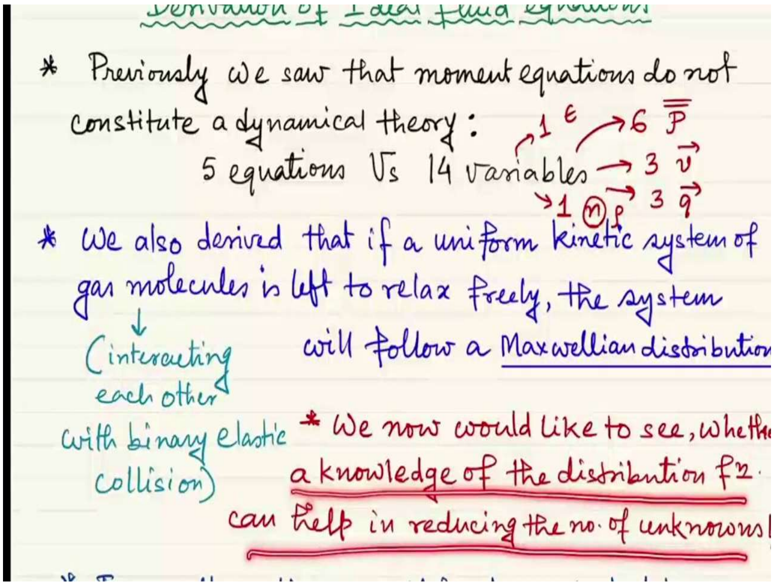
Introduction to Astrophysical Fluids
Prof. Supratik Banerjee
Department of Physics
Indian Institute of Technology, Kanpur

Lecture – 11
Derivation of the ideal fluid equations

Hello and welcome to another lecture of Introduction to Astrophysical Fluids. In this lecture, we discuss the Derivation of ideal fluid equations.

(Refer Slide Time: 00:23)



Previously we saw that, the moment equations are not sufficient to constitute automatically a dynamical theory. The reason was very simple that we had 5 moment equations, one was the continuity equation, the other four were just the three components of the equation of the evolution for the linear momentum and the last one being the equation for the evolution of internal energy ϵ.

Now, we had 14 unknown variables. So, 5 equations and 14 variables. So, in case you cannot remember, we recapitulate that of the 14 variables, we had 6 from the symmetric tensor of pressure, 3 variables from the bulk velocity v, 3 from the heat flux vector q, 1 for the density n and 1 for the internal energy density, that is ϵ.

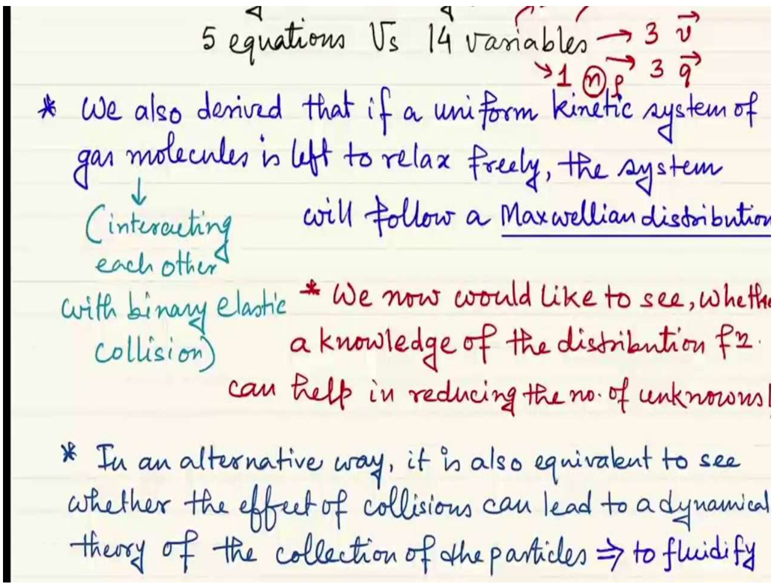
I mean n or ρ, they are just the same. When n is multiplied by the mass of one single particle that will give you the massive density. So, we had total 14 variables, 5 equations. So, this does not make sense because we know that, if we have n number of state variables, we need n number of evolution equation for each of the state variables in order to get a dynamical theory.

That is just a recapitulation from our lectures of the first week. We separately derived that, if a uniform kinetic system of gas molecules, who are interacting just by the virtue of binary elastic collisions then, this type of gas if they are left to relax for a long time, for example, then this is such a system will attain an equilibrium distribution and we showed that this equilibrium distribution is nothing but a Maxwellian distribution.

Now, in this lecture, we will try to see that whether a knowledge of this distribution function can help in reducing the number of unknowns. That is where exactly we ended in the last lecture, where we said that, in general one way to reduce the number of unknown. So, basically how can we expect that we can make a dynamical theory for a continuum, we have to somehow reduce the number of variables.

Now, one way of reducing the number of variables is to find interrelations between them. And then another way to show that some variables are identically vanishing. Then also in a way we can eliminate those variables and that is again reducing the number of total number of variables.

(Refer Slide Time: 03:47)



So, in this lecture we will see that, if we have a distribution which is Maxwellian in nature then how this knowledge can reduce the total number of unknowns. In an alternative way, this is equivalent to see whether the effect of collision can lead to a dynamical theory of the collection of the particles.

So, this is something interesting to understand that the Maxwellian distribution basically becomes the most probable or the equilibrium distribution, when the system is collisional, and not any arbitrary collision but just the binary elastic collision. If the knowledge of the distribution function being Maxwellian in nature, can reduce the number of variables, that means, can find some interrelations between the variables.

That is somehow equivalent to say that the collisions are the reason for the Maxwellian distribution at the fundamental level, actually helps the system attaining a dynamical theory. So, in this way, we will also check that whether the collisions and actually binary elastic collisions can help to fluidify the system.

