Introduction to Astrophysical Fluids
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Lecture – 16
Steady flow, streamlines and stream function

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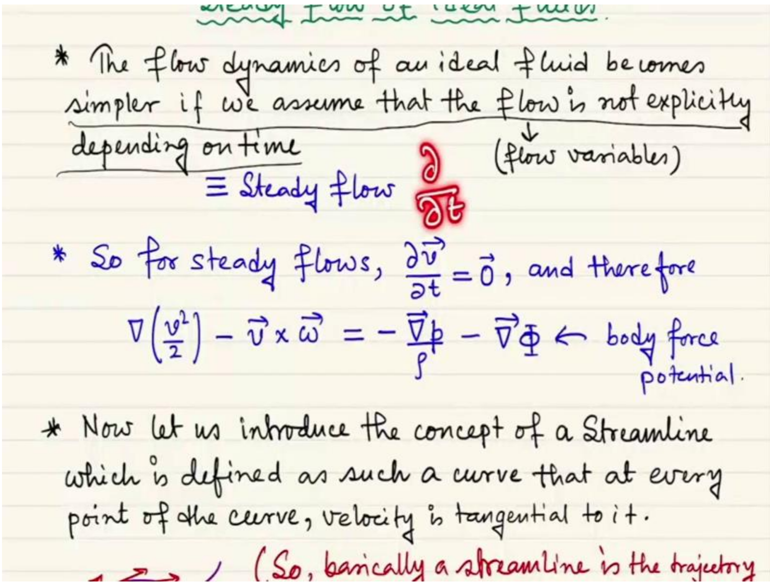

Hello. In this part, we will discuss about the Steady flow of ideal fluids. So, as we know that the flow dynamics of an ideal fluid must become much simpler if we assume that the flow is not explicitly depending on time, that means, that the flow does not have any $\frac{\partial}{\partial t}$ type of thing.

So, whenever we see some $\frac{\partial}{\partial t}$ of something, we will say that is 0. So, for steady flows, of course, that is the same thing for the velocity vector. So, $\frac{\partial}{\partial t}$ of velocity is 0. Simply, we can also say that was the trivial thing but we can also say that $\frac{\partial \rho}{\partial t}$ is also 0.

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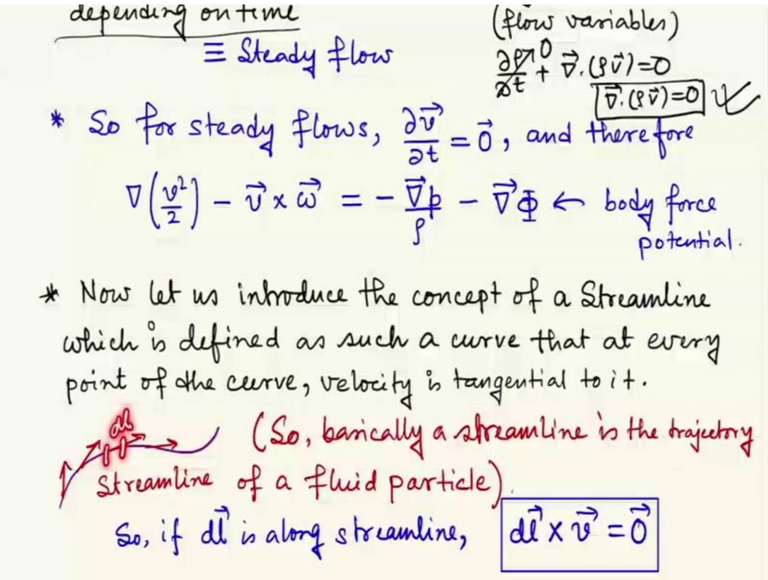

So, here we just say that as in corollary that the continuity equation basically can be written like this $\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0$. This $\frac{\partial\rho}{\partial t}$ is 0. So, $\vec{\nabla}.(\rho\vec{v})$ is 0. So, remember that, in case of incompressible fluids we had $\vec{\nabla}.(\vec{v})$ is 0, but here ρ is just independent of time.

