Introduction to Astrophysical Fluids
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Lecture – 17
Departure from Maxwellian distribution

Hello and welcome to another lecture of Introduction to Astrophysical Fluids. In this lecture, we discussed the derivation of real fluid equations.

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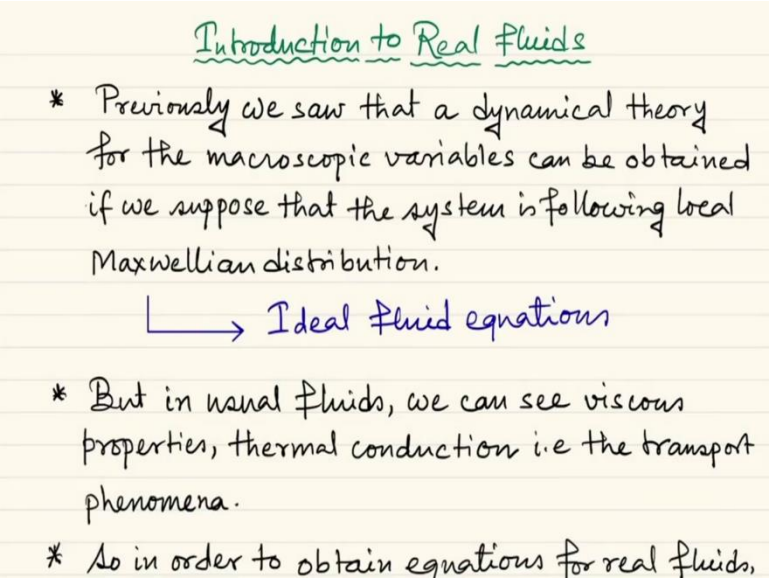

So, previously we saw that the dynamical theory for the macroscopic variables can be constructed, if we start from the equilibrium distribution or the 0-th order approximated distribution which is the Maxwellian distribution and to be in consistency with the fluid picture, we actually need to be precise local Maxwellian distribution.

And from that we showed finally, that we have a set of equations, which corresponds to a fluid where there is no viscosity, there is no energy transport type of thing. So, although we are constructing, starting from the kinetic theory the macroscopic equations, we have to understand that well before these formalisms people already knew the form of the real fluid equations. And, so when they matched these two, they found that some effects are missing and that is why those equations were called the ideal fluid equations, which we derived from Maxwellian distribution, but in practical fluids or in usual fluids which we really see every day,

for example a gas in a container or even a liquid. So, we can always see viscous effects and also we see thermal conduction. So, if you let, for example, a solid ball to go through a fluid a gas or a liquid, it will not go uninterrupted, it will necessarily experience a drag force, a force which will resists its motion.

And not only that actually different layers of the fluids try to reduce the relative velocity between the layers. And that is why, sometimes you can see that the fluid, if you do not put a large force, after sometime so for example, if you throw some fluid from a mug then after some instant it will come to rest.

So, if you really think that actually the principal role played in this whole game is by the viscosity. Again, if you heat one part of a fluid, let us say a gas or a liquid for example, the heat is transported. Even before the convection starts the heat is transported through conduction, from one part to the other part of the fluid.

And if you are considering this type of ideal fluid equations these effects are missing. So, people found and they realized that something was missing. So, that was the sake for which finally, we tried to do something much more general called the real fluid equations.

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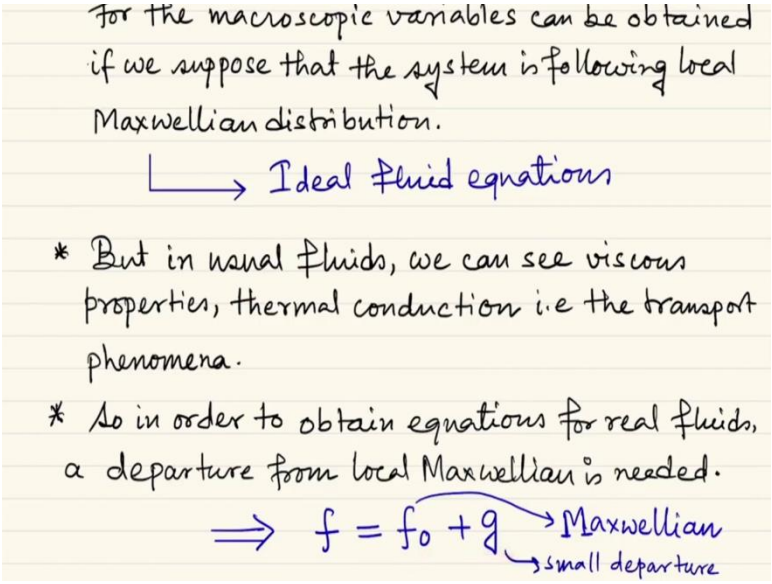

And all these things which we just discussed, viscous effects or thermal conduction, all these things are known as transport phenomena. So, in an ideal fluid transport phenomena is systematically absent.

