Introduction to Astrophysical Fluids
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Lecture – 12
Macroscopic forces on an ideal fluid

Hi. We are continuing our discussion on ideal fluids. So, in the previous discussion, we derived the ideal fluid equations.

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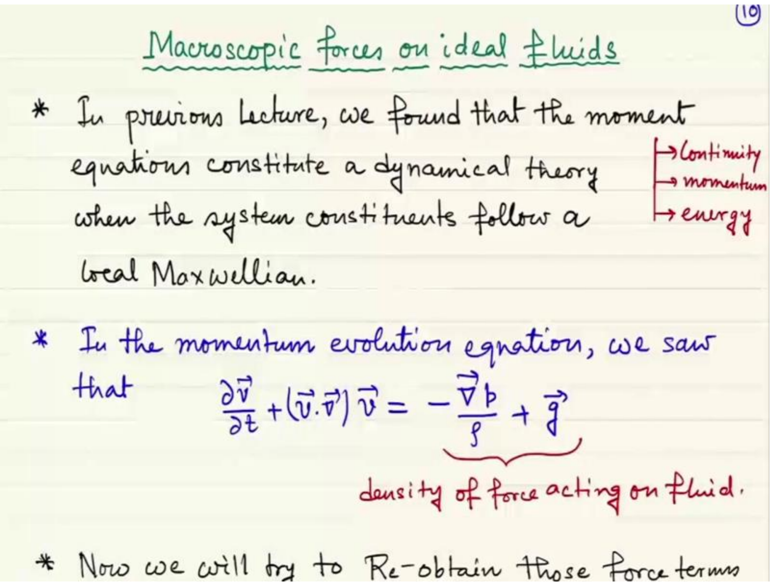

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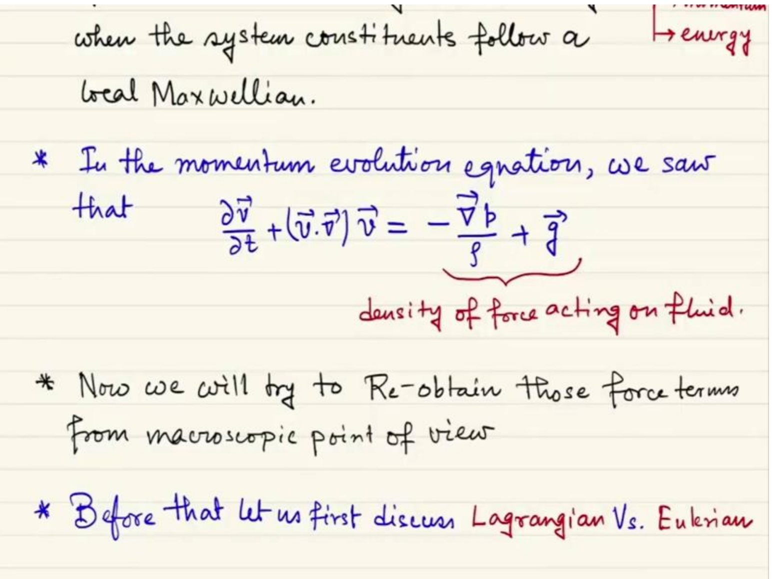

So, we actually saw that the moment equations. So, continuity equation, momentum equation and energy equations, these three moment equations they constitute a dynamical theory. I mean, three means, one is vectorial of course. So, well, if we count the number of scalar equations, we have five equations in total. So, they constitute a dynamical theory finally. When the system constituents they follow a local Maxwellian distribution that we saw in the previous part.

And the momentum evolution equation then simply becomes $\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla}p}{\rho} + g$, which is the gradient of pressure force, so which is actually coming from the inside of the fluid, so that is not an external force. So, p is the fluid pressure basically.

So, p is coming due to the pressure of the fluid. And this is the external forcing agent. So, I said that g is something, which we call the body force. And most of the cases, in astrophysics, this is either I mean external gravitational field, in some cases, it can be an electromagnetic field as well. But most of the cases, this is a gravity field.

So, $\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla}p}{\rho} + g$ is an equation of the acceleration of force. So, here you will see $-\frac{\vec{\nabla}p}{\rho}$ is nothing but the density of force. So, I mean mass density of force. So, basically this simply says that this is force per unit mass which is acceleration.