Now, fluidify means to make a dynamical theory for the continuum. And if you remember that, our primary objective from the beginning was to develop a dynamical theory at different levels and finally, to get a dynamical theory for the continuum of fluids.

(Refer Slide Time: 06:05)

* We recall the Maxwellian distribution as

$$f(\vec{u}) = n\left(\frac{m}{2\pi K_B T}\right)^{3/2} \exp.\left[-\frac{m(\vec{u}-\vec{v})^2}{2K_B T}\right]$$

where, $n, T, \vec{v}$ are constants. This is the distribution function of a uniform gas at equilibrium (globally).
(No flow

* Let us assume that our system is such that the particles are following local Maxwellian distribution

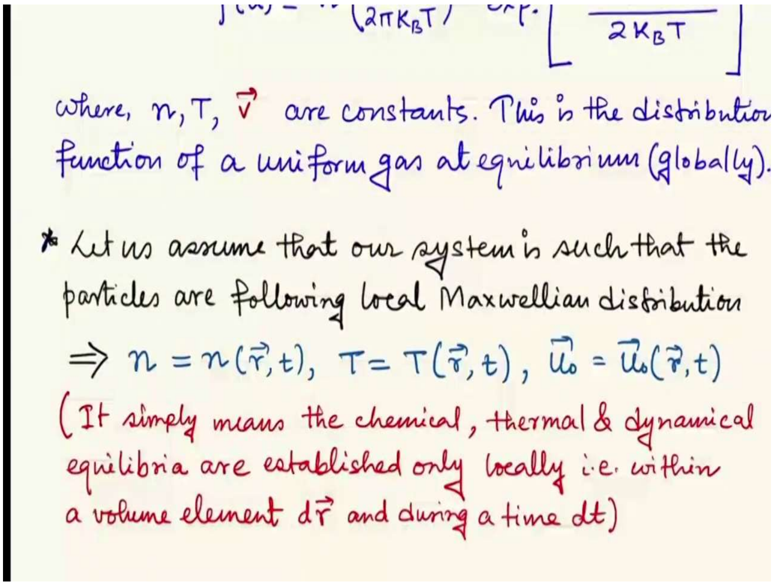
$$\Rightarrow n = n(\vec{r},t), \quad T = T(\vec{r},t), \quad \vec{u}_0 = \vec{u}_0(\vec{r},t)$$

(It simply means the chemical, thermal & dynamical

So, now we just recall the Maxwellian distribution. Now, the Maxwellian distribution is nothing but this f which is a function of the velocity. This is the equilibrium distribution, there is no dependence. So, if it is an equilibrium distribution for a uniform gas, and the system is led to relax itself then we can easily expect that there will be no space and time dependence explicitly.

So, the only explicit dependence will be on velocity and it should look like simply $n\,(\frac{m}{2\pi k_B T})^{3/2} exp\,(-m\,\frac{(\vec{u}-\vec{v})^2}{2k_B T})$. So, here we assumed that this n, T and v all are constants. Now, this is the distribution function of a uniform gas at equilibrium.

(Refer Slide Time: 07:31)



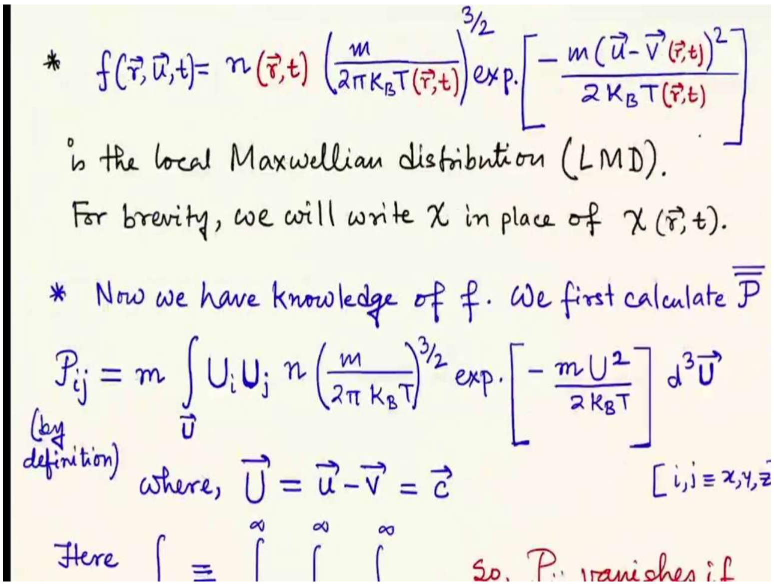
Let us assume that our system is such that, this is not exactly the above equilibrium, but the particles are following local Maxwellian distributions. So, what is the meaning of that? That means that the constants n, T and v, they are no longer absolute constants but they are the functions of r and t, space and time.

Now, what is the physical meaning is of that? That simply says that, the chemical, thermal and dynamical equilibria. So, n is corresponding to the chemical equilibria, T is corresponding to the thermal equilibria and u_0 is corresponding to the dynamical equilibria or the mechanical equilibria.

So, these three equilibria, they are established only locally, that means, within a volume element dr and during a time dt, in the neighborhood of some function of some space point r and time instant t. So, the basic supposition is that in the neighborhood of every r and t, there should be some local Maxwellian distribution.

That means, the velocity distribution is a Maxwellian but these constants are now no longer constants. So, they are basically constant for that specific point but globally they are just the functions of space and time. If we do that, then we can have a local Maxwellian distribution which should look like $f(\vec{r},\vec{v},t) = n(\vec{r},t)\,(\frac{m}{2\pi k_B T(\vec{r},t)})^{3/2} exp\,(-m\,\frac{(\vec{u}-\vec{v}(\vec{r},t))^2}{2k_B T(\vec{r},t)})$. So, you can see there is nothing new, other than just writing these previous constants n, T and v to be explicit functions of r and t.