So, there is no explicit time dependence, but ρ is a function of space. So, here this is $\vec{\nabla}.(\rho\vec{v})$ is equal to 0 that is the property of a stationary flow although it may look very similar, but this is subtly different. Now, what about the case of the momentum evolution equation for steady flows?

So, you will see that similar to density we can say $\frac{\partial v}{\partial t}$ is equal to 0 and therefore, the first term of the momentum evolution equation goes away and we simply have this $\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\varphi$. Now, this is a new way of writing the force density I mean body force density g.

So, since this is a conservative force field usually so g can be derived from a scalar potential let us say φ. Now, let us introduce a very important concept of streamline. Now, the concept of streamlined is not a concept which is specially defined for incompressible or ideal fluid, but this is a general concept of a flow.

It simply says that streamline is defined as such a curve that at every point of that curve velocity is tangential to it. So, let us say this is a streamlined curve the purple one. So, at every point if we just measure the velocity vector at that point the Eulerian velocity vector the velocity field, then we will see that the velocity vector is exactly tangential to this streamline curve. So, what is the meaning of that? The meaning is that this is exactly the same direction as the infinitesimal length elements on this curve.

So, if you take any infinite decimal length element dl on this curve, then that $\overrightarrow{dl}$ and the velocity at that point will have the same direction and that is why the streamlines definition can be given by this $\overrightarrow{dl} \times \vec{v}$ is equal to 0. Now, you know the concept of fluid particle. So, you can easily understand that basically a stream line is nothing but the trajectory of a fluid particle.

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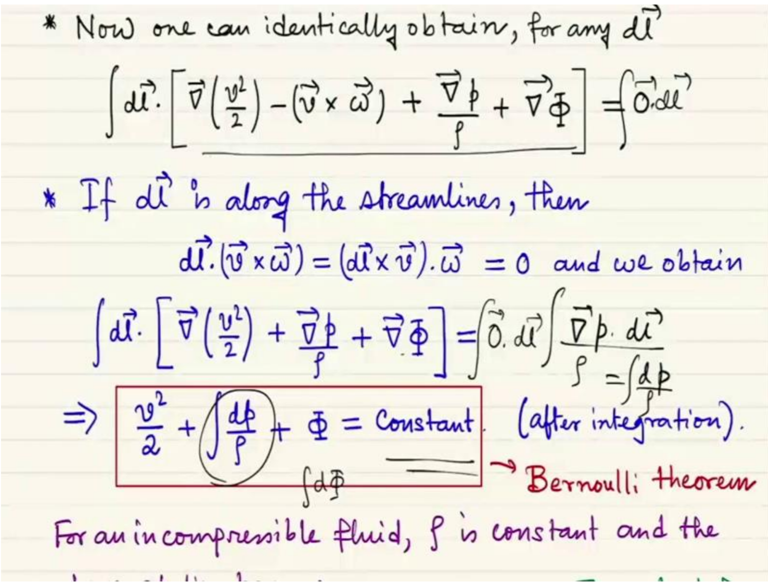

So, just taking these two terms on the other side we can simply write this $\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} + \frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\varphi$ and this is 0.

So, this is the same thing by saying. So, this is this $\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} + \frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\varphi$ identically 0. So, this part $\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\varphi$ when it is contracted with some length element dl and integrated over that length element then this is just equal to 0 and this dl do not need to be an element over a stream line that can be the length element over any trajectory or over any curve.

Now, if dl is along a streamline then what happens? Then this $\vec{dl}.(\vec{v} \times \vec{\omega})$ actually should vanish because you know that if we have this type of scalar triple product then dl can be permuted and that will be equal to $\vec{dl}.(\vec{v} \times \vec{\omega})$ and this is 0.