Now, in this discussion, till now, we were talking about deriving or obtaining the fluid equation or the known equations for the fluids or the continuum starting from the very basic kinetic theory. So, this is called the down to top approach, that means, we are starting from a very basic level, one term, then single particle, then kinetic, then statistical level, and then finally, a continuum level.

Now, it is also true that the fluid equations can equally be obtained, so at least, till now we have talked about the perfect fluid or ideal fluid equation, so that can also be obtained from macroscopic consideration. That means, using such a length scale where we do not have a microscopic view but we just see, right from the beginning the fluid as a continuum.

So, we cannot see their constituents with our acquired instruments or measurement method. Still, we can obtain these equations. So, basically that is the way of obtaining the macroscopic equations macroscopically.

So, in most of the fluid mechanics books or in engineering books, the fluid equations are derived in that way. And historically when people derived them, they actually did this macroscopically, and that was much more interesting.

And then people who wanted to know that really this is consistent with our microscopic theory or rather I mean how can from starting a microscopic theory, we can really find a beautiful down to top approach, and just by saying that from microscopic to macroscopic, how we can develop the different levels of theory or a hierarchy of theory. So, for that people tried this approach what I was discussing.

But now let me do something which is very much conventional. So, let us just try to re obtain force terms from a macroscopic point of view.

But before that it is also useful to say that if we try to handle or study a fluid medium or a flow medium macroscopically, we need to know two about two approaches which are equivalent, but they are priory not the same approach, ok. So, one is called the Lagrangian approach another is called the Eulerian approach.

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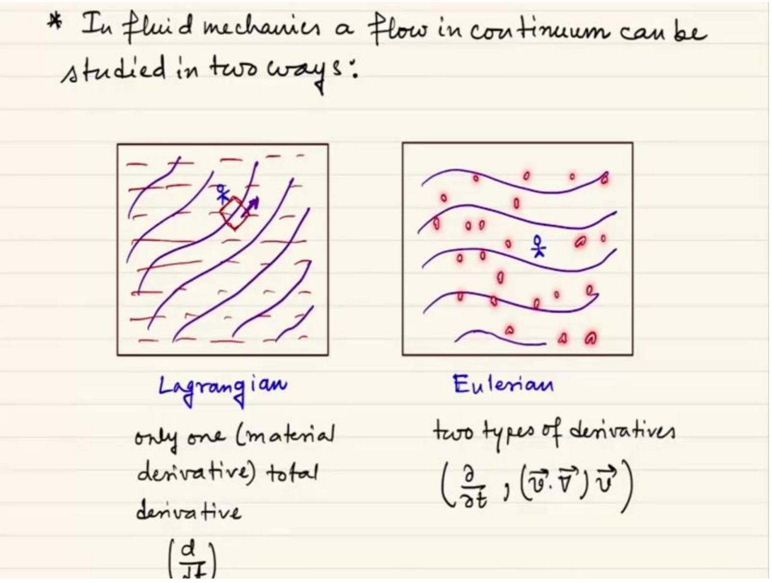

So, in fluid mechanics, a flow in continuum can be studied in these two following ways. The first one is that is called the Lagrangian view, that means, we just assume a very small fluid particle type of thing inside a blob of fluid in a flow that will be of course an imaginary blob, because this is a continuum. You cannot think of a discrete particle like that. And this blob of fluid will be traced throughout the flow, and we will see that where it is going with time.

So, if we just trace this, being on the blob, so let us consider that we are the observer. And we sit on that blob of fluid. And we go at every point. Where we are going? We just measure our positions.

And when we measured our position taking the $\frac{d}{dt}$ of that position, we also measured our velocity. So, in this way, we can measure the position and the velocity of the blob of the fluid. So, this is the way one can study a flow, and this is called the Lagrangian view or Lagrangian approach.

There is an approach, alternative approach which is very useful for physicists at least, and also for studying complex type of fluid flow where you cannot really trace efficiently fluid flow. For example, let us say you have a chaotic flow or something and where it is very difficult to search for a blob of fluid, because this is a mess inside the flow, it is an highly non-linear and you cannot see layers.

So, there is a very high chance that once you start tracing a blob of fluid, you will be lost in some moments. This is highly possible. Then a very effective approach is the Eulerian approach ok. And this Eulerian approach simply says that you are the observer. Again, the same thing, we are the observer and we do not move, and we just sit at one point in the flow field ok.

