Introduction to Astrophysical Fluids
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Lecture – 13
Properties of ideal fluids

So, we continue our discussion on the ideal fluids and in this lecture, we will mostly discuss some interesting properties of ideal fluid equations.

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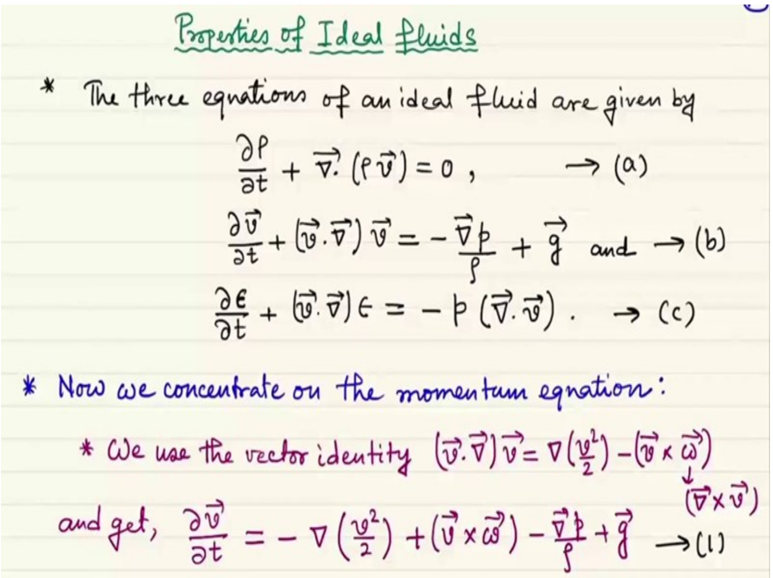

So, basically as you can see that again equations (a), (b) and (c) – these three equations are nothing but the continuity equation, the momentum evolution equation and the energy equation for ideal fluids. So, now you see that we have five equations and we have five unknowns ρ, v and ϵ . Although p is there, but we know that ρ, p and v, they can simply be written in terms of effectively two unknowns – N and T.

Now let us concentrate on the momentum equation. We will see that some interesting things can be brought out of this. So, first we try to rewrite this advective derivative part; which is $(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v}$. Now, you know there is a very popular identity which simply says that

$$(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) - (\boldsymbol{v} \times \boldsymbol{\nabla} \times \boldsymbol{v})$$

and $\boldsymbol{\nabla} \times \boldsymbol{v}$ is nothing but our vorticity vector $\boldsymbol{\omega}$. Then, the original momentum evolution equation can now be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{v^2}{2}\right) + (\boldsymbol{v} \times \boldsymbol{\omega}) - \frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{g} \qquad \textbf{(1)}$$

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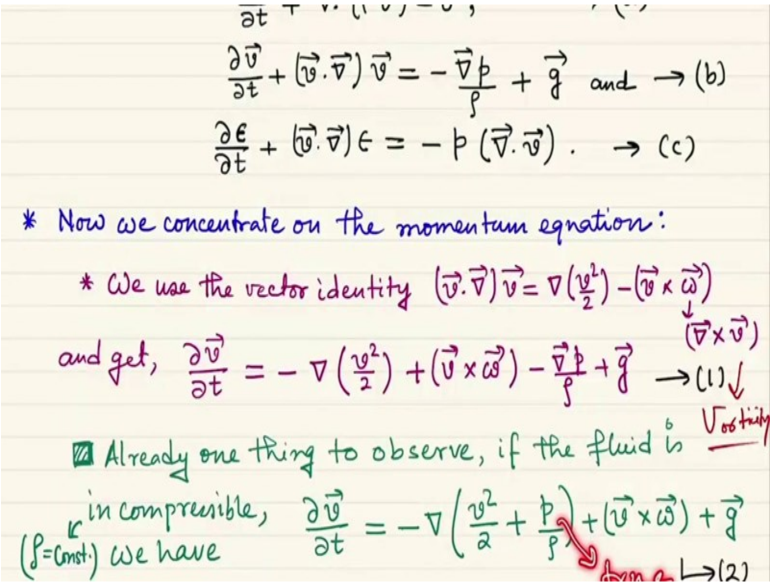

So, for a very simple case where ρ is constant, so this $\frac{\boldsymbol{\nabla} p}{\rho}$ can simply be written as $\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)$ and then this $\boldsymbol{\nabla}\left(\frac{v^2}{2}\right)$ and this $\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)$ can be taken under the same umbrella and this will give you $-\boldsymbol{\nabla}\left(\frac{v^2}{2} + \frac{p}{\rho}\right)$ ok.

