Physics of Turbulence
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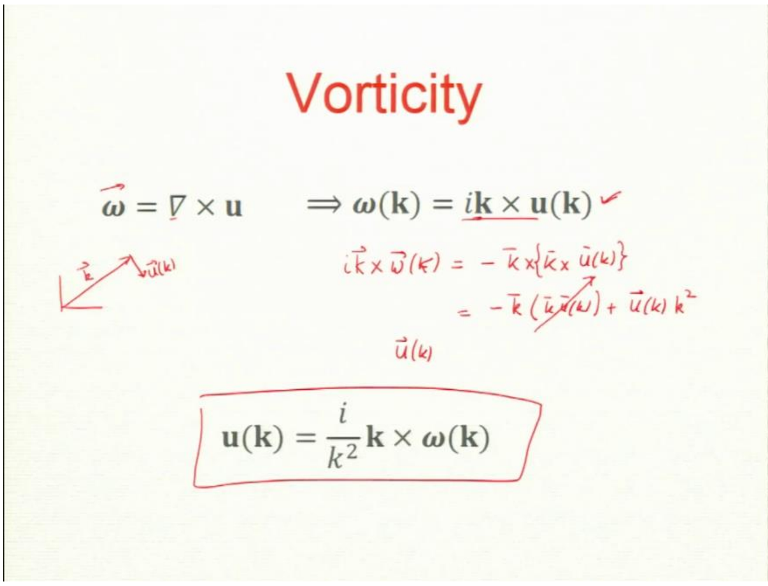
Lecture - 09
Fourier Space Representation
Vorticity, Kinetic Helicity, Enstrophy

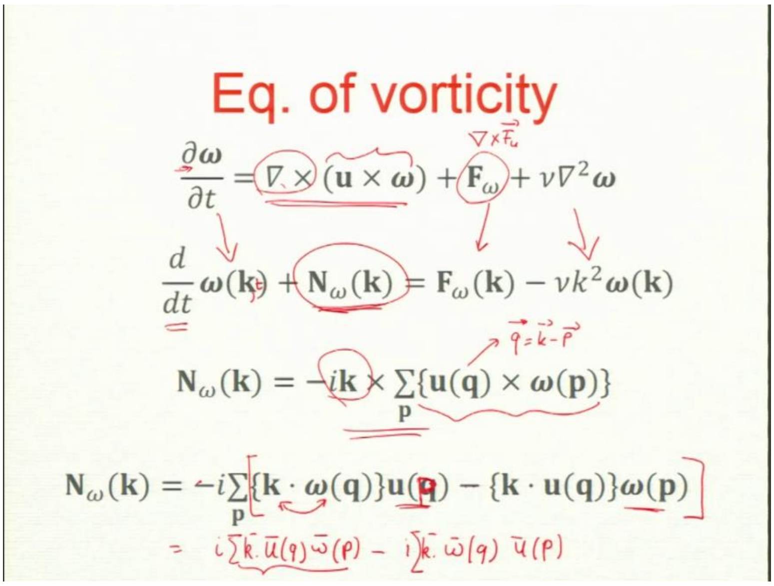
In the last class we discussed Fourier Space Representation. So, we did definitions, then we looked at the Navier-Stokes equation in Fourier space, then energy. Now, we go and derive equations for Vorticity, Kinetic helicity and Enstrophy that is what we will do today.



Vorticity in real space this is a definition you know. See the above figure. vorticity is a vector $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. What happens in Fourier space? curl is a derivative operation this ∇ gives you k, but this curl? is basically $\boldsymbol{ik} \times \mathbf{u}(\mathbf{k})$**.** Now, from this, For given $\boldsymbol{u}$**,** I can compute $\boldsymbol{\omega}$. By the definition, it is perpendicular to k in a plane; it could be anywhere in the plane perpendicular to k in 3D.

Now, from this relation I can compute $\boldsymbol{u}(\boldsymbol{k})$. How do I do it? Which is useful, just multiply, take a cross product of this equation.

Using $\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k}) = \mathbf{0}$**,** incompressibility, we finally can get ***u(k)*** easily. Vorticity is important quantity which we can write in Fourier space as well.