(Refer Slide Time: 09:25)



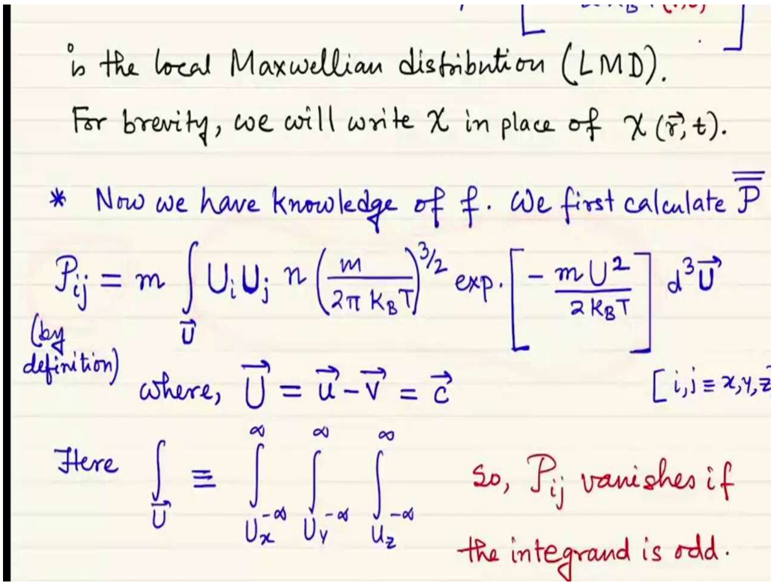
Now, for brevity now onwards, we will simply write χ for any arbitrary variable χ, which is a function of r and t. So, when I write n, I mean this is a function of r and t because we are talking in the framework of local Maxwellian distribution from now onwards.

So, now you see we have the knowledge of f. We know what f is. So, now, we will see how it can help in reducing the number of the unknowns. So, we first calculate $\overline{\overline{\mathcal{P}}}$. So, $\mathcal{P}_{ij}$ is ij^{th} component is given by $m\,\int U_i U_j\, n\,(\frac{m}{2\pi k_B T})^{3/2} ex\,p\left(-m\,\frac{(U)^2}{2k_B T}\right) d^3\vec{U}$. So, this U is nothing but the fluctuation velocity.

So, that is easy to understand, if you just go back to the original definition of $\overline{\overline{\mathcal{P}}}$, you will see that for a Maxwellian distribution, this is nothing but m times this fluctuation $U_i U_j$. So, $\overline{\overline{\mathcal{P}}}$ is the tensorial product of the fluctuation velocities times the distribution function and then d^3U.

So, when this integration, integrated over the velocity space, it gives you the pressure tensor. So, the ij^{th} component will simply be $m\int U_iU_j\, n\,(\frac{m}{2\pi k_BT})^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)d^3\vec{U}$.

(Refer Slide Time: 12:05)



Now, you see that, when I just write this type of single integration with a $\vec{U}$, it basically denotes three integrations over three components of U. And all the integrations will vary from -∞ to +∞. So, from this type of expression, you will simply understand that $\mathcal{P}_{ij}$ vanishes, if the integrand is odd.

And how is that possible to understand? Just a minute, you see that, $n\,(\frac{m}{2\pi k_BT})^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)$ part is an even function of U. Now, this part U_iU_j you have two different components. So, of course, these things we even do not know how should they be but if this part U_iU_j will have an odd part, then the total thing will be an odd function. And as the whole integration will vanish within -∞ to +∞. Then anything which is odd inside will make the whole thing vanish, that is very easy to understand.

(Refer Slide Time: 13:21)

* Since the nature of the integrand (i.e. even or odd) depends on U_iU_j, evidently $\mathcal{P}_{ij}$ vanishes when $i \neq j$, e.g. U_xU_y, U_xU_z, U_yU_z

[Check! $mn\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_xU_y \left(\frac{m}{2\pi K_BT}\right)^{3/2} exp.\left[-\frac{m\overbrace{U^2}^{U_x^2+U_y^2+U_z^2}}{2K_BT}\right] \underbrace{d^3\vec{U}}_{dU_xdU_ydU_z} = 0$

* The integration does not vanish if $i=j$. It becomes,

$$\mathcal{P}_{ii} = mn\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{\vec{U}} U_i^2 \exp\left[-\frac{mU^2}{2k_BT}\right] d^3\vec{U}$$

Let us take, $i=x$, we then have,

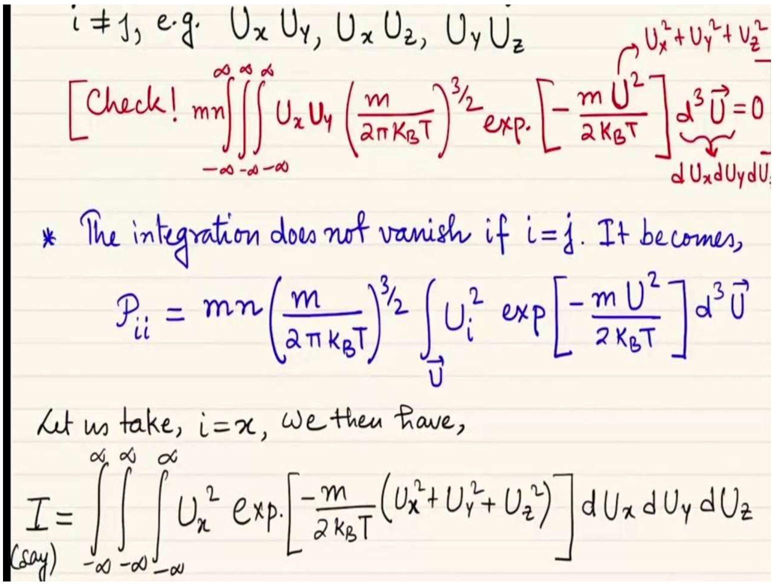
Now, since the nature of the integrand depends on U_iU_j what I just said 5 seconds ago, evidently $\mathcal{P}_{ij}$ vanishes when i is not equal to j that means when we have this type of combinations U_xU_y or U_yU_z or U_xU_z. Now, please check. So, basically due to the lack of time, it is not always possible to do every single step of the lengthy calculations.

