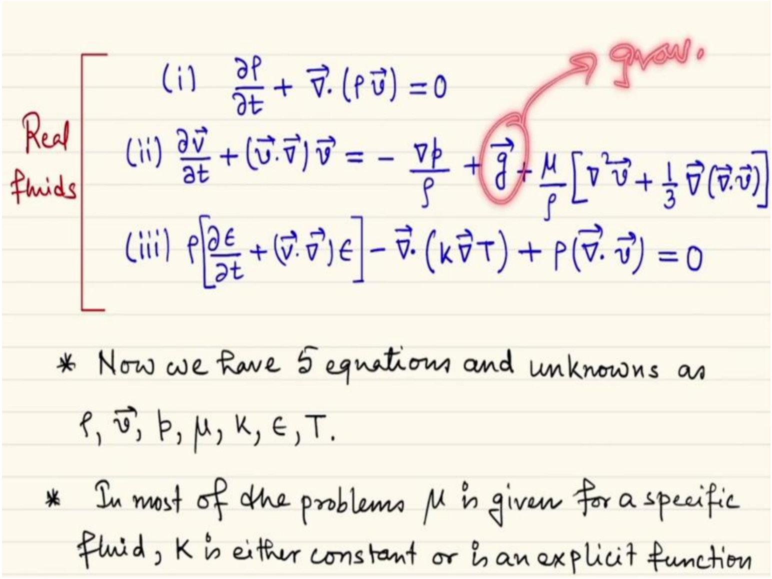
(Refer Slide Time: 37:37)



So, then you we can write this the double contraction term of P tensor and the Λ tensor in the energy equation which was previously just $p(\nabla \cdot \boldsymbol{v})$, now we will have an additional term like $-2\mu\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\nabla \cdot \boldsymbol{v})^2\right]$. So, these terms with μ are due to g, g means the perturbation over the local Maxwellian.

Now, experiments show that the viscous term in (ii) (see above) is usually negligible and actually most of the cases the viscous term $\frac{1}{3}\frac{\partial}{\partial x_j}(\nabla \cdot \boldsymbol{v})$ in (i) is also negligible, but the term in (ii) is much more negligible and this is much more popular actually. Sometimes even in some cases we can keep $\frac{1}{3}\frac{\partial}{\partial x_j}(\nabla \cdot \boldsymbol{v})$, but the term in (ii) is mostly neglected.

(Refer Slide Time: 38:31)

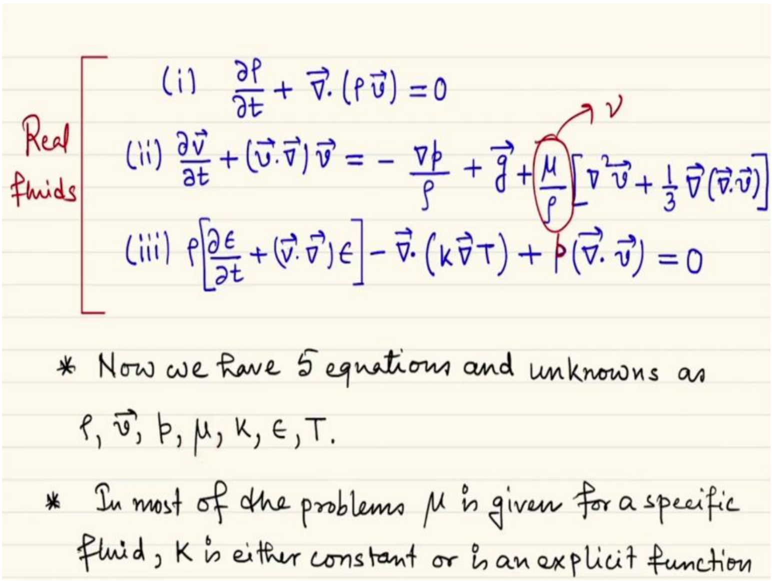


So, finally, we get the equations for continuity as $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0$.

equation the momentum as $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g} + \frac{\mu}{\rho}\left[\nabla^2\boldsymbol{v} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{v})\right]$.

$\boldsymbol{g}$ is the body force, not to be confused with the perturbation of the distribution function. This is just the gravitational acceleration plus the new viscous term.

(Refer Slide Time: 39:11)



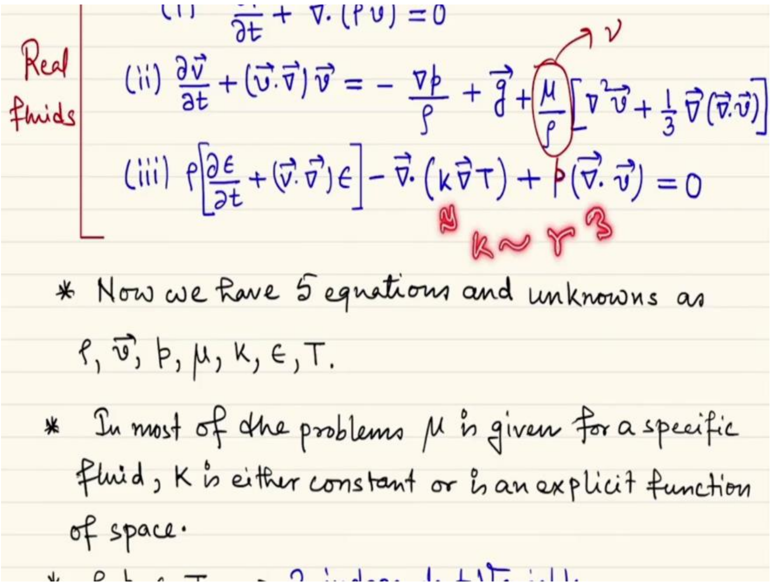
Now, this $\frac{\mu}{\rho}$ which I was calling the coefficient of kinematic viscosity is in general denoted by ν. And finally, the energy equation becomes

$$\rho\left[\frac{\partial \epsilon}{\partial t} + (\boldsymbol{v}\cdot\nabla)\epsilon\right] - \nabla\cdot(K\nabla T) + p(\nabla\cdot\boldsymbol{v}) = 0.$$

Of course, here we have not yet considered K to be independent of space and that is actually not a very good approximation because most of the cases K has a very explicit dependence on space.