So, you see, for an incompressible fluid, where ρ is simply a constant, so $\frac{p}{\rho}$ is, I mean the variable effective variable $\frac{p}{\rho}$ is the pressure itself. So, you can say that gradient term in the momentum evolution equation becomes always as a gradient of a pressure type of thing, then this term $\boldsymbol{\nabla}\left(\frac{v^2}{2} + \frac{p}{\rho}\right)$ is now having two pressure type of thing - one is the true pressure, this $\frac{p}{\rho}$ is the true pressure another is a type of pressure which is $\frac{v^2}{2}$, ok. So, sometimes you can say this $\frac{v^2}{2}$ is not a true pressure, but this is a dynamic pressure or something, ok.

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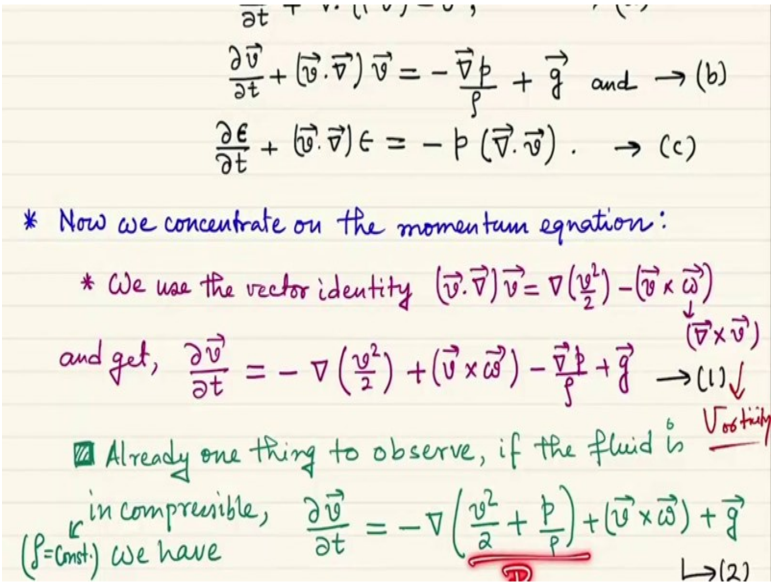

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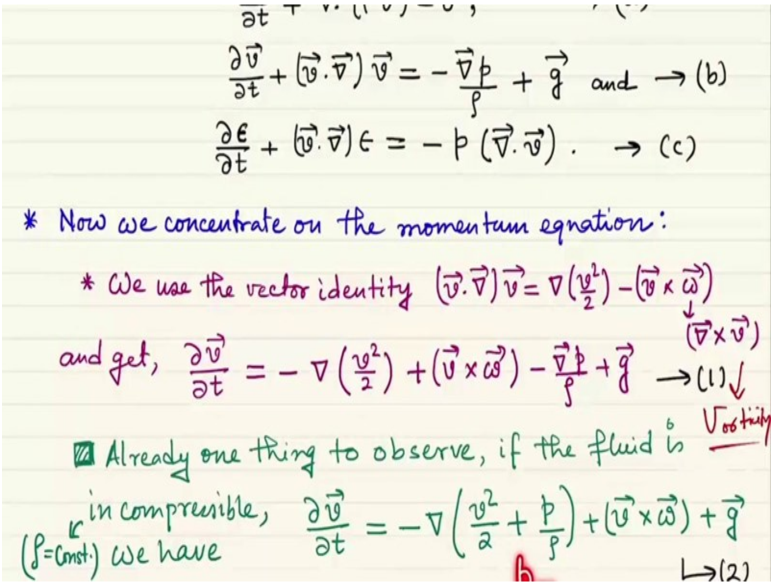

Well, I mean there is no hard and fast name for this, but the thing is that the whole thing can be thought to be as an effective pressure type of thing and this $\frac{v^2}{2} + \frac{p}{\rho}$ is called as a total pressure or you can simply write small p_t, it is a total pressure. So, of course, the dimension or the unit of this $\frac{v^2}{2} + \frac{p}{\rho}$ part is not pressure, but pressure by density, so that part you have to be careful.

Otherwise, you can simply say that just for the incompressible case this $\frac{v^2}{2} + \frac{p}{\rho}$ is effectively a pressure like thing. So, $\frac{\partial \boldsymbol{v}}{\partial t}$ is equal to minus gradient of $\frac{v^2}{2} + \frac{p}{\rho}$, which is an effective pressure type of thing by constant density ρ plus $(\boldsymbol{v} \times \boldsymbol{\omega})$ plus $\boldsymbol{g}$,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{v^2}{2} + \frac{p}{\rho}\right) + (\boldsymbol{v} \times \boldsymbol{\omega}) + \boldsymbol{g} \tag{2}$$