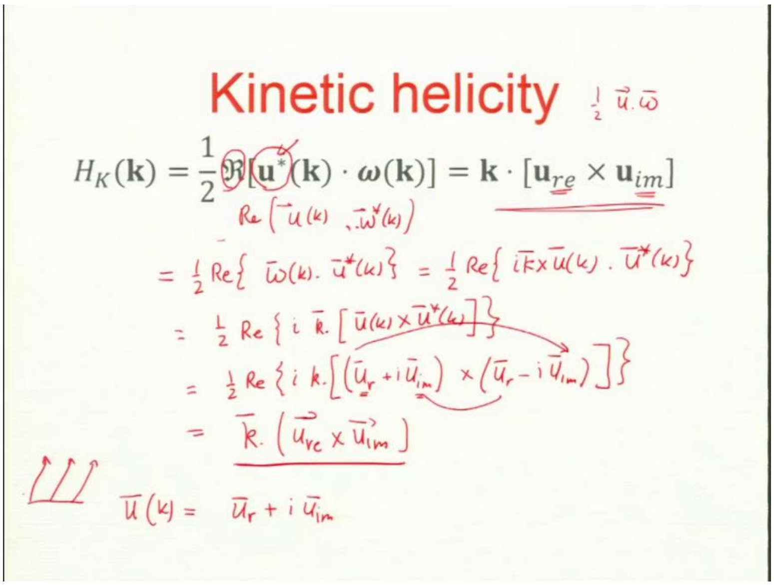


Next is equation for the vorticity, we start always with real space equation, this equation in real space which I have derived. See the above figure. Omega is very similar to magnetic field equation and this $\boldsymbol{F_\omega} = \nabla \times \boldsymbol{F_u}$. **Fu** is the force for the velocity field. Now I can convert this to Fourier space. how does it look like? ∇^2 becomes $-k^2$.

Now, we need to worry about what is $\boldsymbol{N_w}$, Fourier transform of this. See the above figure. This product becomes a convolution, $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{p}$.

So, this is equation for $\boldsymbol{N_w}$ is a convolution; it is not just a product of $\boldsymbol{u}$ and $\boldsymbol{\omega}$. Finally, we get expression for $\boldsymbol{N_w}$. See the expression in the above figure.

These terms have very specific meaning and they have very specific role in energy transfers and dynamics. You do not need to memorize the derivations, but you should know where to look for and after sometime hopefully it will become part of you. We need this equation later, then we consider dynamics of vorticity and how modes interact.

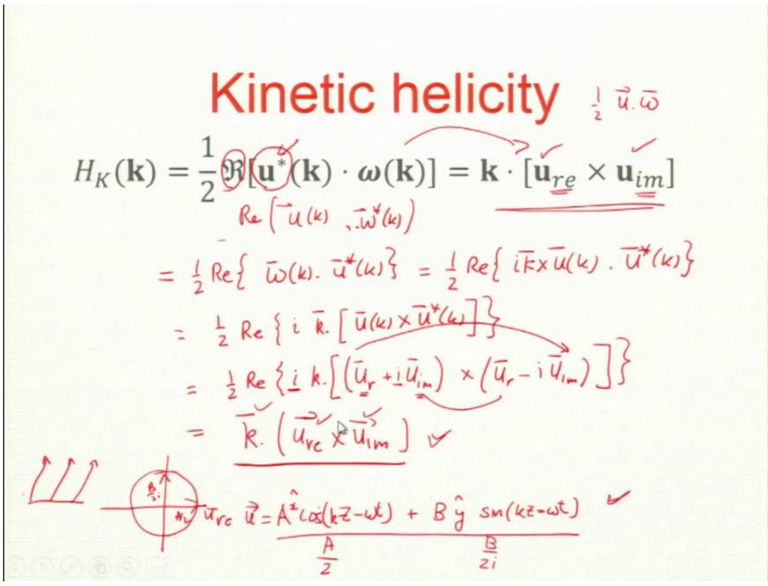


Now, kinetic helicity. This thing is what we did in real space. So, there was some questions how you visualize helicity. So, let us see this in Fourier space. Helicity in real space is $(1/2)\boldsymbol{u}\cdot\boldsymbol{\omega}$. In Fourier space what will that be? See the above figure.

Since it is a product. So, product becomes again a product in Fourier space, but one of them is a complex conjugate. *re* means real part of **u** and *im* means imaginary part of **u.** So, you need both real and imaginary and this for and they should not be aligned real and imaginary. There should be some angle between the two, then we will get non-zero kinetic helicity. See the final expression for kinetic helicity in the above figure and remember it.

Just to connect with what you already know that for linearly polarized waves electrodynamic, linearly polarized wave. A vector field is along a plane so, but it could be complex.