So, this vanishes. So, if we are just studying along the stream lines, then the lamb vector contribution would vanish and we simply have integration $\vec{dl}.[\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} + \frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\varphi]$ is equal to 0.

Now, if we integrate, we will see that $\vec{dl}.\vec{\nabla}\left(\frac{v^2}{2}\right)$ is nothing, but the differential of $\frac{v^2}{2}$. So, when it is integrated over some trajectory over the streamline then it will simply be just giving you $\frac{v^2}{2}$. So, again $\vec{\nabla}p.\vec{dl}$ is nothing, but dp.

So, integration over $\frac{\vec{\nabla}p}{\rho}$ is nothing but integration of $\frac{dp}{\rho}$ and that is exactly what I have written over here. Finally, integration over $\vec{dl}\ .\vec{\nabla}\varphi$, in the same way, this will be simply integration over $d\varphi$ and that is φ. Now, in this part, on the right-hand side you are integrating basically.

So, that integration $\vec{dl}.[\vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \vec{\omega} + \frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\varphi]$, I have written 0, but this is not I mean this is not the correct way of writing. So, that should be some 0 which is integrated over dl. So, that will finally, give us some. So, this will give us some constant of integration.

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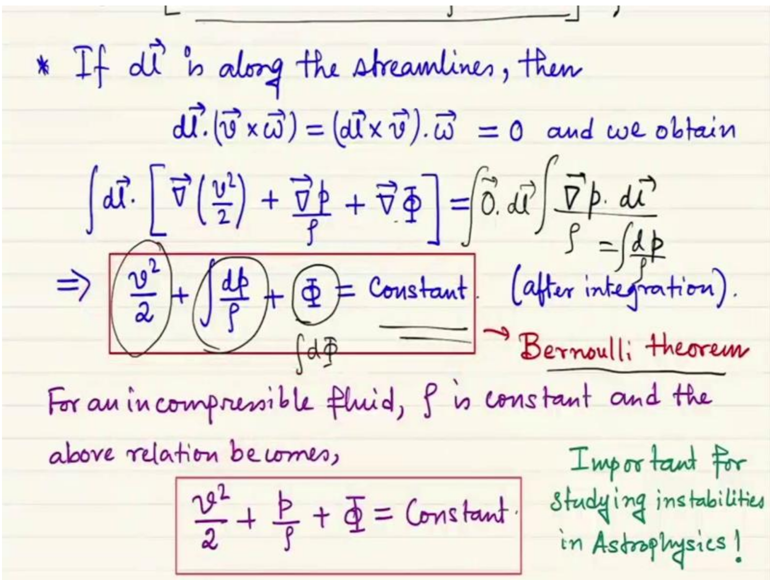

This is nothing but the so-called Bernoulli theorem that is the sum of these three things $\frac{v^2}{2} + \int \frac{dp}{\rho} + \varphi$ is equal to constant. So, in a sense, this $\frac{v^2}{2}$ corresponds to the massive density of kinetic energy, this $\int \frac{dp}{\rho}$ corresponds to the so-called potential energy due to the internal forces and this φ corresponds to the potential energy due to external body force.

So, for an incompressible fluid, actually we have a simpler case and ρ is constant. So, ρ can be taken out of the integration and it will simply be then integration dp, which will be simply $\frac{p}{\rho}$ and we have this simpler form things $\frac{v^2}{2} + \frac{p}{\rho} + \varphi$. So, this type of thing, why I am discussing all these things because this will be very much interesting and important when we will discuss different type of fluid surface instabilities in astrophysics, for example, Rayleigh Taylor instability, Kelvin Helmholtz instability.

So, in order to derive a general treatment or the general formulation for those type of instabilities or stabilities in case of like helioseismology, that we will discuss, or stellar oscillations, we will see that this type of properties are needed there.