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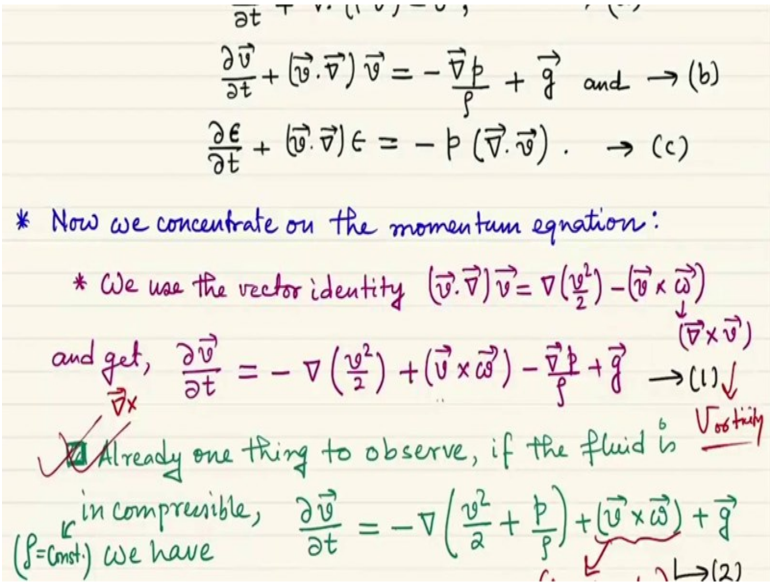

Now, sometimes $(\boldsymbol{\omega} \times \boldsymbol{v})$ is called Lamb vector, this is a name for this vector. This $(\boldsymbol{v} \times \boldsymbol{\omega})$ is the minus of Lamb vector, actually because lamb vector is traditionally $(\boldsymbol{\omega} \times \boldsymbol{v})$.

Again, coming back to this original part, where ρ is a variable itself, and so if we just take the curl on both sides, curl mean this is like partial space operator this $\boldsymbol{\nabla} \times$. So, $\boldsymbol{\nabla} \times$ will always commute with the partial time operator because space and time they are not explicitly depending over each on each other. So, $\frac{\partial}{\partial t}$ and curl will be interchanging, then curl on this term $\boldsymbol{\nabla}\left(\frac{v^2}{2}\right)$ will be vanishing because curl of a gradient is 0. Curl of this term $(\boldsymbol{v} \times \boldsymbol{\omega})$ will be there, and curl of this term $\frac{\boldsymbol{\nabla} p}{\rho}$ will simply be something which we will show and curl of this term $\boldsymbol{g}$ will also vanish due to the simple fact that most of the cases these body forces are conservative in nature.

So, this is seen in nature, the body forces are mostly conservative and the surface forces are mostly non conservative dissipative type of thing. For example, viscous force, I will come into more detail.

But otherwise, so for example, body force which is the gravitational force or the force of gravity basically, so that is a conservative force field. Conservative force field means for fluid is that the acceleration is having a zero-curl ok, so and that is exactly what is used here.

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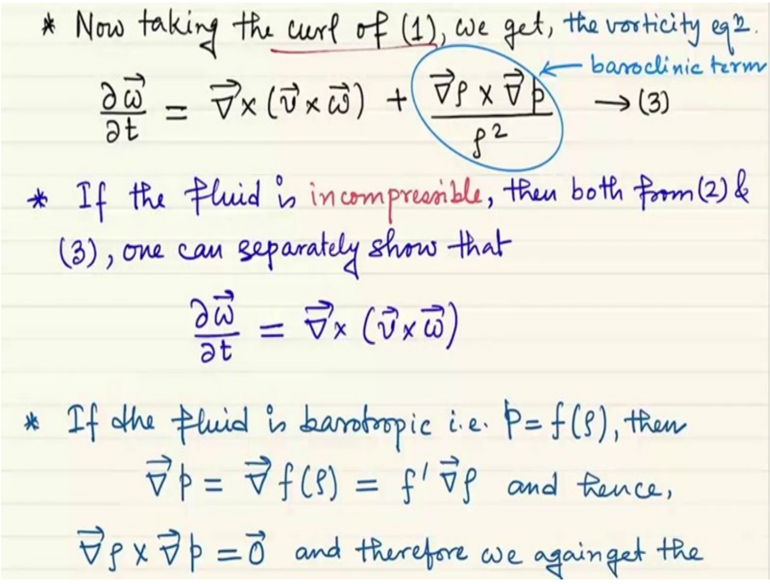

So, you will see if we just take the curl on both sides, you will get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2} \tag{3}$$

So, you just take the curl of this $\frac{\boldsymbol{\nabla}p}{\rho}$ part. There will be two term; one term will be 0 because curl of $\boldsymbol{\nabla}p$ will be 0, again curl of a gradient is 0; the other term where the gradient operator will act on the density that part will be this one $\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}$ and this is known as the baroclinic term. This total term $\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}$ is known as the baroclinic term.

Now, if the fluid is incompressible, from equations (2) and (3), we can show that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) \tag{4}$$

So, if you take the curl of equation (2), so this part $\frac{\partial \boldsymbol{v}}{\partial t}$ will be just $\frac{\partial \boldsymbol{\omega}}{\partial t}$ for an incompressible fluid, once again, this part $\boldsymbol{\nabla}\left(\frac{v^2}{2} + \frac{p}{\rho}\right)$ under a curl will totally go off and this part $(\boldsymbol{v} \times \boldsymbol{\omega})$ will be non-vanishing, this part $\boldsymbol{g}$ will vanish.