But it is highly recommended or highly suggested that, you check all this calculation at home that is the only way to learn. So, please check that. Let us say you just take with this combination U_xU_y, then basically you are calculating nothing but $\mathcal{P}_{xy}$, the xy component of the pressure tensor and this will be given by $m\int U_xU_y\, n\, (\frac{m}{2\pi k_BT})^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right) d^3\vec{U}$.

And you see that in this integration, basically x is not equal to y. So, this is a case, where i is not equal to j. So, you can easily check this is 0. I leave it on you as an exercise.

So, basically you have to remember that, U squared is nothing but $U_x^2 + U_y^2 + U_z^2$. And d^3U is nothing but $dU_xdU_ydU_z$. Then you just use your knowledge of integration and you should find 0. If you do not find 0, that is a problem. Then please let me know. Now, the integration of course, does not vanish in case i is equal to j.

(Refer Slide Time: 15:31)



Why? Because we can show that if i is equal to j then what happens, then it will give you a square of something. So, for example, if x is equal to y, then it will be U_x^2 for example. Now, I showed how it should and what will be the value. If it is non zero, then what should be the value of it.

So, the integration will then be simply $\mathcal{P}_{ii}$ and it will be given by $mn \, (\frac{m}{2\pi k_B T})^{3/2} \int U_i^2 \; exp\left(-m \, \frac{(U)^2}{2k_B T}\right) d^3\vec{U}$.

Now, we again take a particular example, i is equal to x then the integral is nothing but $I = \iiint_{-\infty}^{+\infty} U_x^2 \; exp\left(-m \, \frac{U_x^2 + U_y^2 + U_z^2}{2k_B T}\right) dU_x dU_y dU_z$. And if you have this, now you can simply see that this U_x^2 square will go to this part $exp\left(-m \, \frac{U_x^2}{2k_B T}\right)$, the other two parts $exp\left(-m \, \frac{U_y^2}{2k_B T}\right)$ *and* $exp\left(-m \, \frac{U_z^2}{2k_B T}\right)$ will simply be just integrated within -∞ to +∞.

(Refer Slide Time: 16:45)

Then, $$I = \int_{-\infty} U_x^2 e^{-\frac{mU_x^2}{2K_BT}} dU_x \left[\int_{-\infty} e^{-\frac{mU_y^2}{2K_BT}} dU_y\right]$$

$$= 2\int_0^\infty U_x^2 e^{-\frac{mU_x^2}{2K_BT}} dU_x \left(\sqrt{\frac{2\pi K_BT}{m}}\right)^2 \qquad \int_{-\infty}^{\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$$

$$= \left(\frac{\cancel{4}\pi K_BT}{m}\right)\frac{1}{\cancel{4}}\sqrt{\frac{8\pi k_B^3 T^3}{m^3}}$$

$$\therefore P_{xx} = \cancel{m} n \left(\frac{m}{2\pi K_BT}\right)^{3/2}\left(\frac{\pi K_BT}{\cancel{m}}\right) 2^{3/2}\pi^{1/2}\left(\frac{K_BT}{m}\right)^{3/2}$$

$$= nK_BT$$

So, we will have at the end this I is equal to $\int_{-\infty}^{+\infty} U_x^2\, exp\left(-m\,\frac{U_x^2}{2k_BT}\right)\, dU_x\,\sqrt{\frac{2\pi k_BT}{m}}$. And then for y and z, we will have the same type of integration. So, we will just write one integration of this type and we will square that.

So, that is the square $\int_{-\infty}^{+\infty} U_x^2\, exp\left(-m\,\frac{U_x^2}{2k_BT}\right)\, dU_x\, \left(\sqrt{\frac{2\pi k_BT}{m}}\right)^2$. So, this is exactly what we did and you know that, this type of integration has a very simple formula. So, in integration over from -∞ to +∞, if it is simply $e^{-\beta x^2}dx$, then it should be simply $\sqrt{\frac{\pi}{\beta}}$. And that is exactly what we did here.

What about this integral $\int_{-\infty}^{+\infty} U_x^2\, exp\left(-m\,\frac{U_x^2}{2k_BT}\right)\, dU_x$, that you have to do? So, right from the first assignment, I mean tried to train you in doing this type of integrations. And in the reference books which I told you in the course handout, you can see this type of integrations are done there.

So, if you do it like with proper care and finally, just again replacing the integration value in the $\mathcal{P}_{xx}$. And this $\mathcal{P}_{xx}$, if you just remember this was nothing but this factor $mn(\frac{m}{2\pi k_BT})^{3/2}$ times the integration $\int_{-\infty}^{+\infty} U_x^2\, exp\left(-m\,\frac{U_x^2}{2k_BT}\right)\, dU_x$.

So, $\mathcal{P}_{xx}$, which is now calculated. And if you, do it carefully, you will see that this pressure is nothing but nk_BT. So, this is something you already know may be from your previous knowledge of kinetic theory.