There is another transport phenomenon, which we are not talking a here in this case, because that will be actually taken into account in terms of compressibility, that is the diffusion. Whenever you create a gradient of the density between two parts, the bulk of fluid will move from the higher density to the lower density region.

So, this these are known as three classical transport phenomena. Here we are just talking about two, because these two basically are will be added in this real fluid equations with respect to the ideal fluid equations. Now, if you just see, that if our reference distribution function is Maxwellian or local Maxwellian in nature, then this is simply we can say f is equal to f_0 and we have ideal fluid equation.

So, how to do something different? So, we can always think that our distribution function is something which is different from Maxwellian, but we want to do something analytical and for that what we should do that we should consider a very small departure from the Maxwellian and to see which type of effects it can give.

And you will see interestingly that, already a very small departure is eligible or adequate to account for most of the real fluid features.

So, I mean in another language we can say that f_0 is the 0-th order distribution and g is the 1st order distribution. So, you can always think that g is something of the order of ϵ. And now the total distribution function will be $f_0 + g$.

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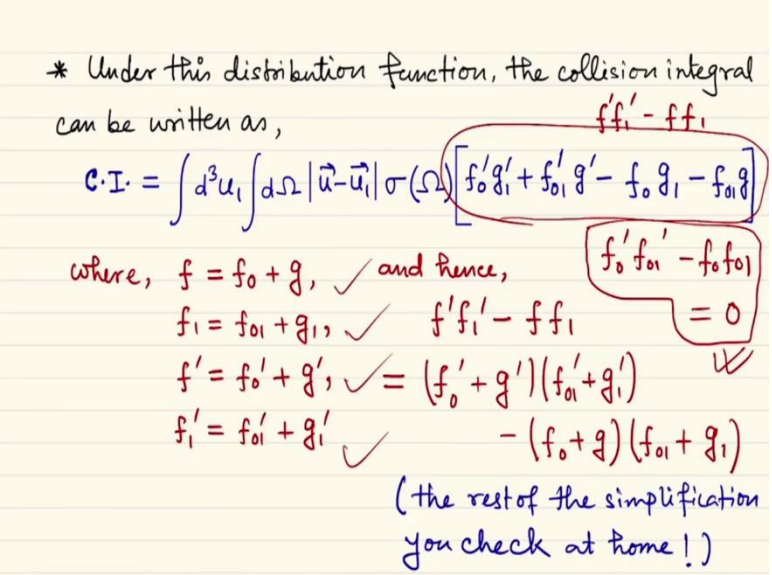

And we do not know exactly what g is and that is the thing we have to determine. Now, when you have written this type of combined distribution function as a sum of two distribution functions, then the collision integral, if you remember, the form that can be written as $\int d^3\boldsymbol{u}_1 \int d\Omega |\boldsymbol{u} - \boldsymbol{u}_1| \sigma(\Omega)(f' f_1' - f f_1)$.

Where, Ω is the solid angle and σ is the differential cross sectional area for the collision. Of course, we are talking in always in terms of binary elastic collisions. And the term $(f' f_1' - f f_1)$ is now written as $(f_0' g_1' + f_{01}' g' - f_0 g_1 - f_{01} g)$.

If you just elaborately write all these distribution functions explicitly and then you expand and of course, we know that from the I mean from the basic consideration you know that $f_0' f_{01}' - f_0 f_{01} = 0$, because what is f_0? it is nothing but Maxwellian distribution.

So, this is equal to 0 that is the basic condition for derivation of Maxwellian distribution. And then if this is 0, we also say that we are we neglect all these terms where 2 times of g appear.

That means, $g' g_1'$ or $g g_1$ because they are of second order roughly. So, we are just keeping the 1st order terms intact and so that finally, the whole thing becomes the term written above. So, here I have just written explicitly all the expressions. So, the calculation of course, always it is recommended that you check at home, so that is the best thing.