So, for an incompressible fluid from (2) or from (3), one can show equation (4). From (3) it is possible since because $\boldsymbol{\nabla}\rho$ is 0 for incompressible fluid, then this $\frac{\nabla\rho\times\nabla p}{\rho^2}$ is again 0 and you will have (4). So, you can show that the vorticity vector satisfies evolution equation (4) ok.

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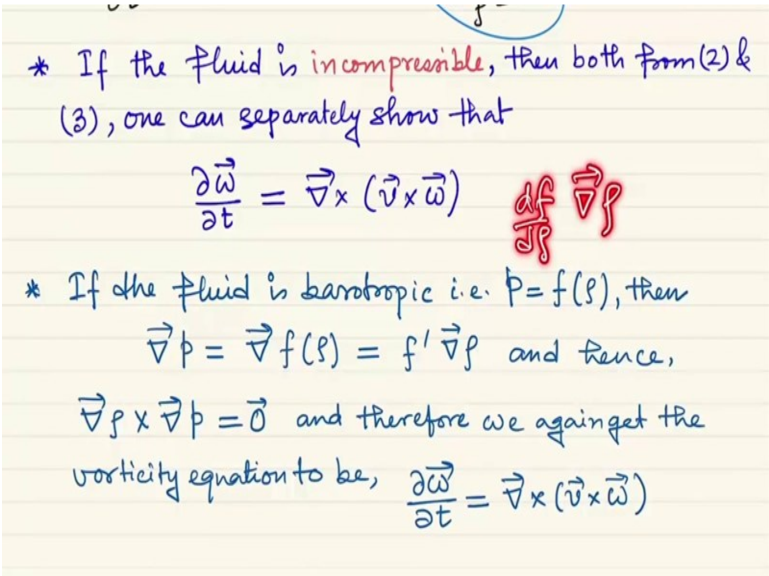

Now, that was the case for simple incompressible ideal fluid. Of course, when we are talking about all these things, they are all inside the framework of ideal fluid because in this part we are only discussing the ideal fluids, we have not at all introduced anything for non-ideal things.

So, if the fluid is now no longer incompressible, but it is compressible but barotropic. What is the meaning of barotropic? So, there are two terms which can be confused one is baroclinic and barotropic.

Barotropic means that pressure is only a function of density i.e., $p = f(\rho)$, then you can always write gradient of p is equal to gradient of that function $\boldsymbol{\nabla}p = \boldsymbol{\nabla}f(\rho)$, which is nothing but $f'\boldsymbol{\nabla}\rho$.

So, what is this $f'\boldsymbol{\nabla}\rho$? This is nothing but $\frac{df}{d\rho}\boldsymbol{\nabla}\rho$ that is the thing and this $\frac{df}{d\rho}$ is nothing but this f', ok. So, $\boldsymbol{\nabla}p$ can then be written as some function times $\boldsymbol{\nabla}\rho$, and then of course, if you just look the baroclinic term that simply says that for a barotropic fluid the baroclinic term is also 0. For an incompressible ideal fluid baroclinic term is 0; for a barotropic ideal fluid, the baroclinic term is also 0 and if the baroclinic term is 0, then again, the vorticity vector satisfies this simple evolution equation (4).

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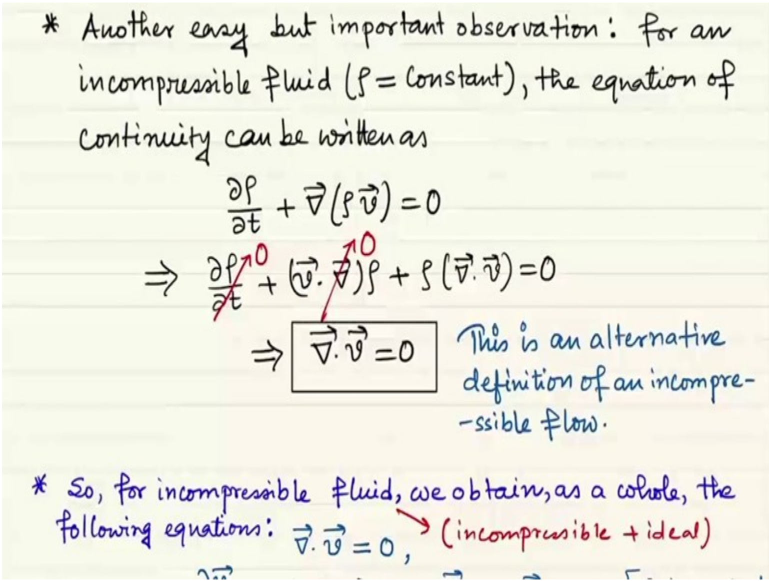

Another easy but important observation is that for an incompressible fluid where ρ is constant, the equation of continuity can be written in a very simple form.

