Introduction to Astrophysical Fluids

Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture – 15
Conservative form and invariants in ideal fluids

We have already derived ideal fluid equations starting from the very basic Maxwellian distributions. Then also we saw that how the different type of forces can be retrieved using macroscopic approach. And in the same way, actually continuity equations can also be derived macroscopically and that you can see any standard book of fluid mechanics. Now, after that we also discussed several properties of ideal fluid equations.

Now, we will discuss a very interesting thing in fluid dynamics, which is the theory or the discussion of the invariants. And then another, I mean the associated concept which will discuss the conservative form.

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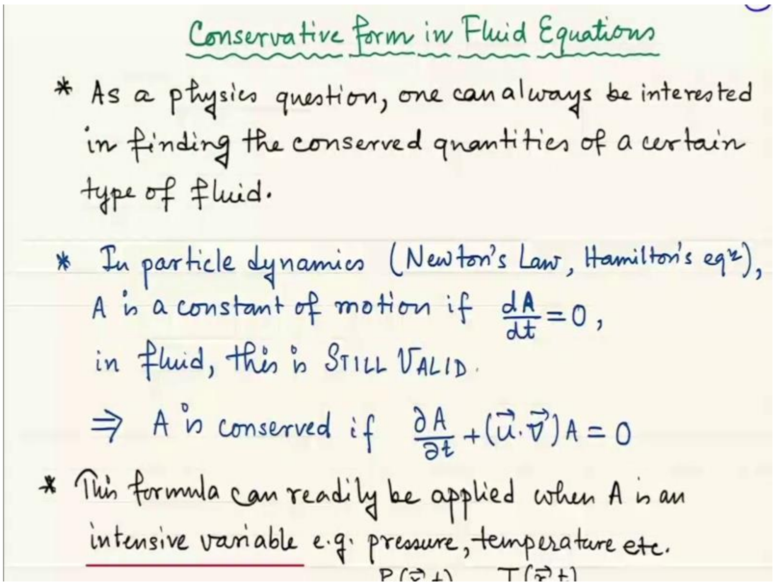

So, as you can always appreciate that we are physicists, and always like a true physicist we can be interested in knowing that whenever we can write the equations of ideal fluids.

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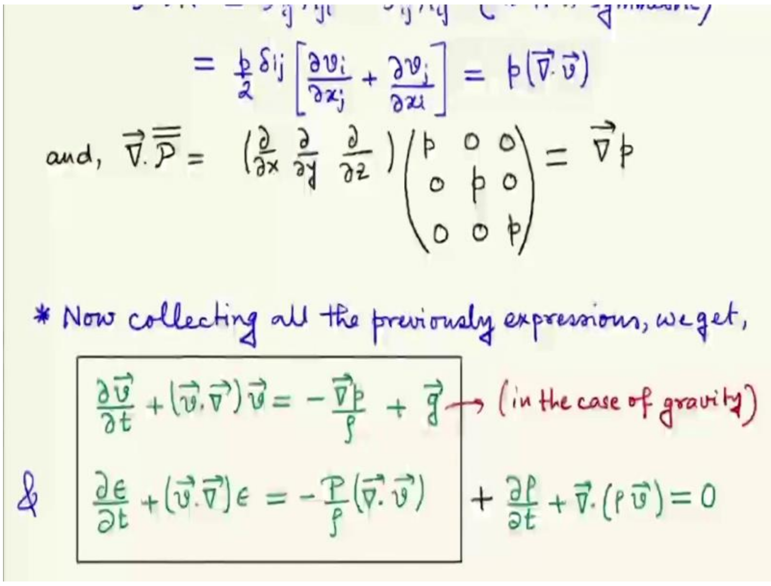

Let us say, the set of equations, these three that the continuity equation, the momentum evolution equation, and the internal energy evolution equation. Then, of course, the question is that whether there is something which are constants of motion or not? And for that, we basically try to introduce this conservative form.