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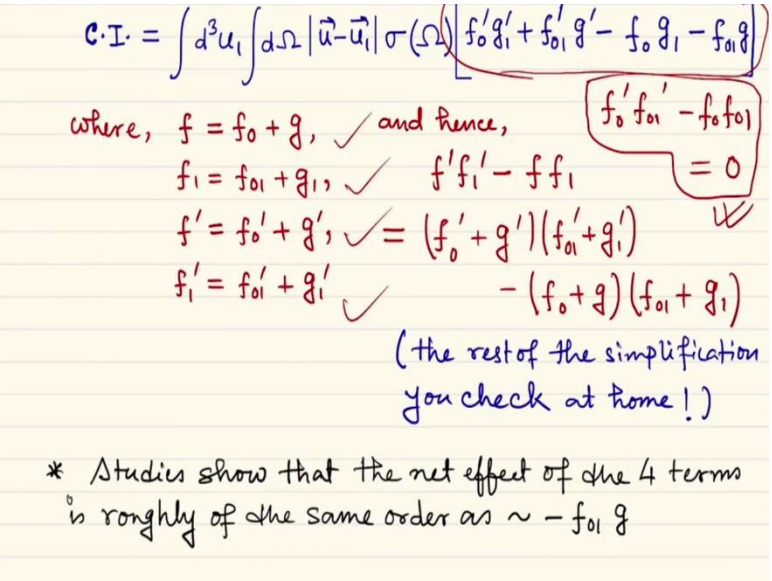

And now you can do the analysis using $(f_0' g_1' + f_{01}' g' - f_0 g_1 - f_{01} g)$, keeping all four terms, but believe me they are really nontrivial and difficult to handle analytically. So, in Chapman's book in a very classical work, you can see this type of works they are done, there is some work by Chapman and Cowling and you can have a look over there.

But other than that, for the practical purpose we have a good news and people also worked on that and they actually showed that, the net effect of all these 4 terms is roughly of the same order if we just approximate the whole thing as $-f_{01} g$. That means, you just take the last term and you are done, how is that? This is of course, I mean to be very honest awesome right.

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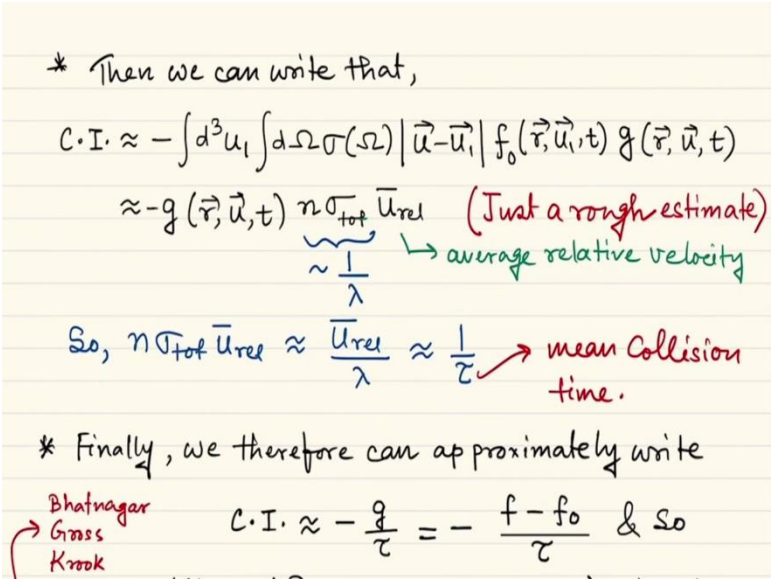

And under this condition, so if you just for the instance admit this fact then you can write your collisional integral as $-\int d^3\boldsymbol{u}_1 \int d\Omega |\boldsymbol{u} - \boldsymbol{u}_1| \sigma(\Omega) f_0 g$. So, the new distribution function will be reflected in this of course, f_0 you know this is local Maxwellian and g is the unknown.

Now, we have to do a number of rough estimates. So, here you can of course, ask me the question that at every step we are doing rough estimate. That is because, finally we are trying to reach at some point which we know already. We know our destination and we know our point of departure, but we are now tracing out the path.

So, as a true physicist there is no problem if we just continue tracing our path in a reasonable way, by reasonable approximations. So, one approximation is to write the whole thing in an average way. So, how to do that? So, $-g$ intact over here, this part please be I mean try to understand carefully.