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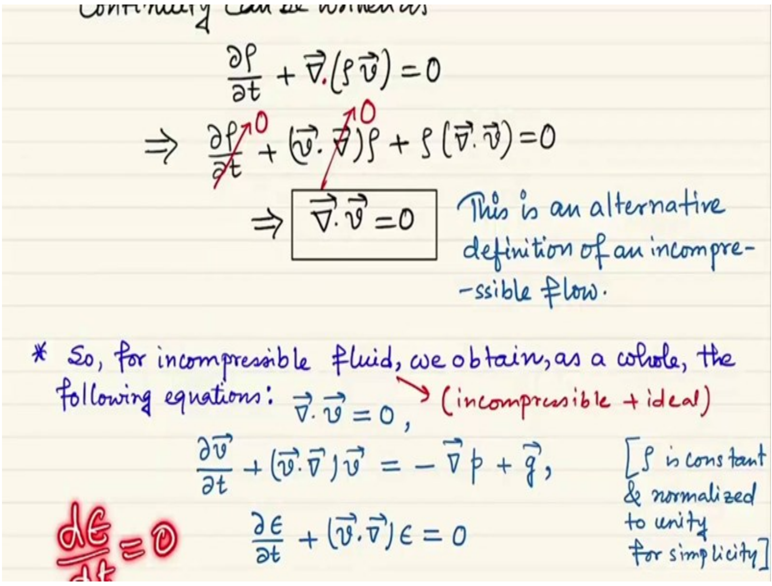

So, here basically we are trying to show something, so our fluid is basically ideal and now we are also taking different particular subclasses, for example, incompressible barotropic etc. For incompressible ideal fluid; actually, there is no question of ideal or non-ideal, just incompressible fluid, ok. The equation of continuity can be written as this

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) = 0 \implies \frac{\partial \rho}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\rho + \rho\boldsymbol{\nabla}.\boldsymbol{v} = 0 \tag{5}$$

and from that, you can say $\frac{\partial \rho}{\partial t}$ is zero because ρ is a constant; $\boldsymbol{\nabla}.(\rho \boldsymbol{v})$ is also 0 because ρ is a constant. So, $\rho\boldsymbol{\nabla}.\boldsymbol{v}$ is zero and ρ is nonzero, so finally,

$$\boldsymbol{\nabla}.\boldsymbol{v} = 0 \tag{6}$$

This is an alternative definition of an incompressible flow. Now, why we are talking about this? So basically, we are just discussing different properties; one property we showed for the vorticity vector and its evolution, now we will show another interesting property that is for incompressible fluid. Finally, if we rewrite all the moment equations, then they should look like this

$$\boldsymbol{\nabla}.\boldsymbol{v} = 0 \tag{7}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{v^2}{2}\right) + (\boldsymbol{v} \times \boldsymbol{\omega}) - \boldsymbol{\nabla}p + \boldsymbol{g} \tag{8}$$

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\epsilon = 0 \tag{9}$$

Why the right-hand side of (9) is 0? Because if you remember the right-hand side was $\frac{p}{\rho}(\boldsymbol{\nabla}.\boldsymbol{v})$ and $\boldsymbol{\nabla}.\boldsymbol{v}$ is zero so right-hand side of (9) is 0. We have also used one fact that ρ is a constant. So, we just say that we are normalizing ρ to unity, so that ρ just do not appear here in order to confuse the reader. So, ρ is no longer there ; ρ is there, but being a constant and we just put that constant value to be 1.

We are saying here that we have equations (7) and (8) is the system of dynamical equations, where we simply have v and p as unknown, and we also have equation (9) as a dynamical equation. Now, from your Lagrangian derivative concept, you can say equation (9) is nothing but total time derivative of energy density is zero i.e., $\frac{d\epsilon}{dt} = 0$.

So, in an incompressible fluid, basically from the three dynamical equations, one is very easy to understand, the third one; that is the internal energy okay, the internal energy density ϵ of a fluid particle is constant with time. Now, we have to extract some other interesting information out of (7) and (8).

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* Now let us take the divergence of both sides,

$$\cancelto{0}{\frac{\partial(\vec{\nabla}\cdot\vec{v})}{\partial t}} + \vec{\nabla}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v} = -\nabla^2 p + \underbrace{\vec{\nabla}\cdot\vec{g}}_{\text{known}}$$

$$\Rightarrow \quad \nabla^2 p = -\vec{\nabla}\cdot\left[(\vec{v}\cdot\vec{\nabla})\vec{v}\right]$$

So, for an incompressible fluid, p can be determined from the knowledge of velocity only

↳ p is not an isolated thermodynamic quantity in case of an incompressible flow.