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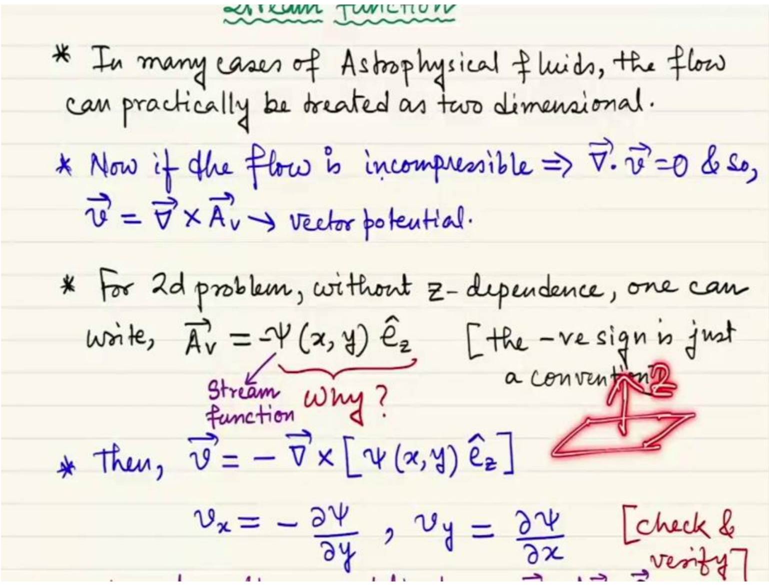

So, that was one thing to discuss, Bernoulli theorem or Bernoulli principle in case of ideal and stationary fluids. So, of course, you see that here, in this case we are actually like mixing two concepts. So, one is streamline and streamline is very much general.

So, streamline is not very much specific about ideal fluid or anything. So, whenever there is a flow, we have streamline, but then we have merged this streamline to steady flow of ideal fluids, that is something to remember every time. In this part we also define or also discuss another important property or another important concept that is stream function.

So, just a moment ago, I said that streamlines they are very much general, whenever there is a flow. So, the trajectory of the fluid particles constitutes the streamlines. Stream function on the other hand has a very specific domain of application. So, that I am coming. So, first let us start by saying that, in many cases of astrophysical fluids, the flow is practically two dimensional. So, this type of thing can happen, for example, when a cyclone is forming.

Let us say when a cyclone type thing is forming or we are talking about the dynamics of a disk type of thing, for example, a galactic disc or something. So, the flow is actually $2d$ I mean very much nearly two dimensional then this type of concept of stream function can be of utter importance. So, if the flow is incompressible then we can actually say that divergence of v is 0 that we already showed.

If the flow is incompressible then we can write the velocity to be derived from a vector potential. So, that we all know from our knowledge of preliminary vector calculus, that whenever a vector field has a zero curl, then the vector field can be derived from a scalar potential and whenever a vector field has a zero divergence this can be derived from a vector potential.

So, a very popular example is, of course, the map for magnetic field which is by virtue of the absence of the monopoles has zero divergence and that is why we know that there is a vector potential of a magnetic field. So, in case of velocity this is not always true only when the velocity is corresponds to the velocity of an incompressible fluid, then only the divergence of the velocity is equal to 0 and then we talk about a vector potential of a velocity which we simply call here as A_v.

So, velocity will be curl of A_v. For $2d$ problem, without z dependence this curl of A_v or the velocity vector potential can be written as minus of a scalar function ψ, which is a function of x and y only times the unit vector in the z direction. The z direction simply means the direction which is perpendicular to the plane of the problem.

So, if this is the plane of the problem then this is the z direction. So, whenever the vector potential can be written in this form, that is only possible in $2d$ case then this scalar function is called a stream function. So, stream function is not a general concept, first of all your flow has to be incompressible, then your flow has to be practically in two dimensional then only you can talk of a stream function.

So, streamline is general but stream function is not that general. The minus sign is chosen only by convention you can just say it is plus that is also possible. Now, if I just take this is as a definition of $2d$ vector potential for the velocity field of an incompressible fluid, of course, then v is equal to minus curl of this whole thing ψ times $\hat{e_z}$ and then you simply know that what will be v_x.