Now the suggestion is that, you can take somehow, the integration over Ω separately and that will give you the total cross sectional area (σ_{tot}) for the binary elastic collision of course. And this integration $\int d^3\boldsymbol{u}_1|\boldsymbol{u}-\boldsymbol{u}_1|f_0$ will roughly give you $n\bar{\boldsymbol{u}}_{rel}$.

Of course, you see here I have just written $\boldsymbol{u}_{rel}$ as the relative speed. Although I wrote here average relative velocity this is simply the speed I have written. So, this is just an order of magnitude calculation that you have to do in various non-linear and complicated cases of physics. In proper astrophysics most of the time, where we do not have analytical results and analytical functions and analytical functional forms to analyze, the only thing which helps us, is the order analysis.

So, this is the tool which must be assimilated of course, with time it is not evident I also know that, but with time it will be becoming evident. So, whether this part is clear or not just to check once again, the collision integral becomes $-g(\boldsymbol{r},\boldsymbol{u},t)n\sigma_{tot}\bar{u}_{rel}$.

Now, we group $n\sigma_{tot}$ and if you remember when we were talking about the dilute gas and what is the condition of the dilute gases? weakly collisional strongly collisional this type of thing, we showed that $n\sigma_{tot}$ roughly should scale as $\frac{1}{\lambda}$, where λ is the average mean free path.

So, you can remember there was some factor of $\frac{1}{\sqrt{2}}$ type of thing, but other than that, roughly you can say always that this is a good approximation where $n\sigma_{tot}$ is almost like $\frac{1}{\lambda}$. So, finally, your $n\sigma_{tot}\bar{u}_{rel}$ can be written as $\frac{\bar{u}_{rel}}{\lambda}$, everything is in scalar form.

And so average relative velocity between two molecules by the mean free path, what is that? This is nothing but a rough estimation also, but reasonable estimation for the mean collision time. That means, this is the average time between two successive binary elastic collisions between two molecules.

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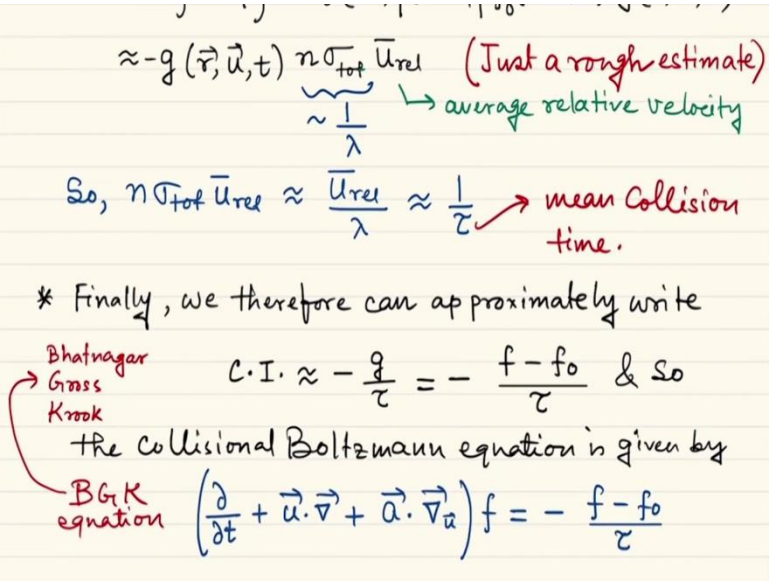

Finally, we can approximate the whole thing. That means, the whole collisional integral as simply as $-\frac{g}{\tau}$ right because $\frac{\bar{u}_{rel}}{\lambda} \approx \frac{1}{\tau}$. So, the collisional integral will be $-\frac{g}{\tau}$.

And what is g? It is nothing but $f - f_0$. And so the collisional Boltzmann equation now takes the form

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla + \boldsymbol{a}\cdot\nabla_{\boldsymbol{u}}\right) f = -\frac{f-f_0}{\tau}.$$

So, there is nothing to do in the left hand side, this is already done. So, the right hand side we have done after some rough estimates we have reached over here. This equation is known as BGK equation and this is given by Bhatnagar, Gross and Krook, these three scientists.