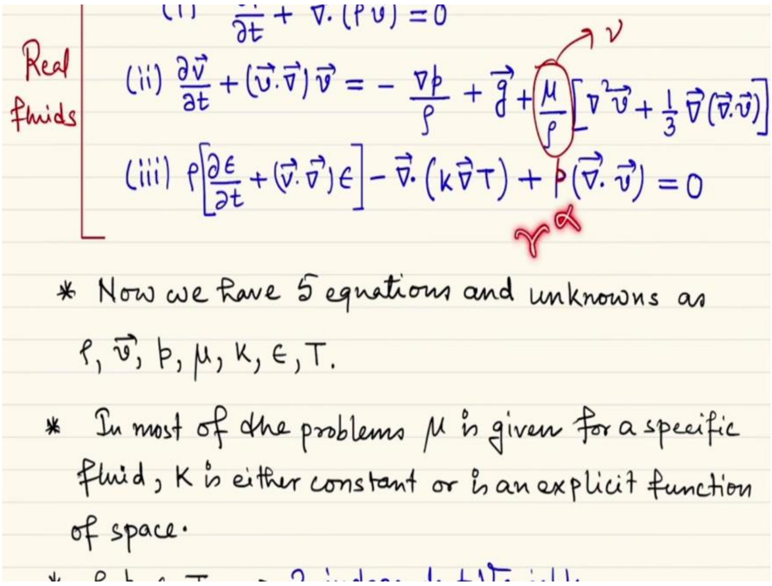
Now, we have again five equations and how many unknowns? ρ, $\boldsymbol{v}$, p, μ. $\boldsymbol{g}$ is known, ϵ, K and T. Now μ, in the most of the problem is given for a specific fluid. So, it is a constant of course, other than this accretion disc problem, but even in accretion disc it is handle able. We will talk about that.

(Refer Slide Time: 40:45)



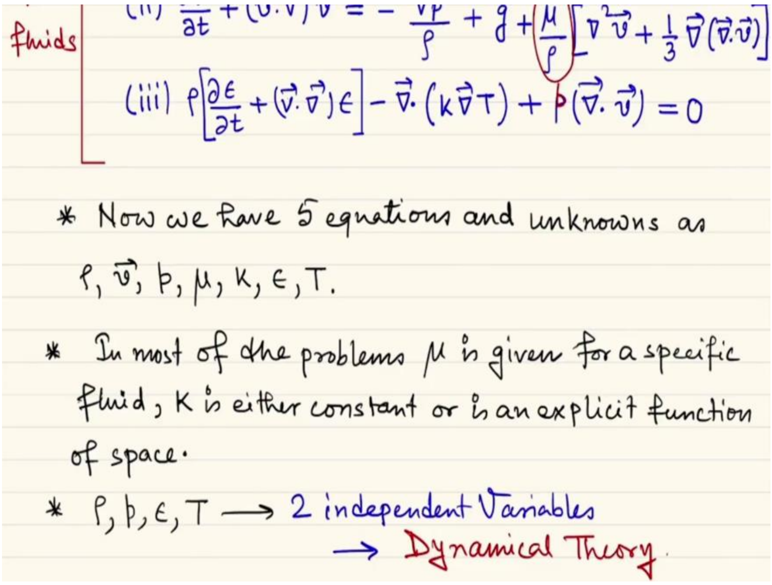
K is either a constant or is an explicit function of space. So, you can always write that K is like let us say proportional to r^α.

(Refer Slide Time: 40:59)



So, then what happens, if you are just doing some integration, this does not add to the complicacy of the unknown variables and so K in most of the cases, is a known explicit function of space or is a constant; because there are some experiments by which you can actually find the functional form and the functional dependence of K. So, it does not really complicate the scenario.

(Refer Slide Time: 41:33)



And finally, ρ, p, ϵ and T we know that they are all related by two independent variables and most popularly the 2 variables are n and T. So, finally, 2 independent variables from them and 3 components of $\boldsymbol{v}$, so 5 equations, 5 unknowns. So, finally we succeeded to constitute a dynamical theory for the real fluids.

So, that is exactly where I should stop for this lecture. So, in the next discussion we will talk about a very interesting application of these equations and we will talk about the hydrostatic condition, the hydrostatic equilibrium of such a fluid where the viscous effect is negligible, but not the conductive conductivity effect.

So, that we will do, but till now what we have seen something very interesting that, starting from a very simplistic assumption that the distribution function is perturbed only by a very small amount with respect to the Maxwell-Boltzmann equation; finally we retrieved all the, under of course certain approximations, all the transport phenomena in a real fluid.

So, now these set of equation is known as the equations of the real fluids.

Thank you very much.