* Two observations: For incompressible fluids,

(a) The continuity equation + momentum eq^n make

How to do that? Now, we will take the divergence of the both sides. Beforehand we took the curl now, we will take the divergence of both sides of this momentum equation (8). If we do that, first we see again, even for divergence there is the same thing like curl. So, $\frac{\partial}{\partial t}$ and divergence operator will commute, so first term we will be $\frac{\partial(\nabla.v)}{\partial t}$ so this will be 0 plus divergence of this advective term $\boldsymbol{v}.\nabla\boldsymbol{v}$ is equal to minus divergence of ∇p whch is nothing but $\nabla^2 p$ plus divergence of $\boldsymbol{g}$. Now, divergence of g is known.

Now, what is not known? This, this one this $\nabla^2 p$ and $\nabla.(\boldsymbol{v}.\nabla\boldsymbol{v})$ are not known and $\nabla.\boldsymbol{g}$ is nothing but a constant, so we can just for instance forget that and we can say

$$\nabla^2 p = \nabla.(\boldsymbol{v}.\nabla\boldsymbol{v}) \tag{10}$$

And actually, using some mathematical technique, you can always know p in case where $\boldsymbol{v}$ is known. So, for an incompressible fluid, p can be determined from the knowledge of the velocity only.

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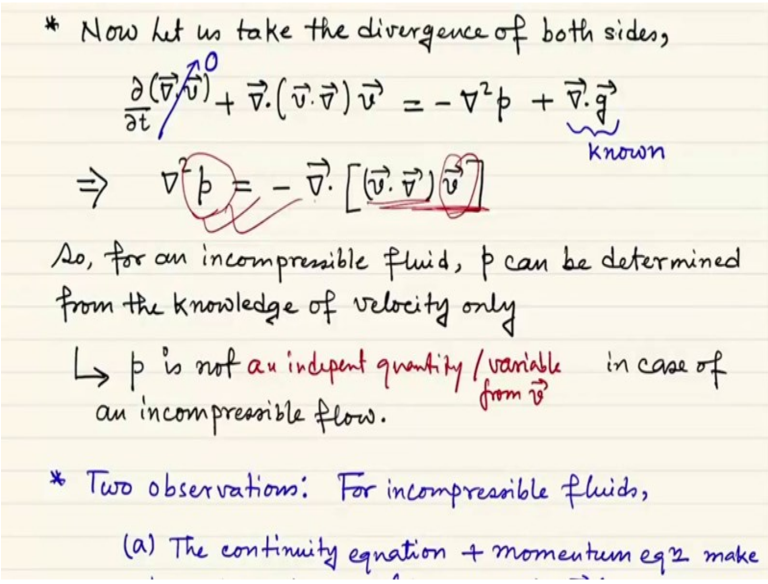

p is not an independent quantity or variable, it can be known from $\boldsymbol{v}$ in case of an incompressible flow.

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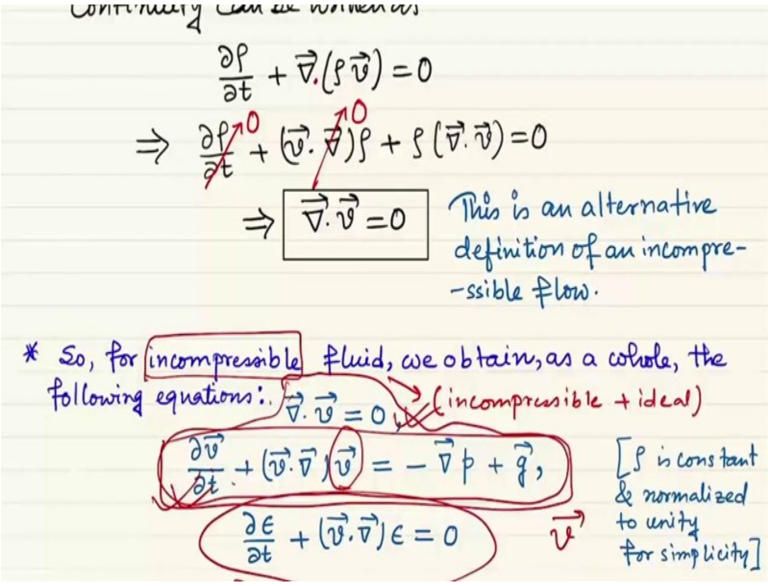

It simply says that in these two equations (7) and (8), practically we have one unknown and that is the velocity field and that is why we say that in the case of an incompressible fluid these two equations they just constitute a dynamical theory themselves, and $\boldsymbol{\nabla}.\boldsymbol{v} = 0$ is just a condition or that is ok.

And if you just simply take equation (8), so one state variable is $v$and its evolution equation is given; so, three scalar unknown and three scalar equations. So, it is done and equation (9) does not really participate in the dynamical theory.