And then we will measure the fluid which is coming and the passing that point, and we will measure the velocity of that particle. So, that is the thing where we are not changing our space, but we are simply measuring the velocities with respect to different fluid particles.

So, here basically we are not interested in one single fluid particle, but we are interested in one point in space. So, that is the part where you will see that how the velocity at one point just changes explicitly with respect to time, and that gives the partial time derivative of velocity or any quantity. It can be a velocity, it can be density, it can be pressure.

So, whenever this fluid particle or some fluid passes through the point where you are sitting, you just measure its density, pressure, velocity like this. So, basically, when at some other time interval another fluid comes, so no change in space, but only time is changing. So, that simply says that this will give you some partial time derivative of that corresponding quantity.

Again, another thing is there that in the same way, you can easily understand that is not all, that is a partial picture of the studying the flow. So, now we can say that many of us we are sitting at different point in space, and all of us are measuring velocities at a given time instant. And that will give us an essence of a partial space derivative of velocity.

Once again, when I just say velocity, that can be any property of the fluid, velocity, density or pressure whatever. And that will be given by its variation and this is called the advective derivative of some quantity. Once again it can be ρ, it can be velocity itself.

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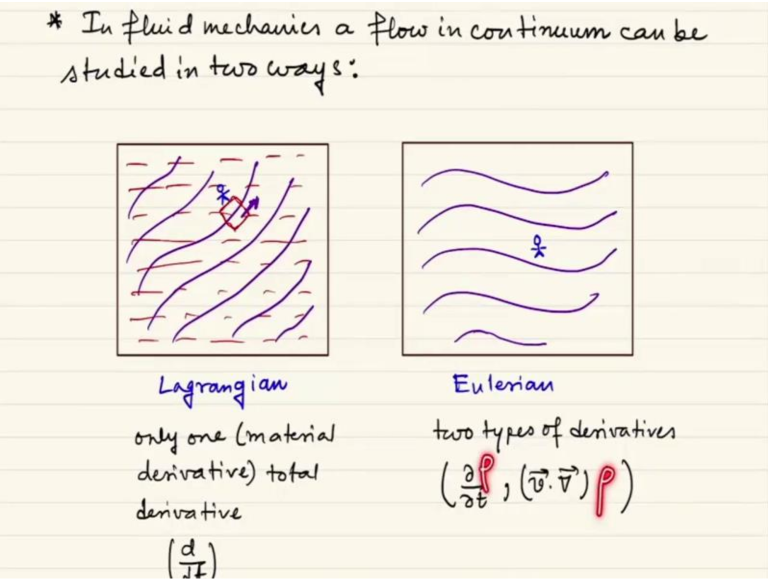

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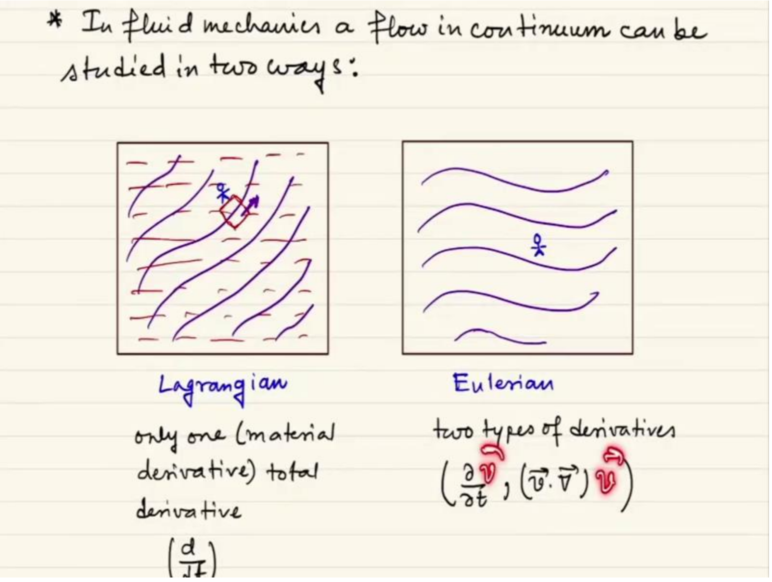