So, v_x will be the x component of this product and that will be simply minus $-\frac{\partial\psi}{\partial y}$. Similarly, what will be v_y? Please calculate. That is a small calculation and I always encouraged to do at home. Here, in the lecture, it is not possible to show every single step of calculations.

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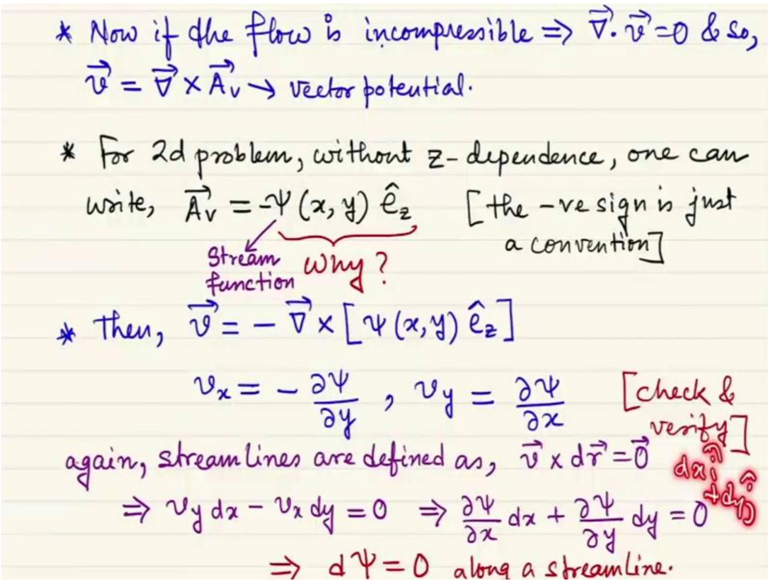

So, I will hope that you will try this at home. So, v_y will be plus $\frac{\partial\psi}{\partial x}$. Now, we know that stream lines are defined as $\vec{v} \times \overrightarrow{dr}$ is equal to 0 where dr is the incremental length elements I mean length vectors or length elements of a stream line. If we can write that then basically from this definition, we can write that the stream lines can also be written like this $v_y dx - v_x dy = 0$.

Just I am doing this cross product and I am writing this because dr or dl is nothing, but $dx\,\hat{\imath} + dy\,\hat{\jmath}$ because we are in $2d$ then we are just doing the cross product and we have $v_y dx - v_x dy$ is equal to 0 and v_y is nothing but $\frac{\partial\psi}{\partial x}$. So, and v_x is nothing but $-\frac{\partial\psi}{\partial y}$.

So, the whole thing will be then what is then this cross product? This cross product will now simply be $\frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy$ and that is equal to 0 now what is this basically? This is nothing but the total differentiation of the function ψ, $d\psi$ and that simply says that along a streamline $d\psi$ is equal to 0. So, how to characterize in a very efficient way a stream function? So, stream function is something which is constant.

So, I mean, which is the vector, I mean, which gives you the magnitude of the vector potential in a $2d$ incompressible fluid flow and which is conserved or which is kept constant along a streamline. So, in this part, we have learnt basically two concepts one is of the steady flow where we can we introduce the concept of stream line and then we also learn the concept of stream functions.

Now, just for your interest, what you can do as a very good exercise at home that you know like velocity fields when they are irrotational in nature, then you can actually also define for the velocity field a scalar potential. Now, let us say you can assume an incompressible fluid which is also irrotational, then can you say something about some properties of the velocity and its corresponding potentials, can you do that? So, it is at the same times incompressible and irrotational.

So, you think about that. So, that is something to think about and I mean whether this is totally an impossible case or it has some mathematical significance, just think and during the forum discussions if you still have some confusion or if you come out with an answer, you can tell me and I will try to make some discussion on that.

Thank you.