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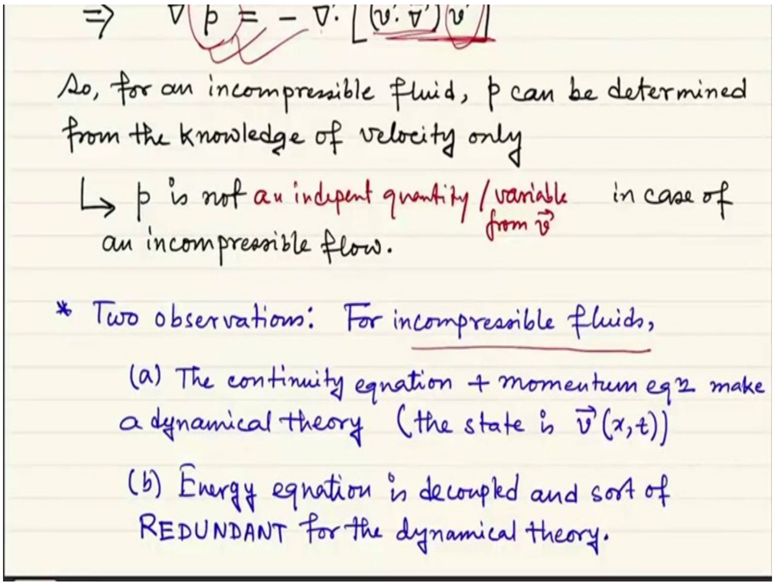

So, what I just said the two observations; for an incompressible fluid what happens, the continuity equation plus the momentum equation make a dynamical theory the state is only $\boldsymbol{v}$, ρ is a state, but, ρ is a trivial because this is a constant. So, it's evolution equation is there and this is nothing but $\frac{d\rho}{dt}$.

Now, you see it is $\frac{d\rho}{dt}$ or $\frac{\partial\rho}{\partial t}$ both are actually zero. So, that is something very interesting to understand. Another condition is that the energy equation is decoupled and sort of redundant for the dynamical theory. It is redundant but this is only redundant when we are talking about incompressible ideal fluids.

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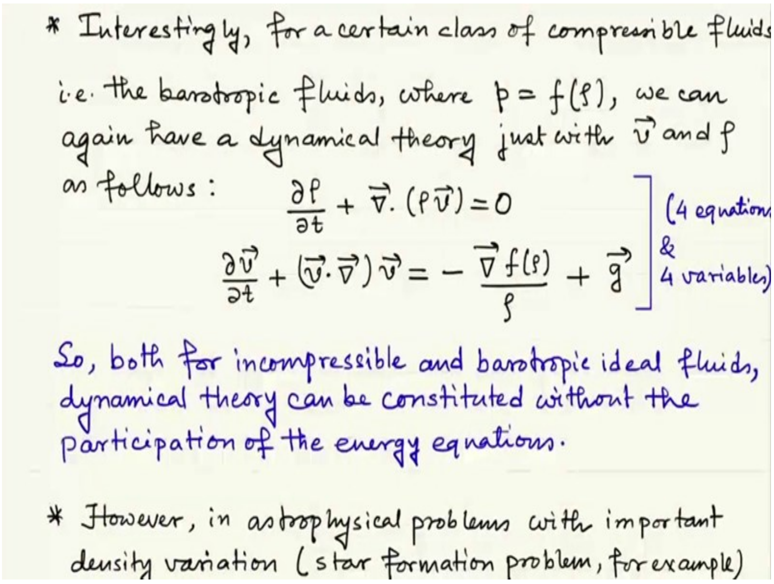

Now, is it the same case for compressible fluids? Well, of course, the answer is no in general. But well once again there is a special family which we already talked about and this is the family of the barotropic fluids, where p is a function of ρ only. Then again, we can write that this momentum evolution equation like this

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} f(\rho)}{\rho} + \boldsymbol{g}$$

and this p is nothing but $f(\rho)$.

So, finally, we now have again these two equations: $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) = 0$ and $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} f(\rho)}{\rho} + \boldsymbol{g}$, but now we have four scalar unknowns, ρ is now a variable and unknown; and three equations: three components of velocity and we have four equations. So, again this continuity equation $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) = 0$ and the momentum evolution equation they constitute a dynamical theory, but now the state needs two things ρ and v vector. So, the conclusion is both for incompressible and barotropic ideal fluids, dynamical theory can be constituted without the participation of the energy equations. Now, why we are talking all about this?

So, in future, if you do some research or some study in astrophysical systems, for example, astrophysical dynamics, the stellar evolution, the equilibrium, the astrophysical turbulence ok, this type of thing, the star formation problem, then the question is that whether when we start to write the equation either to do some analytical treatment or to do some numerical works, you have to set the parent equations: the governing equations.