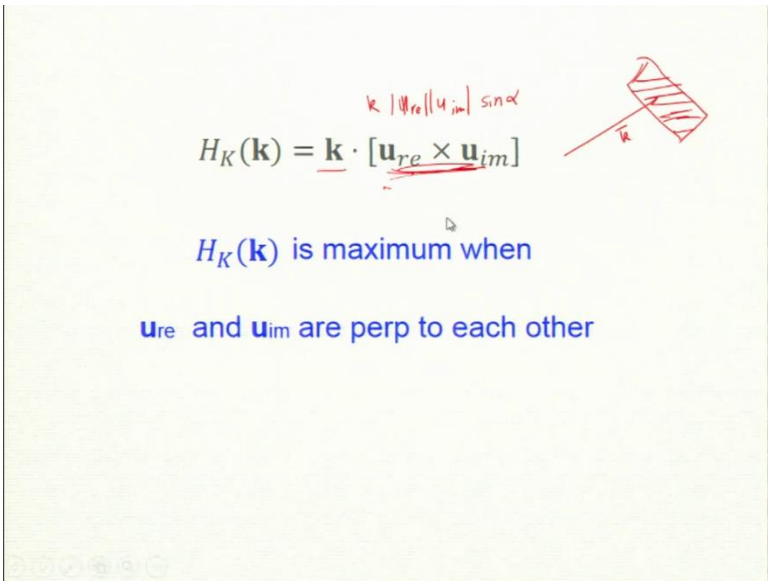
Fourier transforms are complex number. So, real and imaginary part are in the same direction. So, if I take the real part and imaginary part of a vector, there is a same direction. So, cross product will be 0. So, linearly polarized waves have zero cross helicity.



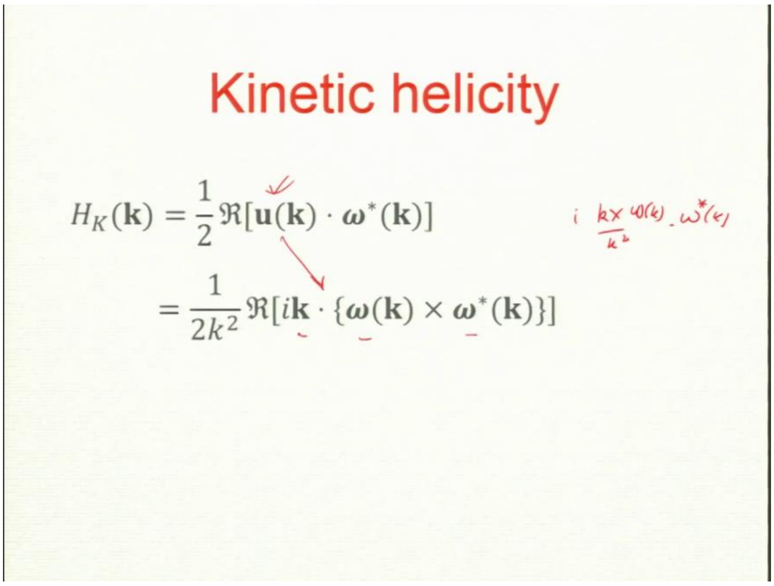
So, normally you represent like this. This is a wave travelling along z direction with polarization with x and y. I mean it has component along both x and y.

If I do the Fourier transform of this, I will get one of them is real. You do the Fourier transform of this object, then you will find that this velocity vector or electric field vector u vector has both real part and imaginary part, but they are not in the same direction. This is a nice formula for kinetic helicity; you just need real and imaginary part of **u.**

So, this is a intuition for kinetic helicity and we will come back to this again. We will look at kinetic helicity again in another basis, but I presume that the circular polarized wave you are getting the idea.

