Introduction to Astrophysical Fluids
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Lecture – 14
Kelvin’s vorticity theorem

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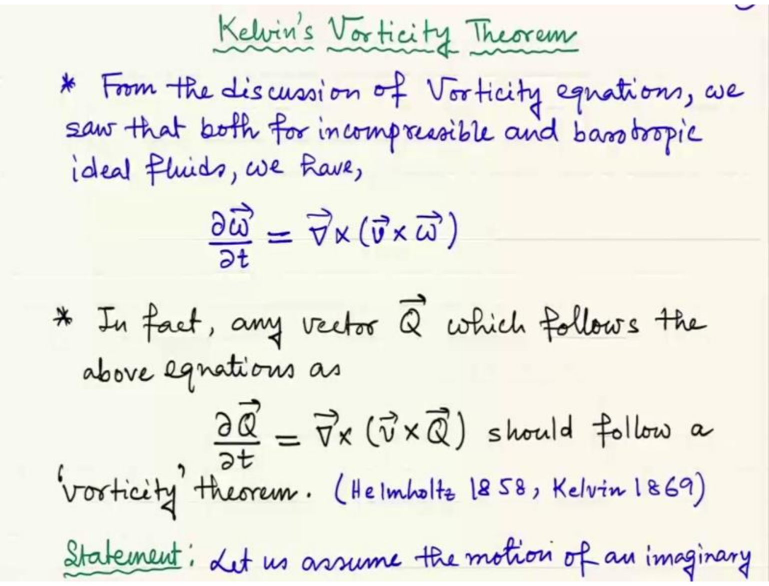

So, we continue our discussion on different properties of ideal fluids. So, in this part, we discuss a very interesting theorem it is called Kelvin’s Vorticity Theorem. So, from the previous discussions we already saw that if we are taking an ideal incompressible and barotropic fluid, then the vorticity equation or the evolution of the vorticity vector is given as $\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega})$. Vorticity vector, once again, just in case you forgot the curl of the velocity is known as the vorticity vector.

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ideal fluids, we have)

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) \qquad \vec{\nabla} \times \vec{v} = \vec{\omega}$$

* In fact, any vector $\vec{Q}$ which follows the above equations as

$$\frac{\partial \vec{Q}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{Q})$$ should follow a 'vorticity' theorem. (Helmholtz 1858, Kelvin 1869)

Statement: Let us assume the motion of an imaginary fluid surface S. If $\vec{Q}$ satisfies the above equation, the total flux of $\vec{Q}$ across the surface $\iint \vec{Q} \cdot d\vec{S}$ will be a constant (in time)

So, the vorticity vector basically follows this type of relation, this $\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega})$, where v is the bulk velocity. So, if any vector Q satisfies an evolution equation of this type, for example, $\frac{\partial \vec{Q}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{Q})$, where v is the velocity or bulk velocity of the fluid. So, then basically this vector should follow a specific property and that is called vorticity theorem. What that is exactly, we are trying to discuss now.

So, this theorem was, for the first time, discovered by Helmholtz in the year 1858. And the current form of this was first worked out by Kelvin 11 year later, but some people call this now more popularly as Kelvin's vorticity theorem. So, in some places you can also hear the name Kelvin Helmholtz's vorticity theorem.

So, the proof of this theorem is not really very much essential for our course or for our purpose, but once again if you are interested, you can just check any standard fluid mechanics book or over internet. There are so many literatures on that.

So, let us assume the motion of an imaginary fluid surface. So, in a flow field you assume an imaginary fluid surface. What is the meaning of fluid surface? That means, a 2D surface, a 2D plane type of thing, which is made up of fluid particles. So, this is not something where, no fluid particle that is not possible. So, this is an imaginary 2D entity constituted by the fluid particles.

If the vector Q, satisfies the above equation, $\frac{\partial \vec{Q}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{Q})$, then the total flux of Q across this surface is $\iint \vec{Q}.d\vec{s}$ and will be a constant in time. What is the meaning of that?

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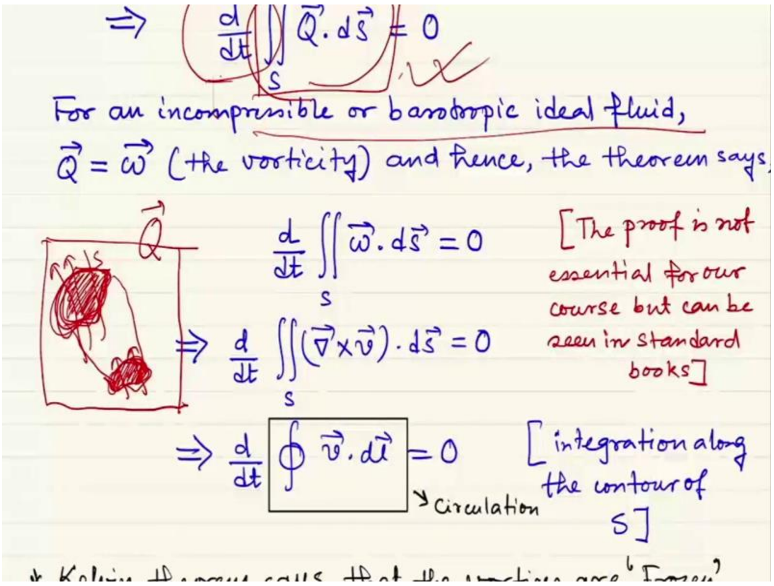