So, we know that in particle dynamics when we talk about Newtonian mechanics or Hamiltonian mechanics, A is a constant of motion if $\frac{dA}{dt}$ is equal to 0 right, for example, energy.

If the total energy of a particle in a conservative force fluid is constant, then $\frac{d}{dt}$ of the sum of its kinetic energy and potential energy is 0. Now, in fluid, this thing is still valid. But now try to understand one thing, this thing should be valid for a fluid particle.

So, always we will remember that whenever we need to draw some analogy with the particle mechanics or particle dynamics to fluid, the best way is to talk to replace a Newtonian particle or a traditional particle by a fluid particle, then only we can write the Newton second law for that fluid particle.

So, here you can see that if we just say that this $\frac{d}{dt}$ is nothing but the Lagrangian or the material derivative, then also in fluid mechanics, we can say that A is conserved if $\frac{dA}{dt}$ is 0. And if you just expand $\frac{d}{dt}$, you know that you will have $\frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}$.

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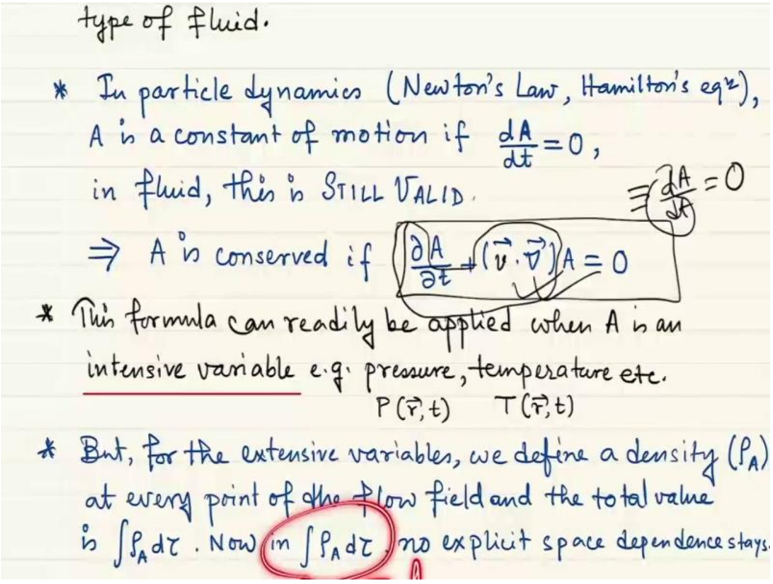

Where v is the bulk velocity, yeah, bulk velocity of the fluid. And then this equation $\frac{\partial A}{\partial t} + (\vec{v}.\vec{\nabla})A$, which is exactly equivalent writing $\frac{dA}{dt}$ is equal to 0. So, then this $\frac{d}{dt}$ is equivalent to $\frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}$. So, this is true also for fluids.

So, just once again do not get confused for fluid nothing actually changes. But then we have to understand one simple thing that finally we have to write everything in terms of Eulerian things, and in fluid actually we have subtle things to add. Actually, we have additional considerations, which we should not forget.

So, if you simply now see, this formula can readily be applied when A is an intensive variable. For example, now for fluids of course, there are subtle $T's$.

When we are talking about a simple particle, well, there is nothing called intensive or extensive. But when we are talking about a system –broader system having some volume, then you know maybe from thermodynamics that we have two types of variables. One is called extensive variable; another is called intensive variable.

Intensive variables are such that they are actually for a system, let us say the temperature. So, if you have a container of gas, so intensive properties are such that if the total container is now divided into n number of subparts, then, of course, imaginarily divided. Then for every subpart the value of the intensive variable is just that simple variable.

That means, let us say if your gas is at rest and there is a uniform temperature throughout the body, so your system is in thermal equilibrium. If the system is in dynamical equilibrium or mechanical equilibrium, throughout the whole system you will have one pressure then at that point if you just imaginarily divide the whole system, then for every subpart the value of the pressure and temperature is the same.