(Refer Slide Time: 19:35)

$$= 2\int_0 U_x^2 e^{-\frac{mU_x^2}{2K_BT}} dU_x \left(\sqrt{\frac{2\pi K_BT}{m}}\right)^2 \qquad \int_{-\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$$

$$= \left(\frac{\cancel{4}\pi K_BT}{m}\right) \frac{1}{\cancel{4}} \sqrt{\frac{8\pi k_B^3 T^3}{m^3}}$$

$$\therefore \mathcal{P}_{xx} = \cancel{m} n \left(\frac{m}{2\pi K_BT}\right)^{3/2} \left(\frac{\pi K_BT}{\cancel{m}}\right) 2^{3/2}\pi^{1/2} \left(\frac{K_BT}{m}\right)^{3/2}$$

$$= nK_BT$$

and also, one can show that $\mathcal{P}_{yy} = \mathcal{P}_{zz} = nK_BT$

$$\therefore \mathcal{P}_{ij} = p\,\delta_{ij} \quad \text{where, } p = nK_BT \qquad \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$$

If you do not know, you learn here. So, in the same way, if you calculate $\mathcal{P}_{yy}$, and $\mathcal{P}_{zz}$ you can actually see. Even without calculation if you think of it, you will see that will also give you the same thing and this will be again nk_BT. So, the conclusion is for our case, the pressure tensor is such that, it is 0 when i is not equal to j and it is equal to p, which is equal to nk_BT, n is the density, T is the temperature, k_B is the Boltzmann constant.

So, basically if you want to write the pressure tensor roughly, it should look like $\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$. All the off-diagonal elements should be 0 and all the diagonal elements should be p, where p is equal to nk_BT. So, a compact way of writing such type of tensor is nothing but this $\mathcal{P}_{ij}$, is equal to $p\ \delta_{ij}$, where δ is the chronicle delta.

(Refer Slide Time: 20:51)

* Now we have to calculate $\vec{q}$. We know,

$$\vec{q} = \frac{1}{2} m \int_{-\infty}^{\infty} U^2 \vec{U} f \, d^3\vec{U}$$

for a Maxwellian distribution,

$$\vec{q} = \frac{1}{2} m \int_{-\infty}^{\infty} U^2 \vec{U} \, n \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-\frac{mU^2}{2K_BT}} \, d^3\vec{U}$$

Let us just take the x component, for example.

$$q_x = \frac{1}{2} m \int_{-\infty}^{\infty} U^2 U_x \, n \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-\frac{mU^2}{2K_BT}} \, d^3\vec{U}$$

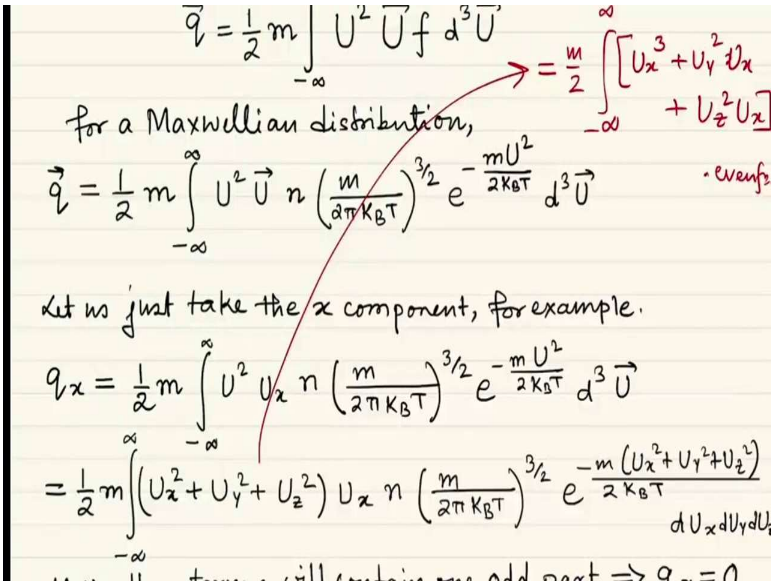
$$= \frac{1}{2} m \int_{-\infty}^{\infty} (U_x^2 + U_y^2 + U_z^2) U_x \, n \left(\frac{m}{\quad}\right)^{3/2} e^{-\frac{m(U_x^2+U_y^2+U_z^2)}{2K_BT}}$$

So, for p basically what we see that, we had 6 unknown variables. Now we have only 1 unknown variable to handle with. Now, we have to calculate the heat flux vector $\vec{q}$. Again, we go back to the definition. So, the definition is q is equal to $\frac{m}{2}\int_{-\infty}^{+\infty} U^2 \vec{U} f d^3\vec{U}$.

For a Maxwellian distribution, it should look like simply $\frac{m}{2}\int_{-\infty}^{+\infty} U^2 \vec{U} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)\, d^3\vec{U}$. Now, you see this part $n\left(\frac{m}{2\pi k_B T}\right)^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)$ is again you can see is an even function of U, for all components.

What about this part $U^2\vec{U}$? This U^2 is an even function but from $\vec{U}$ you will always have an odd function. So, how to check that? You again take one component. So, let us choose q_x then what it should be, I mean how it should look like, it will simply be $\frac{m}{2}\int_{-\infty}^{+\infty} U^2 U_x n \left(\frac{m}{2\pi k_B T}\right)^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)\, d^3\vec{U}$.

(Refer Slide Time: 22:27)