That means, that let say we are considering a flow field, and inside this flow field we consider some arbitrary surface, fluid surface S. Now, as the fluid particles are moving then basically this imaginary surface will also move.

Of course, in the course of the motion, the surface can change its area, but we are just stressing but all the particles which were primarily constituting this surface and if we trace that in that collection of particles then we will see that this area is now this area.

And now we calculate the ω at every point of that previous surface, and we calculate not only ω actually, now we are talking about some arbitrary vector Q. So, for Q, we calculate $\vec{Q}.d\vec{s}$ at every point and then we integrate over that surface S. if you calculate value of this Q vector at every point at the surface and then we do a surface integral. We see that in both cases the total surface integral has the same value.

So, in another way of writing we can simply say that $\frac{d}{dt}\iint \vec{Q}.d\vec{s}$ is equal to 0. Now, remember this $\frac{d}{dt}$ means Lagrangian or material derivative. Now, this simply says, because we are now cracking the particles, not one single particle, but one area or surface of particles.

So, that is why this $\frac{d}{dt}$ is the Lagrangian time derivative of the whole flux that is 0, that means, this is a Lagrangian invariant or it is invariant over time along the flow.

So, for an incompressible or barotropic ideal fluid, ω behaves like this Q and that is why the theorem simply says that for this type of incompressible or barotropic ideal fluids $\frac{d}{dt}$ of $\iint \vec{Q}.d\vec{s}$ is equal to 0. So, that is so called Kelvin Helmholtz vorticity theorem.

Now, I can do some mathematical reductions from this theorem, which simply says ω is nothing but $\vec{\nabla} \times \vec{v}$. So, this is nothing, but fluids $\frac{d}{dt}$ of $\iint \vec{Q}.d\vec{s}$ is equal to 0, so that is the statement of Kelvin's theorem.

And by Stokes law, this is equal to the close surface close line integral. So, this close line integral will be basically done on the contour of the surface s, the closed line integral of $\vec{v}.d\vec{l}$. So, basically $\frac{d}{dt}$ of that closed line integral is 0. So, these are two equivalent statements of Kelvin's vorticity theorem.

This line integral $\oint \vec{v}.d\vec{l}$ is known as circulation vector. So, another equivalent way of saying Kelvin Helmholtz vorticity theorem is that in an incompressible or barotropic ideal fluid the circulation across any fluid surface is a constant in time.

So, it is given that we are tracing the surface along the path of the flow. So, basically this means that we are tracing the fluid particles along their real motion.

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$\vec{Q} = \vec{\omega}$ (the vorticity) and hence, the theorem says,

$$\frac{d}{dt}\iint_S \vec{\omega}\cdot d\vec{s} = 0$$

[The proof is not essential for our course but can be seen in standard books]

$$\Rightarrow \frac{d}{dt}\iint_S (\vec{\nabla}\times\vec{v})\cdot d\vec{s} = 0$$

$$\Rightarrow \frac{d}{dt}\underbrace{\boxed{\oint \vec{v}\cdot d\vec{l}}}_{\text{Circulation}} = 0$$

[integration along the contour of S]

* Kelvin theorem says that the ~~vortices~~ vortex lines are 'Frozen' along the fluid flow. (This theorem will also have importance for plasmas where, $\vec{Q}$ is the mag. field).

And what is the physical meaning of that? that is something very important, and actually will be used when we will even talk about like plasmas. Because you will see that there will be something called magneto hydrodynamics when we talk about plasma fluids. And then B or the magnetic field will actually satisfy this type of equation for ideal magneto hydrodynamics case.

So, what is the meaning of that? When some vector ω or Q basically satisfies this vorticity theorem the meaning is that the vortex lines, they are actually frozen along the fluid flow.

What is the meaning of that? That means, the total lines of vortex which were passing through this surface S is actually not changing. So, they are the same number of vortex lines are now passing through this surface.

And that is why basically you can say that this is the flux of vortex. So, if that does not change, they simply say that whenever the fluid is flowing along the fluid flow the vortex lines are moving, so that they are stuck to the fluid flow or they are frozen in the fluid flow. So, in a sense, this is also called frozen in theorem.

So, this thing has very importance I mean very deep importance in various things in astrophysics. So, maybe when we will discuss different type of instabilities or some other astrophysical examples, we will come into that.

Thank you.