On the other hand, if you just take the mass for example, and if you divide the whole system in imaginary subparts, the mass of the subparts will be different. And when you add up the mass of the subparts, then we you will finally get the total mass of the system.

So, something which is in a thermodynamic static system and when there is an equilibrium of some type, for example, some microscopic quantities like temperature, pressure, they are uniform throughout the bulk.

So, this type of thing will have the same value even for any sub volume when it is divided in some n number of sub volumes imaginarily. Then this property is called an intensive variable, I mean this variable is called.

Now, when the system is in flow and not in equilibrium, for example, in a fluid, so at every point it has some pressure it has some temperature. But when you imaginarily divide the system, when you just imaginarily divide the system in n subparts, then it is true that at every subpart, then value of the temperature and pressure, they are not the same. But then in that case also, you can talk about intensive variables.

If you just see that, they will have some value, but the total value of the temperature or the representative value of the temperature of the fluid does not have any physical meaning, when this is added over the number of sub volume. That means, for example, if you now take a fluid or a flow field something is flowing, and you divide the whole sub whole system into n subparts, then you calculate the temperature for all the subparts.

Now, if you add all the temperatures, so does this sum represent the total I mean the actual temperature of the system, of course not, that does not have any meaning. On the other hand, the same thing for pressure. On the other hand, if you were talking about number density, then actually this will be added to give you the total number density of the particles inside this volume right.

So, this is something you have to be careful that not number density, but I am saying that the mass. Because number density is intensive, it is number per unit volume. What I am just trying to say is the number itself, so just the number of particles inside every sub imaginary subpart. If you add them, they will give you the total number, of course, not the number density. Number density, pressure, temperature, they are just basically intensive variables.

But the total number, the mass, the entropy, all these things where if you just subdivide imaginarily the total system, whether this is at rest or in flow, if you add the value of the subsections, they will not give you the value of the total section. So, if you add the thing, they will give you the value of the total system if they are extensive variable.

Now, coming back to our original discussion. So, if we are talking about an intensive variable, let us say pressure, temperature, then, for example, if I say that pressure is constant, then $\frac{dP}{dt}$, Lagrangian derivative of P is equal to 0. And then you can say from that $\frac{\partial P}{\partial t}+(\vec{v}.\vec{\nabla})P$ is equal to 0.

But for extensive variables, so basically that means that if we say that an intensive variable is constant, that means, we are talking about the local values. But when we talk about the constancy of the extensive variable, then we actually talk about the total value of the extensive variable.

Then we actually define a density. So, if we just let us say we talk about an extensive variable A. Then we define a density ρ_A for that variable at every point of the flow field. And the total value will simply be the integration of ρ_A over the volume.

Now, when we are saying that A is conserved, then that means, integration over $\rho_A d\tau$ that is conserved because that is nothing but equal to A. And remember the mass if A is the mass, then ρ is simply the density. So, integration $\rho d\tau$ is nothing but the mass.

So, the only thing is that for an extensive variable, when we talk about the conservation or I mean conservation of thing or something is concern is constant, then we say this should be a constant in time. So, $\frac{d}{dt}$ of this thing should vanish. Now, it is true that, this is already something which is integrated over space, so that does not have any explicit space dependence.