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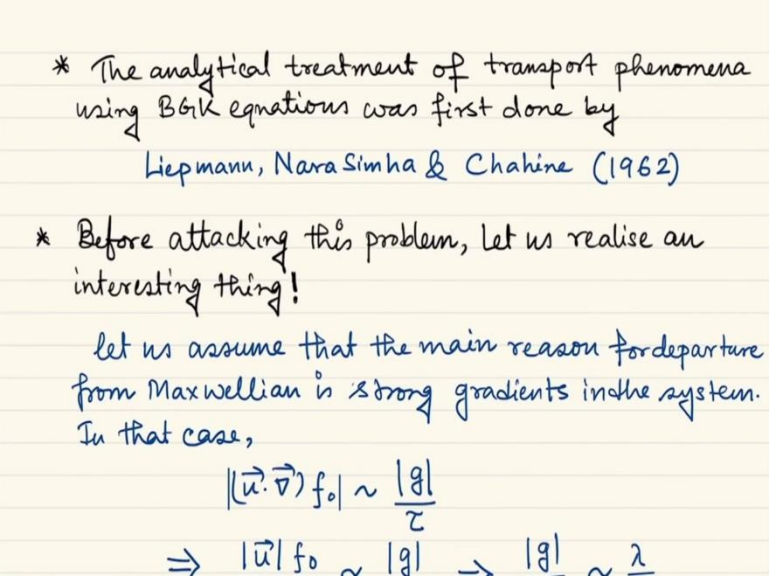

And what was the marvel of these three scientists Liepmann, Narasimha and Chahine. In the year 1962 they showed that the total analytical treatment of the transport phenomena can be done sufficiently or adequately using BGK equations.

So, BGK proposed this simplified form, but Liepmann, Narasimha and Chahine, these three people, they showed that it is sufficient to just use this BGK form of the collisional Boltzmann equation, in order to account for all the known transport phenomena of a real fluid.

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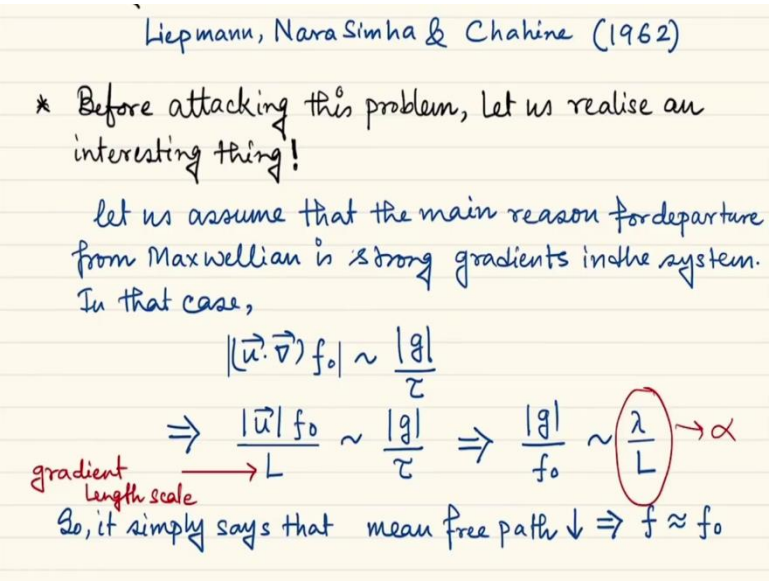

So, in this course we will just follow their pathway. Of course, if you want to see some once again the detailed work Chapman and Cowlings monograph this is the best thing, so you can search your internet.

Now, before attacking the problem, let us also think of a very interesting thing and that will somehow reconcile some of the aspects of collisionality and the fluidification. Which I already told while discussing the macroscopic theory and the dynamical theory for the macroscopic variables basically.

So, let us now assume that the main reason for departure from Maxwellian is strong gradients in the system. Of course, if your system does not have gradients everything is very uniform then you do not have to have transport phenomena. And so you do not have to have any departure from Maxwellian. So, Maxwellian is very happy to have everything uniform.

Only when the gradient, that means, for example, when you travel over space then the value of the velocity is changing very rapidly (for example). Then only you can talk about the considerable departure from Maxwellian.

And in that case the second term on the LHS of the Boltzmann equation is very important. That means, that the spatial gradients are very very strong of the distribution function. So, that the system is now deviating from its equilibrium distribution, then what happens? We will see.

Again, this is a rough order analysis. So, then of course, we can say that if this is very important then this term can actually counter balance this term roughly by magnitude ok. So, $\frac{\partial f}{\partial t}$ and $(\boldsymbol{a} \cdot \nabla_{\boldsymbol{u}})f$ both are actually subdued by or dominated by $(\boldsymbol{u} \cdot \nabla)f$.