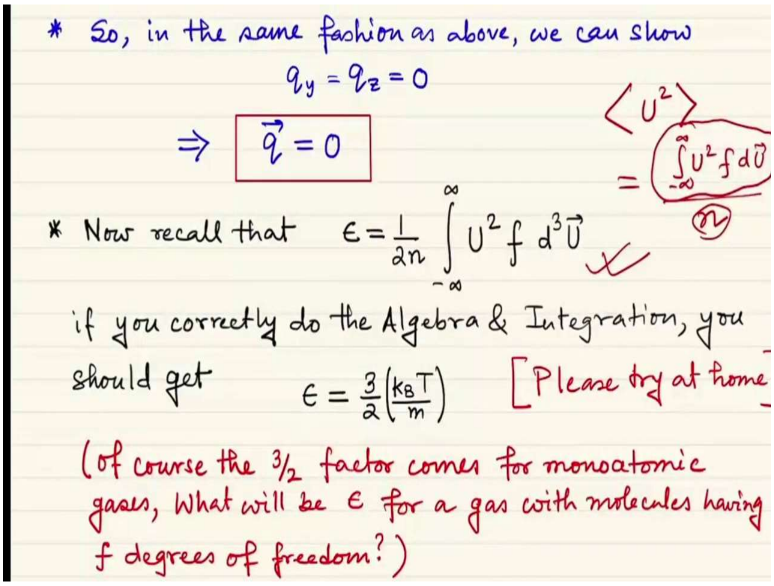
And U^2U, you can write as $U_x^2 + U_y^2 + U_z^2$ times U_x. So, now we will have the products of three integrations. So, for U_y and U_z, you will have nonzero things because that will be globally an even function. But for U_x, you will have U_x^3, which is an odd function and from $exp\left(-m\,\frac{U_x^2+U_y^2+U_z^2}{2k_BT}\right)$ you will have U_x^2 all as a function of U_x^2. So, it is an even function.

So, all the three terms will contain one odd part. So, basically you will have three integrations. So, if you just want to check it. So, that will be $\frac{m}{2}\int_{-\infty}^{+\infty}(U_x^3 + U_y^2U_x + U_z^2U_x)\, n\, (\frac{m}{2\pi k_BT})^{3/2} exp\left(-m\,\frac{(U)^2}{2k_BT}\right)\, d^3\vec{U}$. So, $exp\left(-m\,\frac{(U)^2}{2k_BT}\right)$ is the even function.

So, U_x^3 is an odd function. So, when I just take the first one, you will see that this U_x^3gives me the odd function and for the U_y and U_z, I do not have any odd thing. But finally, I have one odd part and that will give me 0, for this integration, so that will be a sum of the three integrations. For this integration again $U_y^2U_x$ gives me an odd part and this will give me 0.

Again, for the third one $U_z^2U_x$ give me an odd contribution. So, which makes again the integration to be 0. Finally, the thing is that q_x is 0. For the same reason q_y and q_z can also be shown to be 0.

(Refer Slide Time: 24:49)



So, it is there is a big relief that we have somehow eliminated a total vector. So, already for p, we have reduced 6 variables to 1, and for q, we have got rid of 3 components, because it is identically vanishing for Maxwellian distribution. Now, try to understand the last one, which is epsilon ϵ.

By definition this is $\frac{1}{2n}\int_{-\infty}^{+\infty} U^2 f d^3\vec{U}$. Here I am again suggesting that when I am writing this type of equation, please check that to the equation in the previous lectures and try to match them, whether they are consistent with each other or not.

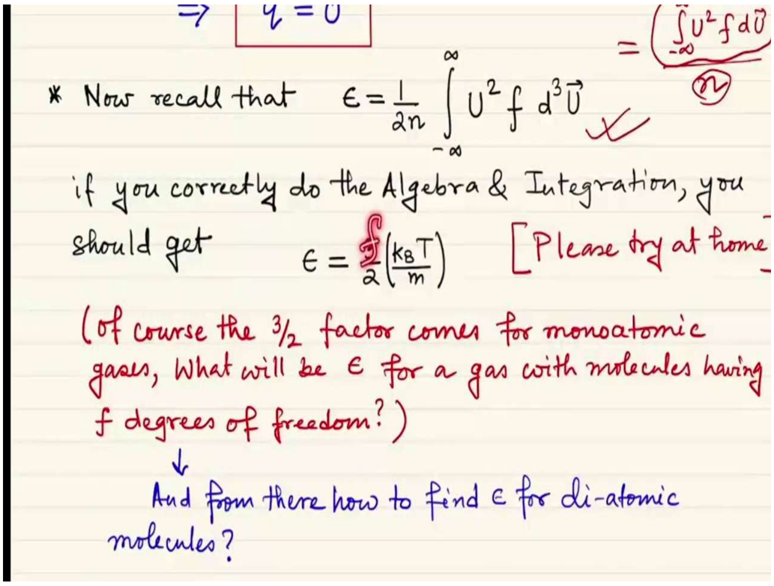
Let us say I am writing $\langle U^2 \rangle$ and this is nothing but $\frac{1}{n}\int_{-\infty}^{+\infty} U^2 f \, d\vec{U}$. So, if I just want to, for example, make a reference for this integration, it will simply be $n\langle U^2 \rangle$.

So, just keeping this in mind, try to check whether this is consistent with its primary definition or not. So, if you correctly do the algebra and integration, then you should get ϵ is equal to $\frac{3}{2}(\frac{k_B T}{m})$. And you see, that is something we did not expected. The $k_B T$ has a unit of energy. So, $\frac{3}{2}(\frac{k_B T}{m})$ is unit of energy per unit mass. So, this is a unit of velocity squared.

And that is very easy to check from here as well because $\int f \ d^3\vec{U}$ has a dimension of density. So, has a unit of density. So, this by n density will go off and you will have a unit of U squared that is the energy per unit mass.

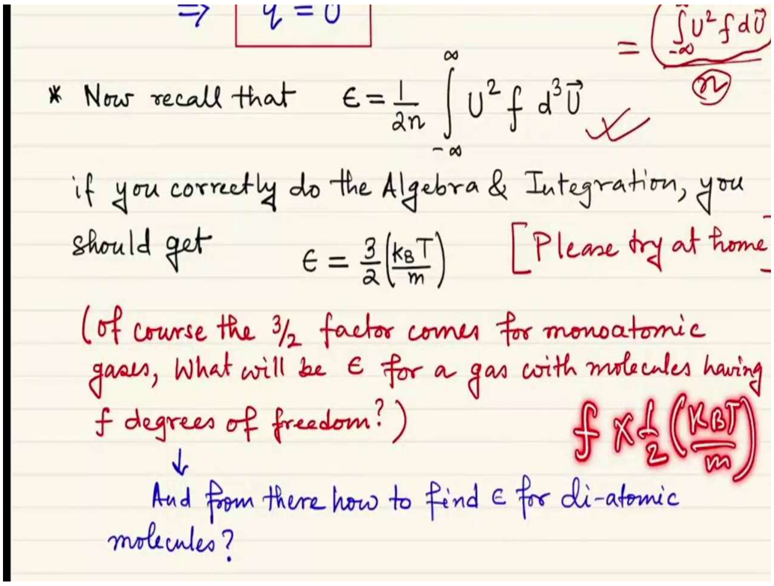
So, this is a fabulous exercise to do at home. So, try by taking f as Maxwell Boltzmann distribution and calculate integrations carefully and you will see that epsilon ϵ will be equal to $\frac{3}{2}(\frac{k_B T}{m})$.