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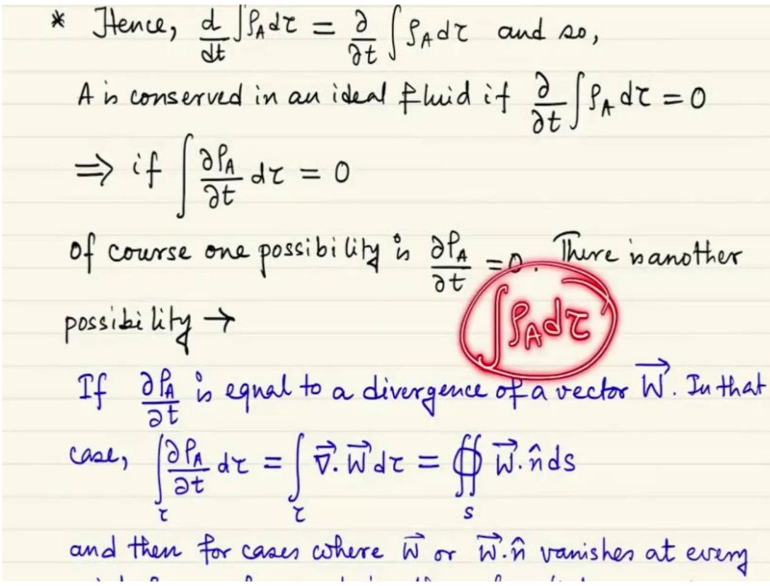

And as it does not have any explicit space dependence, basically we can simply write that $\frac{d}{dt}$ of integration $\rho_A d\tau$ is equal to $\frac{\partial}{\partial t}$ of integration $\rho_A d\tau$, because this integration $\rho_A d\tau$ can only have time dependence. So, $\frac{d}{dt}$ and $\frac{\partial}{\partial t}$ for this is equivalent. Then we can say A is conserved in an ideal fluid if $\frac{\partial}{\partial t}$ of integration $\rho_A d\tau$ is equal to 0.

So, of course, one possibility is that if this is 0, then this $\frac{\partial}{\partial t}$ will be commuting with $d\tau$ because this is a space operator space integration, and this is an explicit time dependence derivative, that means, partial time derivative. So, they will commute. So, you will see that $\frac{\partial}{\partial t}$ will enter and you will have this one should be equal to 0 if A is conserved.

So, there are two possibilities. One possibility is that $\frac{\partial \rho_A}{\partial t}$ is 0, that is true. So, it simply says that if some quantity density or some extensive variables is explicitly time independent, then for that quantity actually we can say that this is the corresponding quantity which is nothing, but the integration of that density over $d\tau$ is a constant of motion.

There is another possibility that is more interesting if $\frac{\partial \rho_A}{\partial t}$ is equal to a $\vec{\nabla}.\vec{W}$ And then integration over $\frac{\partial \rho_A}{\partial t} d\tau$ is nothing but integration over $\vec{\nabla}.\vec{W} d\tau$.

And you can simply remember this type of term should appear in Gausses divergence theorem which should be then equal to closed surface integral of $\vec{W}.\hat{n}ds$ where the integral is done on a closed surface which encloses the volume τ.

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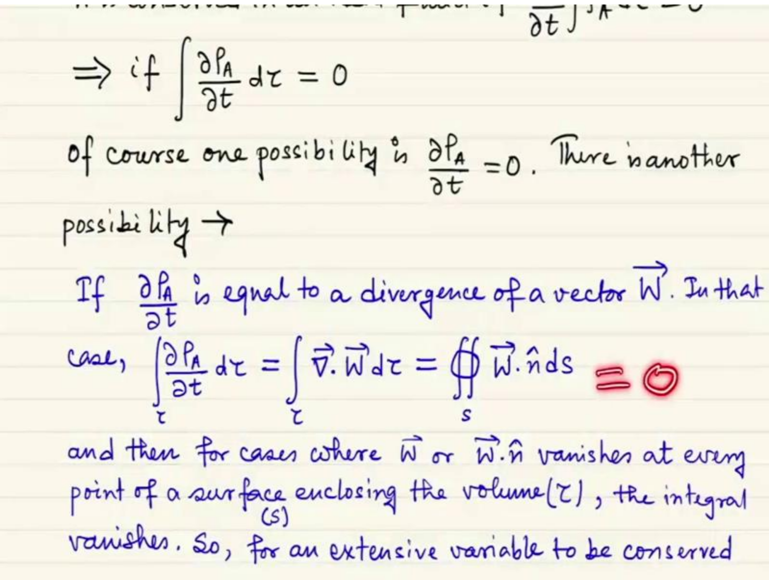

So, then actually you can see that we can always choose such a surface extending at infinity. So, at every point on the surface either W itself vectorially or the $\vec{W}.\hat{n}$ vanishes, and finally, we will have this closed surface integral of $\vec{W}.\hat{n}ds$ equal to 0, so that is another way.