Then the question is that, can you just write your two equations? If you simulate mostly or do some analytical treatment, will then just continuity equation plus the momentum equation will be enough or you need the energy equation? The answer is if your system is such that your system is fairly incompressible, then you do not need the energy equation or if your system is barotropic, that means, you can sufficiently say that I can think of an isothermal system, isothermal system is a barotropic system, I will come to that or an adiabatic system; simple adiabatic system that is another case of a polytrophic system. For example, if you draw a profile of pressure with respect to a function of the density and you see that pressure only varies due to density, then polytropic relation is a very good approximation for your system and for that, you do not need the energy equation.

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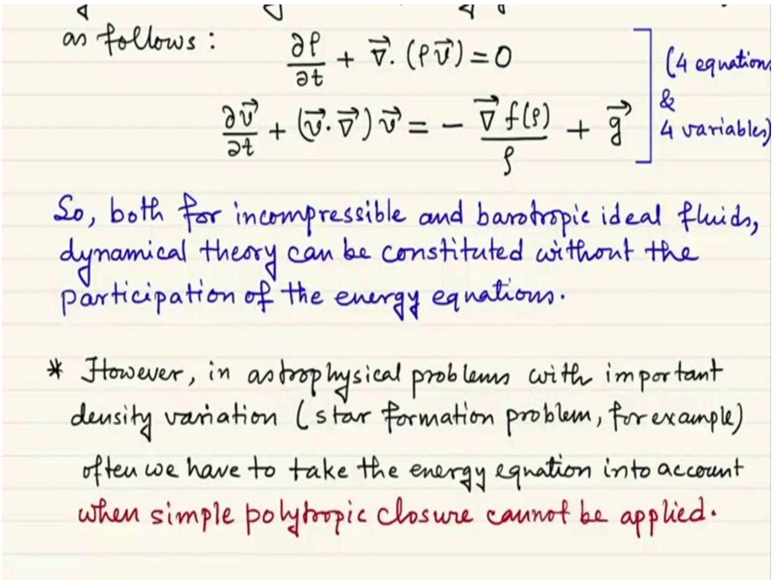

But in several astrophysical problems: with important density variation that means you cannot consider the incompressible assumption, for example, the star formation problem ok. Often, we have to take the energy equation into account when simple polytropic closures cannot be be applied or cannot be valid.

So, that is something, so you understand that is why I am saying the modus operandi should be like that: first you have to check whether a polytropic closure is a good approximation for your system or not. Fisrt of all important density variation, no chance of incompressibility; second is that you check whether you have like the pressure as a well-defined function of density only or not. If yes, if you can find, no problem. You just simulate or do analytical treatment just using continuity equation and momentum equation.

Otherwise, you must include the energy equation. When we are talking the mixing problem in astrophysics, convective problem and instabilities, then of course, the energy equation must be taken into account, I will discuss later about this.

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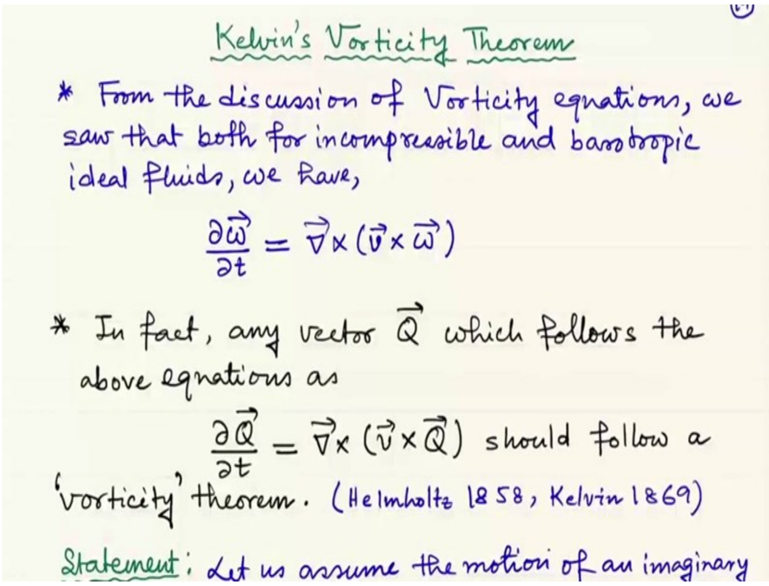

So, in the next part we will discuss. So, that was some discussions about some of the properties of ideal fluids and we saw actually how we can derive the evolution equation of the vorticity and also how the dynamical theories or development of dynamical theories are getting simplified in case of incompressible ideal or barotropic ideal fluids, ok.

So, there are so many information, now you have to really think and all these information are actually very very interesting for several circumstances. So, these are the general discussions or the general exposure of different type of properties of fluids.

Now, specifically from case to case, we will just study the nature of the corresponding fluid we will try to apply the corresponding property, that is how it is. So, in the next lecture, we will discuss about very interesting theorem which is also a property of ideal fluid, of course, it is called the Kelvin vorticity theorem ok.

Thank you very much.