And, when we are writing $(\boldsymbol{u} \cdot \nabla)f$, I mean f_0 is 0-th order and g is 1st order you can I mean sufficiently approximate $|(\boldsymbol{u} \cdot \nabla)f_0| \sim \frac{|g|}{\tau}$.

Now, I can write again roughly as $\frac{|\boldsymbol{u}|f_0}{L} \sim \frac{|g|}{\tau}$, this is nothing but, the average speed times the f_0 by some typical length scale, called the gradient length scale, that is the length scale over which a fluid or a continuums property is expected or even not even continuum if you

have a kinetic system for example, the systems property is expected to change considerably.

So, if the length scale chosen, is below then for example, temperature or velocity or density is changing considerably only over this length scale. So, we then write that $\frac{|\boldsymbol{u}|f_0}{L} \sim \frac{|g|}{\tau}$ and from which you can simply see that $\frac{|g|}{f_0}$ is of the same order of $\frac{\lambda}{L}$.

So, again you see this is the interplay between the mean free path and the typical gradient length. Do you agree? and it simply says that when mean free path goes down. So, $\frac{\lambda}{L}$ becomes less and less and $\frac{|g|}{f_0}$ will also become less and less. So, f will be very close to f_0, so that means, when your system is very much collisional then your systems distribution function will tend towards Maxwellian.

On the other hand if λ is non negligible and is basically considerably bigger and it is smaller than maybe L, but it is considerable with respect to L, then the departure with respect to f_0 is also considerable. So, then the system cannot be approximated very well using a local Maxwellian distribution, you must have to consider the departure.

So, now how much? So, for example, in our approach we have already assumed that this is of the 1st order, but this can be actually many things. So, for weaker and weaker collisional systems you can have higher orders.

And then, you can simply see that this factor $\frac{\lambda}{L}$ you can call this as α and you can just take this as a smallness parameter, and then you can just write the total very general case, the total distribution function as a linear expansion, of all the orders of α.

And then according to your necessity, you can keep the degrees of α. So, for example, if the system is such that the collisionality is so weak that you have to consider up to the second order of α, that is you have to go up to α^2, that is possible.

Here in this case we start we just do the simplest way, the 1st order and we will just neglect all the other orders. We are just thinking that the collisionality is there, so it is not perfectly collisional to have a equilibrium distribution like this Maxwellian type of thing, but the collisionality is sufficient. So, that it is just sufficient to consider up to the 1st order of α.

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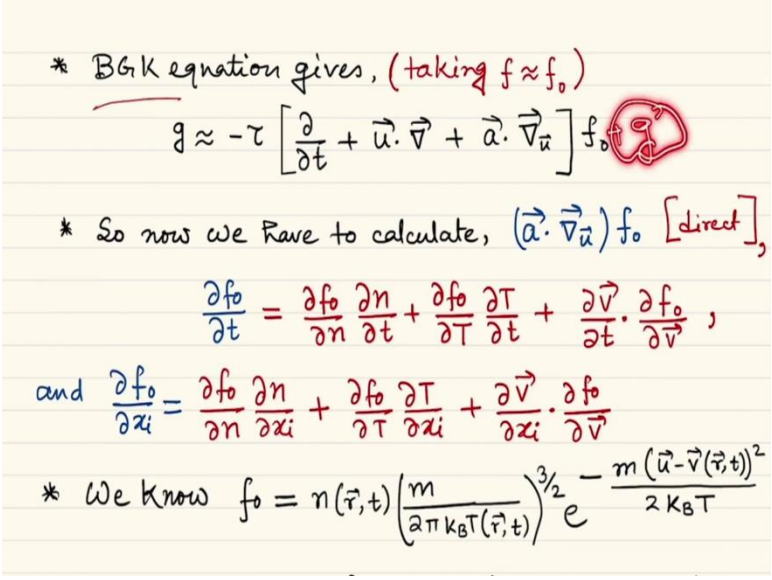

So, just so using all these things now finally, if we can see that the BGK equation gives g

$$g \approx -\tau\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla + \boldsymbol{a}\cdot\nabla_{\boldsymbol{u}}\right) f_0$$

Everything operating on f_0, now that should be f, but once again here you can just neglect g in front of f_0. So, a very good approximated form of g is given above. So, in the next part of our discussion we will continue to talk about this and to show that how using all these things finally we can reach to real fluid equations.