And in the most frequent cases, this is the way by which we show the conservations in normal fluids including ideal fluids. It is true that I said normal fluid is not yet introduced, just we are talking about ideal fluids now.

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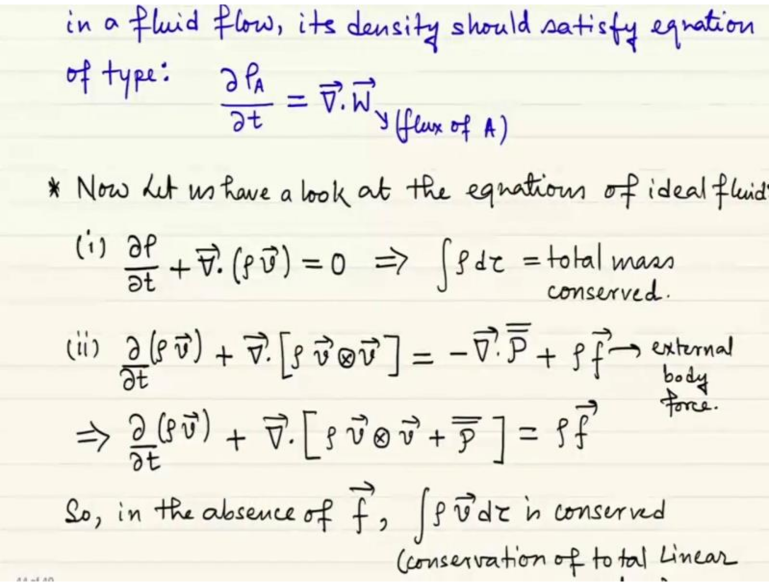

So, for an extensive variable to be conserved in a fluid flow, its density should satisfy the equation of type this $\frac{\partial\rho_A}{\partial t} = \vec{\nabla}.\vec{W}$. And this W will then be called the flux of A. So, ρ_A is the density of A. So, if you just remember the case of the continuity equation, you see that this is simply the mass density, and this is the divergence of another vector $\rho\vec{v}$. So, $\rho\vec{v}$ is called the mass flux. And this signifies the conservation of integration over $\rho d\tau$ which is the total mass.

Then what about the momentum equation? So, that was a scalar equation. Now, we have a vector equation. But the essence is the same $\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}.[\rho\vec{v}\otimes\vec{v}] = -\vec{\nabla}.\bar{\bar{\mathcal{P}}} + \rho\vec{f}$, that $\vec{f}$ is the external body force. And you can actually take this thing to the other side and take everything inside this divergence.

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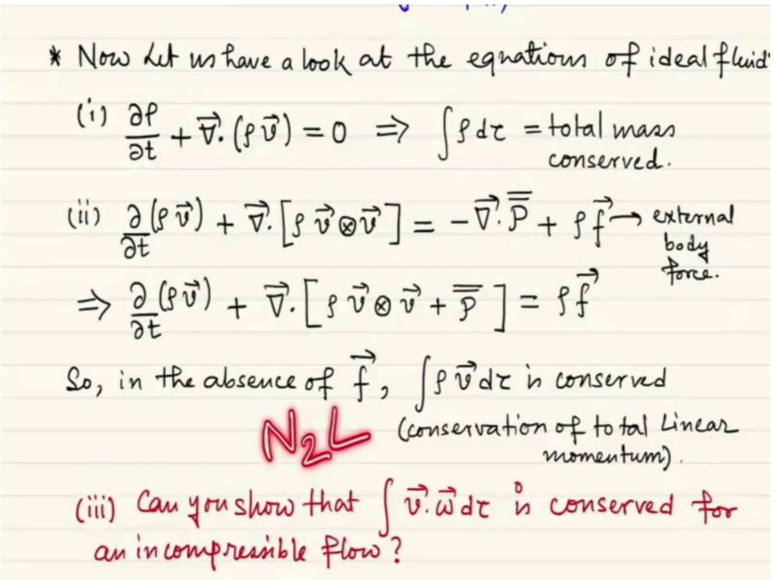