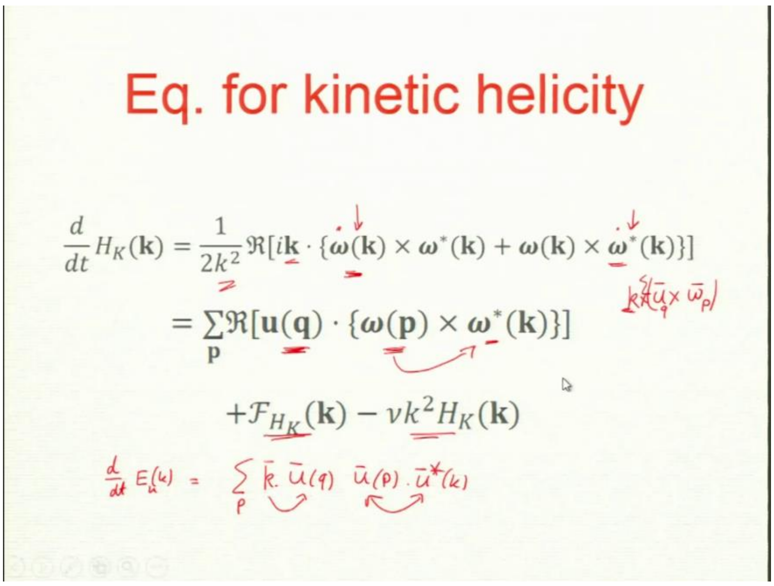


Now we can make some statement about maximal helicity. When is kinetic helicity maximum? $\boldsymbol{u_{re}}$ and $\boldsymbol{u_{im}}$ must be perpendicular to $\mathbf{k}$. In 2D you cannot have any kinetic helicity. When $\boldsymbol{u_{re}}$ and $\boldsymbol{u_{im}}$ are perpendicular to each other, then this is maximum. Now this can be negative or positive kinetic helicity and that depends on the sign of this. So, we can get both maximal and minimal.



When they are 90 degrees to each other just appropriate phase and the examples we will again revisit today in Fourier space and you can see that why it should be positive, maximally positive or maximally negative. Now we will look at the equations.

So, this is what I wrote; know this is a formula. Now we replace this u. So in fact I need to do the by formula I derived in the second slide or first slide of this. See above figure.



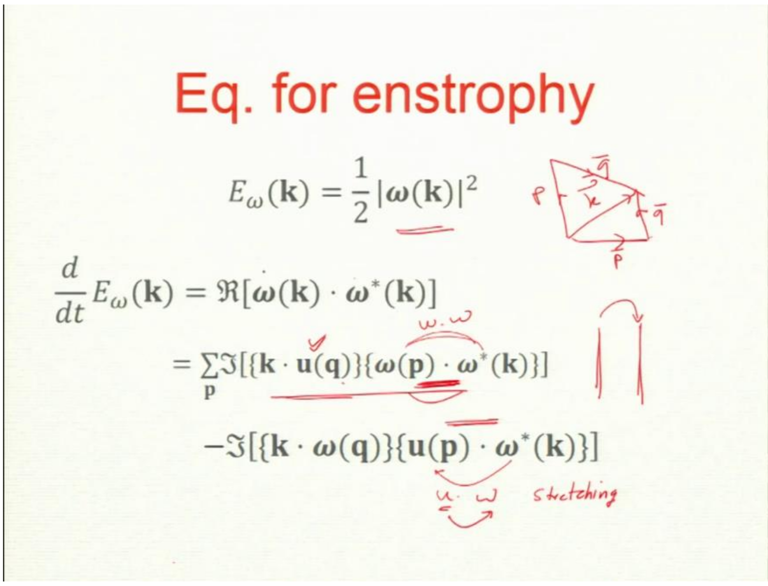
Now we derive evolution equation for kinetic helicity. See the above figure.

This is coming from the non-linear part, force part and this is a viscous part. the non-linear part which we should focus on, there is a convolution here

This is kind of nice trick which gives you this formula. This is known before, but we it is nice to do them this nice form.

This is where we will understand how energy is flowing from one Fourier mode to other Fourier mode or how kinetic helicity is flowing from one Fourier mode to another Fourier mode, ok. That is why we labored on deriving these equations very carefully.

So, this is for kinetic helicity and we will visit them when we want to derive the formulas for energy transfers and helicity transfers and so on.



So, what is enstrophy? See the above figure. Enstrophy by definition is $(1/2)|\boldsymbol{\omega}|^2$. Now you can write down time derivative of this. See the above figure.Vorticity will interact basically the vorticity to vorticity transfer. So, there is just ω to ω transfer. So, there is energy transfer from one ω mode to another ω mode, this is the term where u to ω transfer kinetic energy is supplying to ω. Second term will stretch the vorticity because u is stretching the vorticity field. So, the second term is the stretching term and first term is in fact is transport the ω by this u. So, these are interpretations one can make by looking in the structure and similar interpretations were for magnetic field as well. So, these are transport of magnetic field by the velocity field, but this is stretching of the magnetic field by the velocity field.

They will exchange among themselves, but overall if you look at the full field $|\boldsymbol{\omega}|^2$will not change by these interactions.

If the neighboring modes contribute, then it is called local interaction. If the far away modes contribute, then it is called non-local interaction. Enstrophy is also dissipated by viscosity.

This ends my discussion on the derivation of Helicity, Enstrophy, Vorticity.

Thank you.