(Refer Slide Time: 27:37)



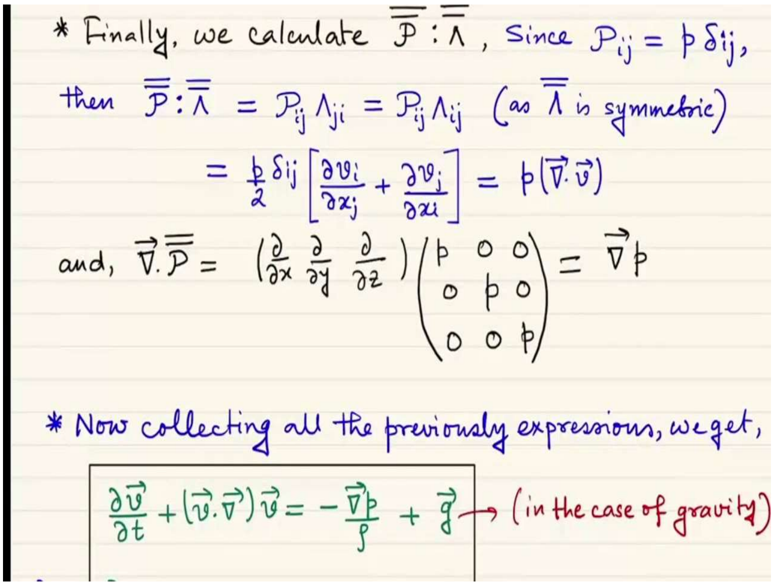
Of course, $\frac{3}{2}$ factor comes for mono atomic gases. Now, what will be ϵ for a gas with more molecules having f degrees of freedom? That will be simply $\frac{f}{2}$ because for mono atomic gases, they have just 3 degrees of freedom, so $\frac{3}{2}(\frac{k_B T}{m})$ that is the formula. When f degrees of freedoms are there, so $\frac{f}{2}(\frac{k_B T}{m})$ that will be the formula.

(Refer Slide Time: 28:01)



So, now I have a question for you, this is a small task, from there how to find this epsilon ϵ a for diatomic molecules? Then you have just to find the number of degrees of freedom, that is quite simple.

(Refer Slide Time: 28:27)



So, finally, you see that we have calculated also ϵ. So, what else? Finally, if you remember there was a term of double contraction in the energy equation and that was the double contraction of $\boldsymbol{\mathcal{P}}$ tensor to Λ tensor, where Λ is the symmetric part of the velocity gradient.

So, now in case our $\mathcal{P}_{ij}$ is equal to some small p times Kronecker delta δ_{ij}. For this simplified case of Maxwellian distribution, of course local Maxwellian distribution then this contraction is simply equal to $\mathcal{P}_{ij}\,\Lambda_{ij}$. And from $\mathcal{P}_{ij}\,\Lambda_{ji}$ finally, you can also write lambda $\mathcal{P}_{ij}$ Λ_{ij} because lambda is symmetric.

So, if I now expand Λ_{ij} and put the value of expression for $\mathcal{P}_{ij}$ for our current case, we will see this is nothing, but $\frac{p}{2}\delta_{ij}[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}]$, which is equal to p times $\vec{\nabla}.\vec{v}$ simply. Because δ_{ij} will survive only when i is equal to j.

So, simply you will have here $\frac{p}{2}\vec{\nabla}.\vec{v}$. And what is this $\vec{\nabla}.\bar{\bar{\mathcal{P}}}$? When $\boldsymbol{\mathcal{P}}$ is diagonal then it has the three identical values p. So, the $\vec{\nabla}.\bar{\bar{\mathcal{P}}}$ will simply be the $\vec{\nabla}p$. So, $\vec{\nabla}.\bar{\bar{\mathcal{P}}}$ will equal to, will be equal to $\vec{\nabla}p$, where p is the eigenvalues of the pressure tensor.

(Refer Slide Time: 30:41)

$$= \frac{p}{2}\delta_{ij}\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right] = p(\vec{\nabla}\cdot\vec{v})$$

and, $\vec{\nabla}.\bar{\bar{\mathcal{P}}} = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} = \vec{\nabla}p$

* Now collecting all the previously expressions, we get,

$$\boxed{\begin{aligned} &\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p_0}{\rho_0} + \vec{g} \\ \& \quad &\frac{\partial \epsilon}{\partial t} + (\vec{v}\cdot\vec{\nabla})\epsilon = -\frac{p}{\rho}(\vec{\nabla}\cdot\vec{v}) \end{aligned}} + \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

(in the case of gravity)