Now, if we say that we do not have any external force, so in the absence of any body force, $\frac{\partial(\rho\vec{v})}{\partial t}$ is equal to $-\vec{\nabla}.\left[\rho\vec{v}\otimes\vec{v} + \bar{\bar{\mathcal{P}}}\right]$ this thing. So, then, if $\rho\vec{v}$ is the density of something, then this $\rho\vec{v}\otimes\vec{v} + \bar{\bar{\mathcal{P}}}$ will be the corresponding flux.

And what will be the corresponding conserved quantity? It is simply integration $\rho\vec{v}d\tau$. And what is that? This is nothing, but the total linear momentum. So, you see this is what? This is simply Newton's second law. That means, in the absence of any net external force the total linear momentum is conserved. Now, I have a very interesting and slightly twisting, slightly non-trivial exercise for you that is if we now consider so that was just for compressible normal ideal fluid.

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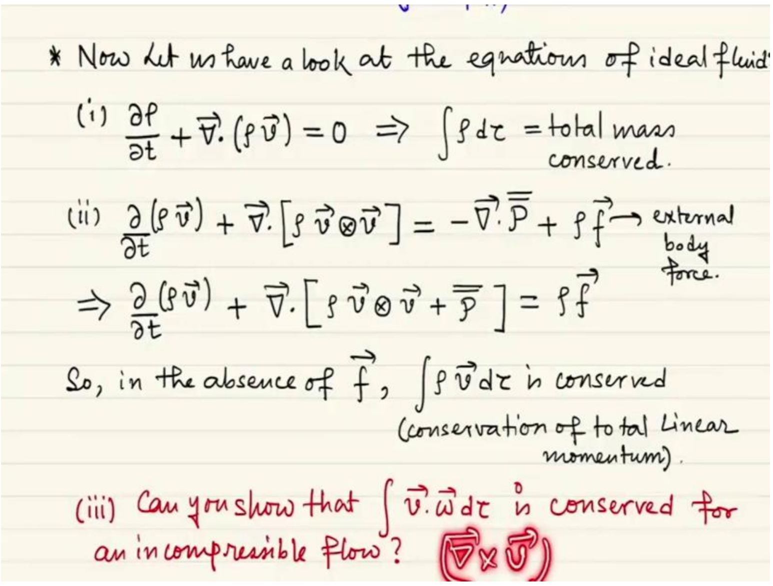

Now, if we are talking about ideal plus incompressible fluid, then can you show that integration over $\vec{v}.\vec{\omega}$, where omega is nothing but $\vec{\nabla} \times \vec{v}$. So, then can you show that $\vec{v}.\vec{\omega}d\tau$ integration over τ is a conserved quantity?

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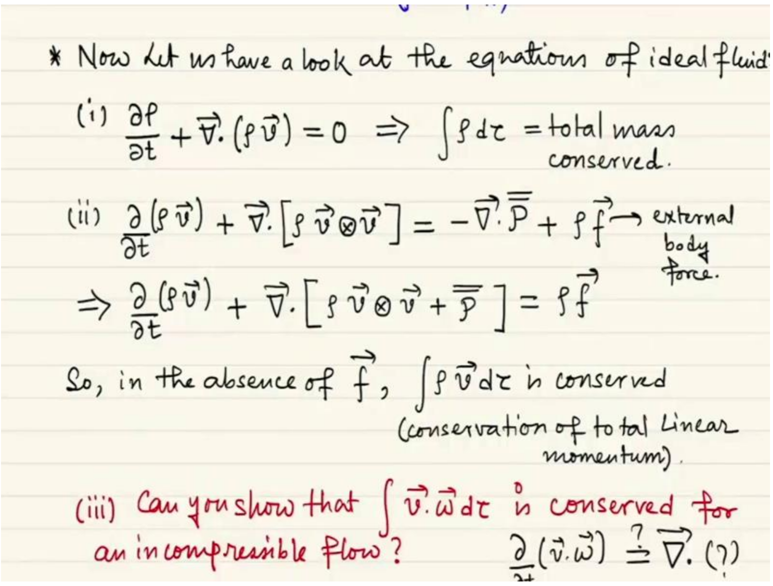