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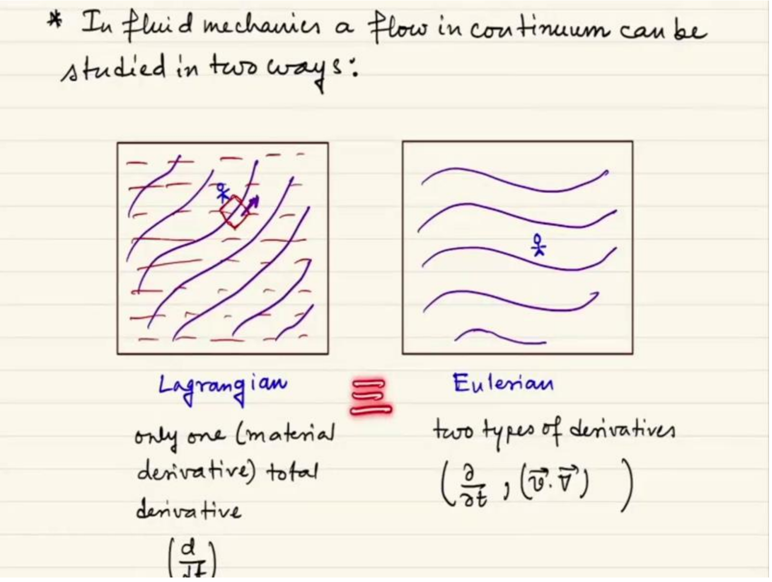

So, these two approaches are equivalent and that is why basically Eulerian is a combination of these two parts of Eulerian approach, that means, in one case you are not changing space, but changing time and in one case, study is being done at different point in space, but at a given time of course, and more than one observer are needed ok. The combined result will finally give us something.

So, basically, I mean some particle which is coming here and at a given time instant, it will have some. So, let us say I am interested in this blob.

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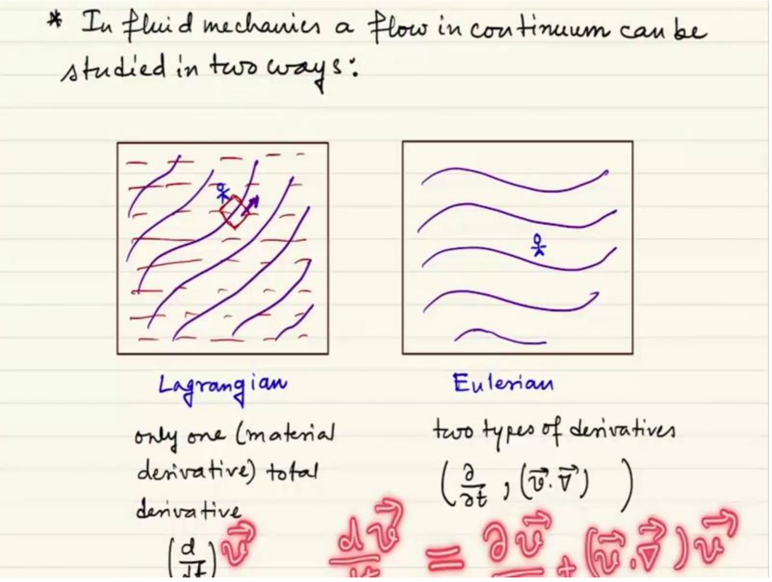

And in this blob, now I am just tracing its $\frac{dv}{dt}$. Now, these $\frac{dv}{dt}$ can be obtained just by the combination of $\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v}$.

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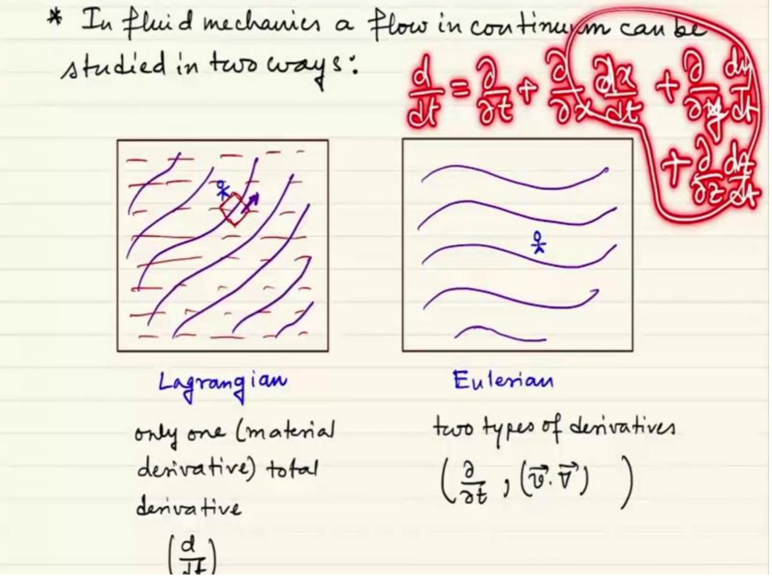

That is simply because $\frac{d}{dt}$ is nothing but $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt}$. And this part is nothing but the advective derivative part ok.