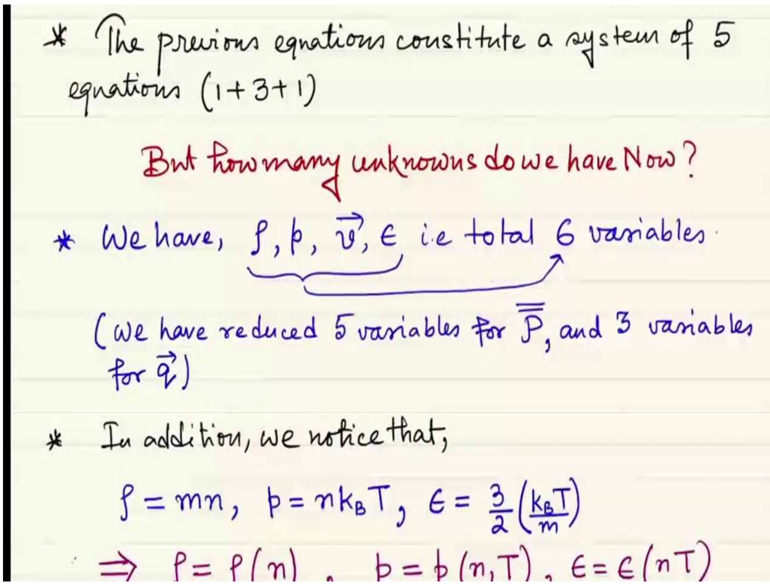
Now, collecting all the previous expressions, finally we can write the momentum equations to be $\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho}+g$, g is nothing but the acceleration due to the body force. I wrote g because most of the cases in astrophysics, we use the case or the framework, where the body forces the gravity. So, this is just the gravitational acceleration. you can write any other symbol.

And the internal energy evolution equation $\frac{\partial\epsilon}{\partial t}+(\vec{v}.\vec{\nabla})\epsilon = -\frac{p}{\rho}(\vec{\nabla}.\vec{v})$. Now, you see we should not forget this continuity equation $\frac{\partial\epsilon}{\partial t}+\vec{\nabla}.\rho\vec{v} = 0$, that is the third equation. So, now, you will see, we have 3 equations. $\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + g$ is a vectorial equation. So, this basically means 3 scalar equations, so we have 5 scalar equations.

So, we have 5 states, for example, v_x, v_y, v_z, ϵ, and ρ. And effectively how many variables or unknowns we have? We have v, we have p, we have ρ. So, g is given. So, this forcing is known, that we do not worry about.

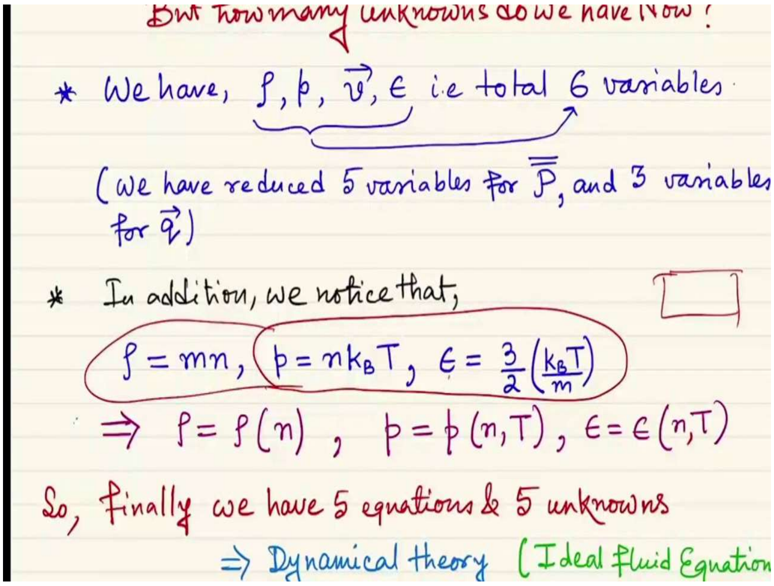
So, v has 3 unknowns and then p, ρ, ϵ. So, we have 6 unknowns. Now, we are almost there. So, we had 14 unknowns, 5 equations. We have 5 evolution equations, 6 unknowns, we are almost there. Can we be achieving the dynamical theory? The answer is well, let us see.

(Refer Slide Time: 33:05)



So, you see the previous equations. So, after this basically the previous equations constitute a system of 5 equations I just said and so this gives you 6 variables. But we really need 5 variables, how to do that?

(Refer Slide Time: 33:27)



That is exactly if we just write the expressions, explicit expressions of ρ, p and ϵ, we will see that these three basically are related by only 2 variables n and T. So, ρ, p and ϵ, all are a function of n and T.

So, you can actually just express p in terms of ϵ. Just I mean in terms of ϵ and ρ to be very precise, that is a small homework for you. So, just eliminate n and T to relate p, ρ, and ϵ in a small simple algebraic relation, ok do that.

But at least without calculating any further thing, it is now evident that we have 5 equations and 5 unknowns. So, we have a dynamical theory finally. We have 5 state variables and we have 5 evolution equations. And this is known as then the system of the ideal fluid equations. Why it is ideal fluid? So, till now, this is the first time we have developed a dynamical theory for a continuum.

So, we even do not know whether this is called an ideal fluid or not but the equations which we get from the previous knowledge of the fluid engineers or fluid mechanists, this is somehow very evident that, these are nothing but well-known ideal fluid equations.

Ideal fluid means, a fluid which does not have any type of transport properties. So, what are the transport properties that we will try later. So, for example, I am just trying to tell you something, which is easier to understand. For example, a fluid which does not have any type of transfer, I mean transfer of density from one point to the other, neither any form of viscous effect ok, in case you already are familiar to fluid dynamics.

And also, it does not have any conductive property, for example, thermal conductivity type of thing. So, if we have a fluid like that, which is not a practical fluid in general but it is called an ideal or perfect fluid. So, $\rho = \rho(n), p = nk_B T,\ \epsilon = \frac{3}{2}\frac{k_B T}{m}$ are then the equations of the perfect fluid.

So, the basically here we end this current part. So, in the next part, we will discuss some properties of the ideal fluid and we also try to understand that, how one can look the ideal fluid and its equations from a macroscopic point of view.

Thank you very much.