So, you have to actually show that the density this the evolution equation of the density, let me just use the true ink $\frac{\partial}{\partial t}(\vec{v}.\vec{\omega})$ can whether this should be written as a divergence of something or not? So, that is something very interesting you can try it at home.

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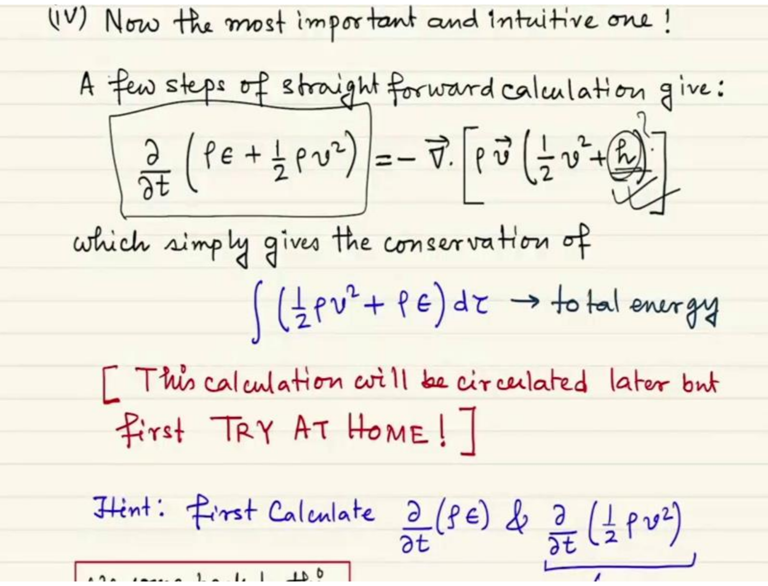

Finally, the most important and the most intuitive one, so this is the conservation of energy. So, a few steps of straight forward calculation actually just a minute, a few steps of straight forward algebra and calculation can give that $\frac{\partial}{\partial t}(\rho\epsilon + \frac{1}{2}\rho v^2)$.

And what is this? This is the density of total energy kinetic plus internal energy. This is equal to $\vec{\nabla}.[\rho\vec{v}(\frac{1}{2}v^2 + h)]$.

So, please, you check what h is over here. So, that is your task to do. So, you try to find first to evaluate this thing and to check whether this is equal to some divergence or not? If it is some divergence then what will be h?

So, if you can show that, then basically you will be able to show the conservation of total energy. That is again the same thing this calculation will be circulated later, but first this is your work to try it at home, because this is the most important thing. This is because this is conservation of energy. If there is no conservation of energy, then there is some problem. So, that is the most interesting thing and that is a very good exercise.