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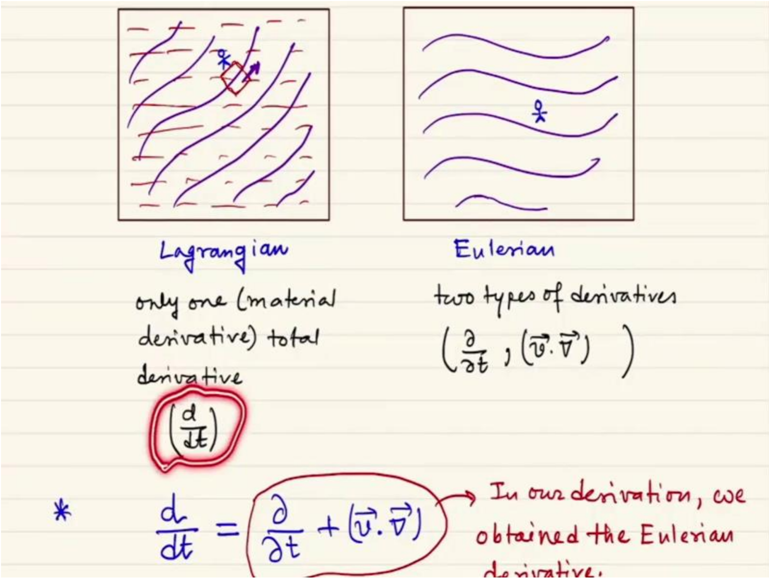

So, finally, due to the equivalence you can see that this $\frac{d}{dt}$ is equivalent equal to $\frac{\partial}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)$. In our derivation, we obtained what we did I mean starting from the kinetic theory, we obtain the Eulerian derivatives of course. Now, these derivatives sometimes are called the material derivative or the total derivative.

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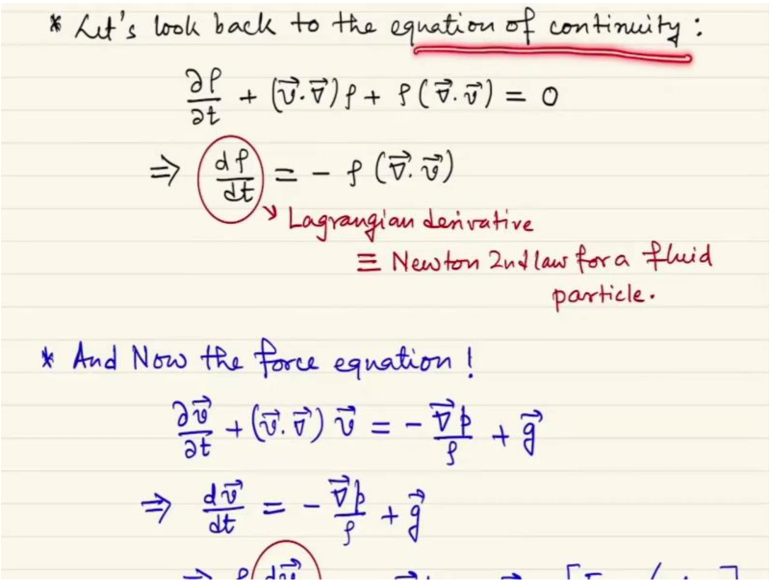

So, once again, this is sometimes important when we are talking in terms of fluid particles. Now, let us look back to the equations of continuity and we will see, these two approaches were important to be introduced just simply because the approach is to study fluid motions or a flow macroscopically.

So, if the equation of continuity you can see $\frac{\partial \rho}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\rho + \rho(\vec{\nabla}.\vec{v})$ is equal to 0, then this $\frac{\partial \rho}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\rho$ is nothing but $\frac{d\rho}{dt}$ by definition this is a Lagrangian derivative, and that is equal to $-\rho(\vec{\nabla}.\vec{v})$.

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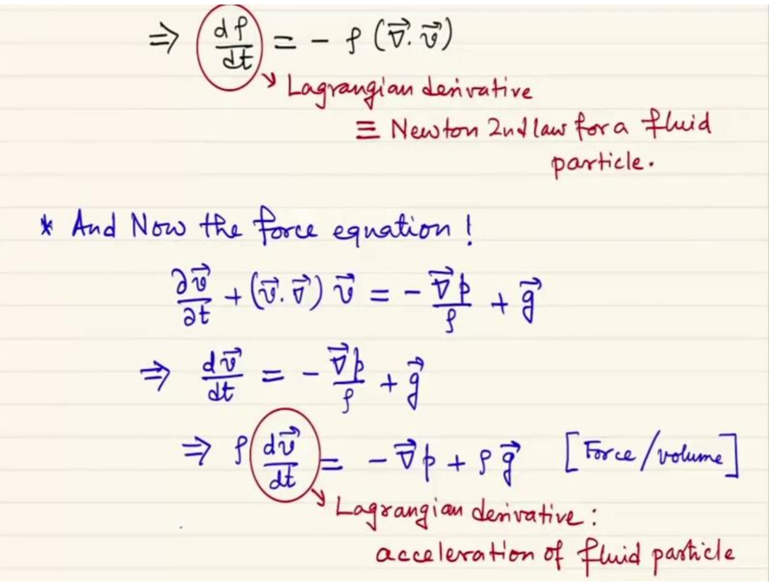

So, that was just to tell you how the Lagrange's derivative will look like. So, this was the Eulerian time derivative. So, this is the Lagrangian time derivative. Now, just have a look at the force equations. The force equation is written like this $\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v}$ is equal to $-\frac{\vec{\nabla}p}{\rho} + \vec{g}$.

So, what is this? This is nothing but $\frac{d\vec{v}}{dt}$. So, this is the acceleration of a fluid particle and finally, you see that the total force or total force per unit mass which is acting on that fluid particle is given by the sum of these two terms. One comes from the inside or once from the internal of the fluid that is the gradient of $-\frac{\vec{\nabla}p}{\rho}$, another comes from the external part.

So, finally, we can say that we have something like a Newton's equation, but of course, in density sense, because we are talking in terms of fluid flow. So, $\rho\frac{d\vec{v}}{dt}$ will be equal to $-\vec{\nabla}p + \rho\vec{g}$. So, this is a force equation if you want. Now, we want to check. So, this one we already got, but we have not derived it till now macroscopically.

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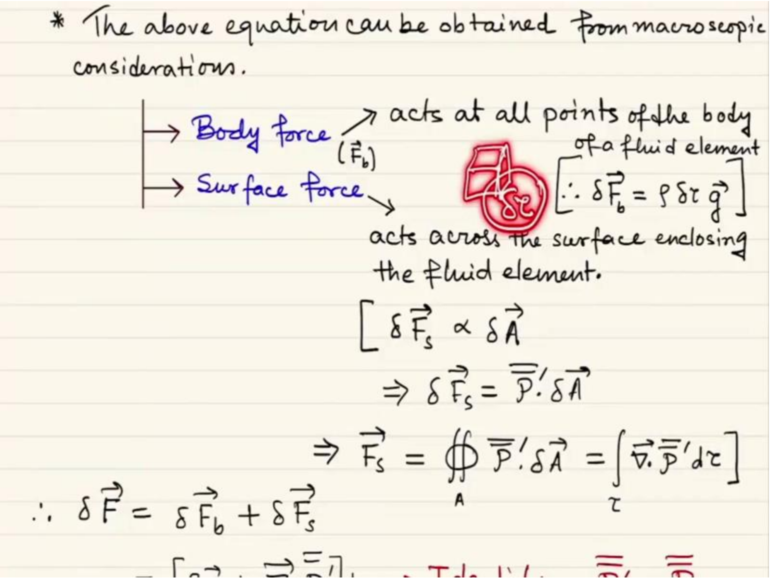

Now, to obtain these equations from macroscopic considerations, we have to define two type of forces; one is called body force, another is surface force. Whenever we are talking about this body force and surface force, basically we have to introduce the concept of this fluid particle.

Whenever we are saying that the force is acting on something then that something is given by the fluid blob or fluid particle and then this Lagrangian derivative comes into play. That is why I introduced both the Lagrangian and the Eulerian point of view. Eulerian is the one which we are using throughout.