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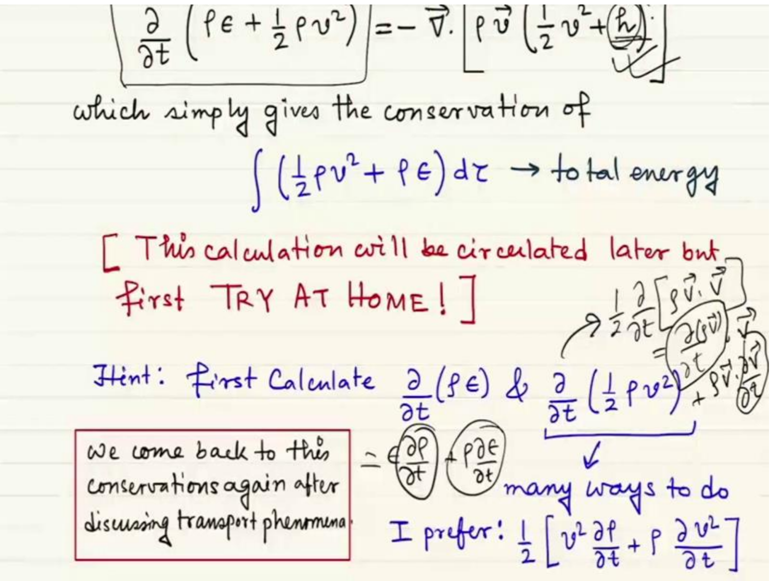

But I am giving you some hints, so you have to calculate separately $\frac{\partial}{\partial t}(\rho\epsilon)$ and $\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2)$. So, this $\frac{\partial}{\partial t}(\rho\epsilon)$ is nothing but the sum of two terms one is $\epsilon\frac{\partial}{\partial t}(\rho)$. And this $\frac{\partial}{\partial t}(\rho)$ can be obtained from continuity equation, plus you have $\rho\frac{\partial}{\partial t}(\epsilon)$ and that is from energy equation – internal energy evolution equation.

And then for the second one you will have two term, so there are so many possibilities actually. So, one can write like this $\frac{\partial}{\partial t}(\frac{1}{2}\rho\vec{v}.\vec{v})$, so then you can actually do it like $\frac{\partial}{\partial t}(\rho\vec{v}).\vec{v}+\rho\vec{v}.\frac{\partial\vec{v}}{\partial t}$ that is a possibility. What I personally prefer that to just write it as ρ times v^2, not $\rho\vec{v}.\vec{v}$ type of thing.

And then I write it like this $\frac{1}{2}v^2\frac{\partial\rho}{\partial t}+\rho\frac{\partial v^2}{\partial t}$, then actually you just need only one form of the momentum equation that is $\frac{\partial\vec{v}}{\partial t}$. Otherwise, you need $\frac{\partial}{\partial t}(\rho\vec{v})$ and $\frac{\partial\vec{v}}{\partial t}$ – these two. So, well that is up to you to decide which one you will choose. You can do both.

So, if you do that correctly and carefully, you should be able to show the conservation of the total energy. We will come back to this problem of conservation again when we will discuss transport phenomena and real fluids because then you will see that there are some types of effects.

For example, you all know that some viscous effects will be coming there like dissipation. So, viscous effect is nothing but a source of dissipation. And then you will see that the energy conservation is only possible, when the viscosity is negligibly small or 0, actually 0.

So, I mean here we do not have all these problems. Here we are in an ideal situation that is why energy is actually conserved. So, as a very interesting exercise, even for a compressible fluid, now you can actually check whether $\vec{v}.\vec{\omega}$ is also conserved or not.

So, one was just for incompressible ideal fluid, another is for compressible ideal fluid. Now, you can check that whether barotropic type of fluid closures can give you something additional or not. So, all these things can be done actually.

So, I mean you can always think again that there are interesting things. For example, if your flow is now incompressible and then your flow is ideal and two-dimensional, then which type of conservations are there? I mean in this course of lecture, I can only tell you a very limited number of examples, but it is you to actually investigate more over internet and then during the discussions you can also ask me questions.

And so, for example, for a $2d$ incompressible ideal flow, is there anything which is conserved like $\vec{v}.\vec{\omega}$? So, actually we will see there is something very, very interesting which is conserved, which is related to ω. So, just try to find it out on your own.

So, I mean that is all about for instance. That is all about the conservation principle of ideal fluids. So, here basically we learned to write the equations in a conservative form. If that is true, then only we can think of the conservation of an extensive variable. For intensive variable, this is very easy and immediate; the form is immediate to write.

Thank you.