Now, what is body force? So, body force is something, which acts at all points of the body of a fluid element. You see we still use the motion of the fluid element and so, this is known as the body force or the bulk force sometimes people say, so this is given by $\vec{F}_b$. So, if we are just taking fluid particle's mass then the mass will be $\rho\delta\tau$, where $\delta\tau$ is the volume.

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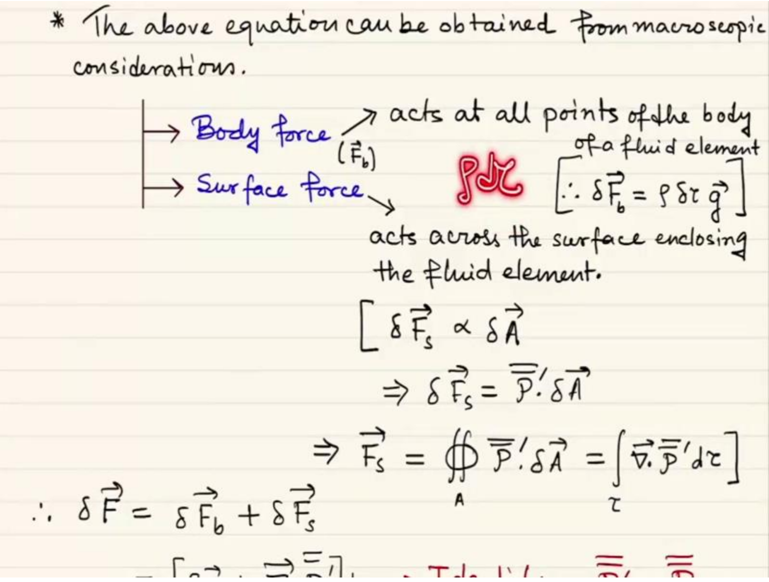

Then the total force will be given by $\rho\delta\tau\ g$ because g is nothing but the force per unit mass or the body force per unit mass. Then what is the surface force? So, this body force acts at all points of the body of a fluid element.

So, body force will be proportional to the volume. And surface force is something this acts across the surface enclosing the fluid element. So, for body force, this acts at every point of the volume of the fluid element, and surface force acts through the across the surface enclosing the fluid element.

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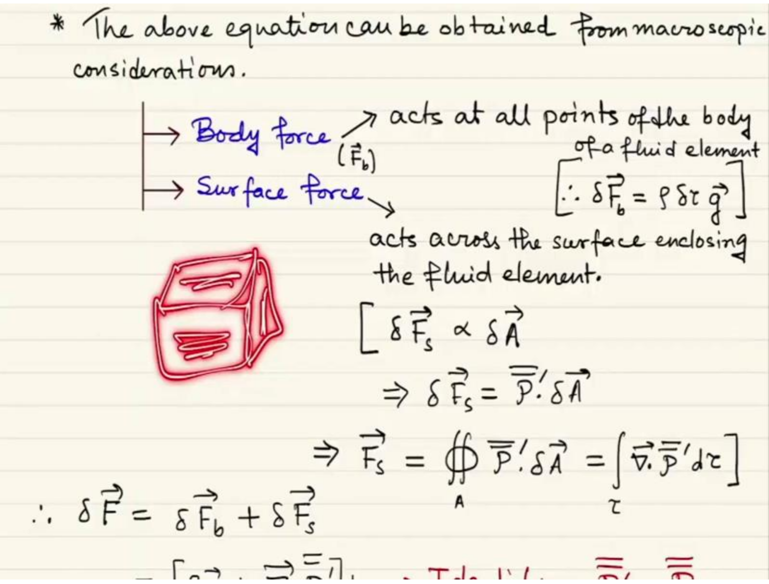

So, some small incremental $\delta\vec{F_s}$, surface force will be expected to be proportional to the incremental area, some small area $\delta\vec{A}$.

And if this is proportional to $\delta\vec{A}$ then one can simply write that when two vectors are proportional, of course, then the proportionality constant can be a scalar or there is another possibility that the both of them are joined by a tensor of rank 2 which is given by a matrix right.

And that is the most general possibility and that is what we are writing over here. So, $\delta\vec{F_s}$ will then be equal to some tensor of rank 2 and area, $\bar{\bar{\mathcal{P}}}'.\delta\vec{A}$. And then the total $\vec{F_s}$ will be some incremental or some infinite decimal surface force.

Now, the total surface force will then be a surface integral over the whole surface. And if you now remember the famous Gauss divergence theorem, then the closed surface integral $\bar{\bar{\mathcal{P}}}'.d\vec{A}$ is nothing but volume integral $\vec{\nabla}.\bar{\bar{\mathcal{P}}}'\,d\tau$.

Now, you see if you write this incremental total force is equal to incremental body force plus incremental surface force, then that will be simply written by $[\rho\vec{g} + \vec{\nabla}.\bar{\bar{\mathcal{P}}}'\,]d\tau$.

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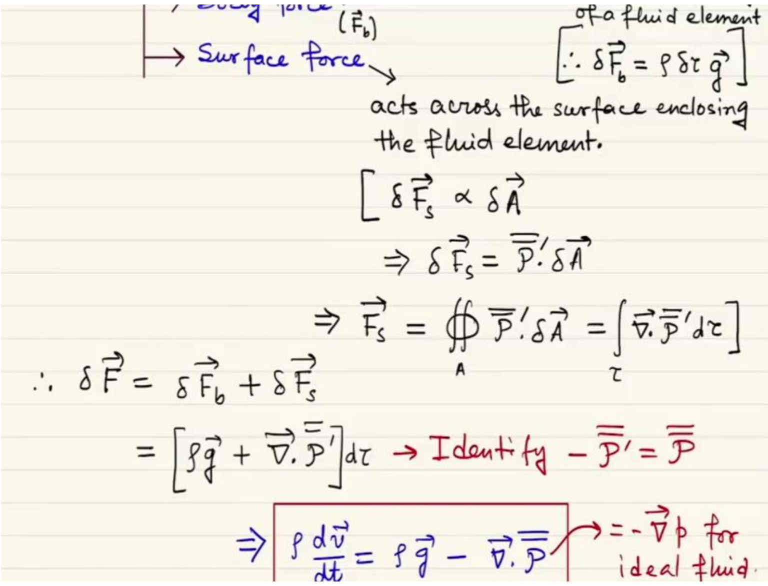

And if you simply identify your pressure term, this $\bar{\bar{\mathcal{P}}}'$ term then this new tensor to be minus of the pressure what we just introduced in our original definition, original derivation. Then we simply get back our ideal fluid equation. We know for ideal fluid this $-\vec{\nabla}.\bar{\bar{\mathcal{P}}}'$ is even simpler for Maxwellian case for $-\vec{\nabla}p$.

So, we again get back our ideal fluid equation starting from macroscopic considerations. Once again whenever we are talking macroscopic considerations, we start from fluid elements, and that is the reason why I introduced this Lagrangian point of view. Now, fluid element is the smallest entity of the macroscopic framework.

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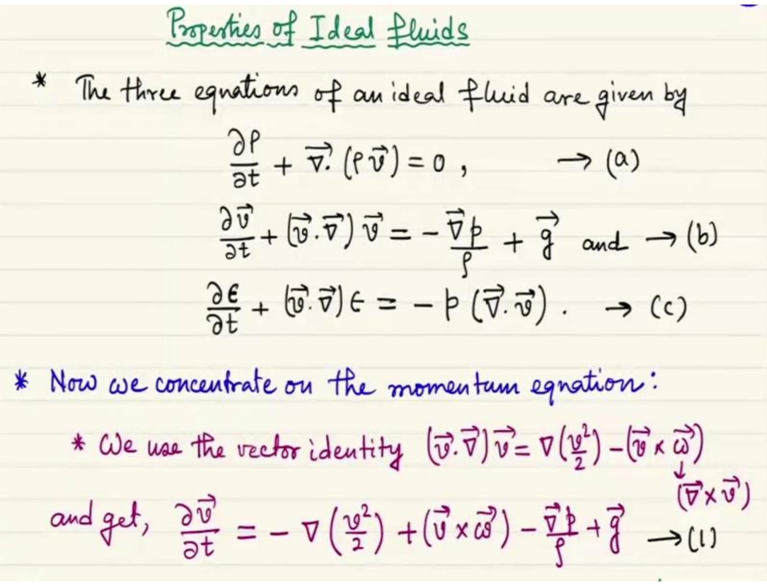

So, in the next part, I will discuss the different properties of ideal fluid.

Thank you very much.