Physics of Turbulence
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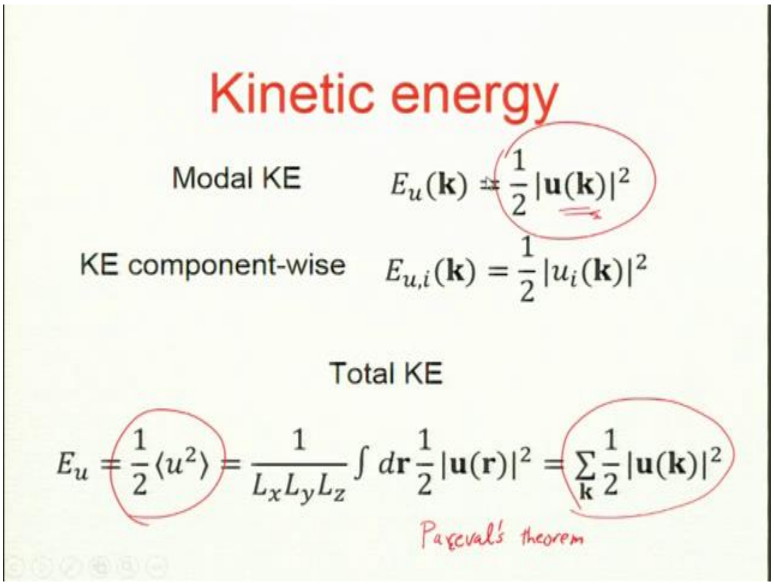
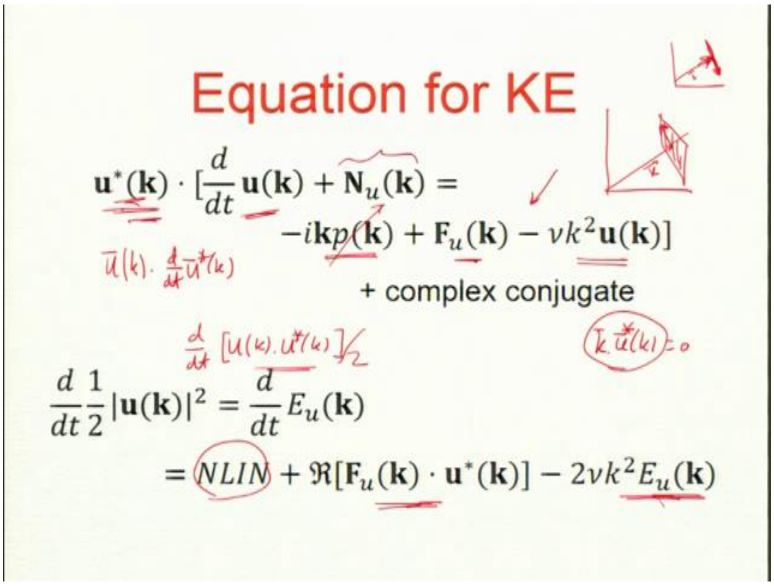
Lecture - 08
Fourier Space Representation Kinetic Energy

So, the third part of Fourier Space is Kinetic Energy. We did flow equation, now we have velocity field; I construct kinetic energy from the velocity field. I already told you about a product $f(r)g(r)$. In fact, we did that $\boldsymbol{u} \cdot \boldsymbol{u}$ becomes some u^2. $u^2(k)$ is a Fourier space energy of a Fourier mode. See the figure below.



I call it modal kinetic energy. Mode means $\boldsymbol{u}(\boldsymbol{k})$ that is your Fourier mode, energy I put 1/2. If I sum over all ks then I will get the total kinetic energy. So, there are many modes but each mode has its energy. So, we call it modal kinetic energy. Now we think kinetic energy component wise. So, this is $u_x^2 + u_y^2 + u_z^2$. I just put a symbol i. So, this we also need it often $u_i^2(k)$.

So, what is total kinetic energy; it is as said sum over $u^2(k)$. I showed by proof that this one is equal to $1/2 < u^2 >$; it is proven in the first module, is called Parceval's theorem. So, these are important relation and we need this in our simulations and so on. We need to write one equation for energy, for this modal kinetic energy.



We derive this equation for modal energy. See the derivation in the above figure. You have equation dot product with $\boldsymbol{u}^*(\boldsymbol{k})$. Do not dot product with $\boldsymbol{u}(\boldsymbol{k})$**,** you have to put $\boldsymbol{u}^*(\boldsymbol{k})$ because I want mod square. Now add complex conjugate to this. What happens to the first term after I add complex conjugate? So, we will have $d\big(u^*(k)\big)/dt$.

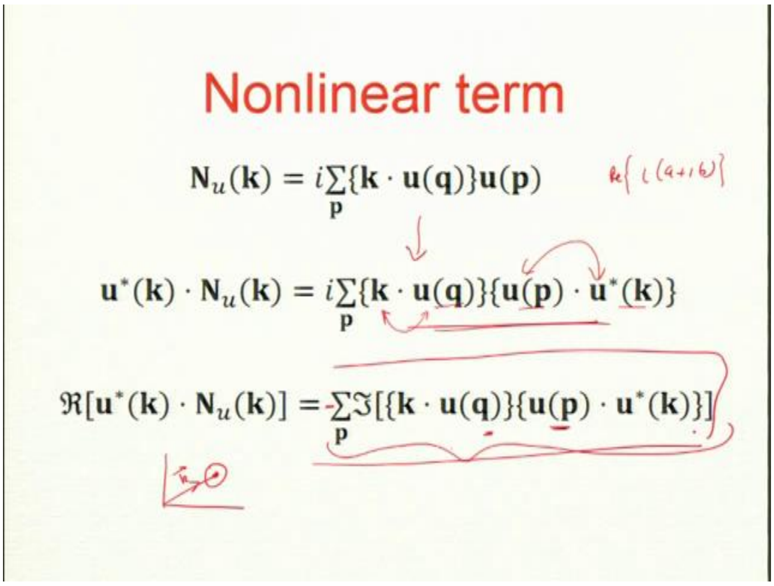
Note that double conjugate gives you the same function. So, you sum it up by product rule you will get this. What is $\boldsymbol{u}(\boldsymbol{k}) \cdot \boldsymbol{k}$?, 0.

0; incompressible condition, for u star k and you take a complex conjugate that should also be 0. So, $\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k})$ is 0 for every mode, please keep in mind. If I take a complex conjugate that also should be 0. Pressure term does not give anything but r one multiplied by $F_u(k)$ is non zero in general and that is a energy supply rate by the force at wave number k.

So, these are coming from energy supply rate; what about the viscous term? So, $\boldsymbol{u}^*(\boldsymbol{k}) \cdot \boldsymbol{u}(\boldsymbol{k})$ becomes $|\boldsymbol{u}|^2$, is like $E_u(k)$. Now look I just divide it by half. This is coming from the force term, this is coming from the viscous term and this term NLIN, I need to compute in the next slide.

So, it will be $\boldsymbol{N_u}(\boldsymbol{k}) \cdot \boldsymbol{u}^*(\boldsymbol{k})$**,** real part because I have to also do two complex conjugate so this function real part. External force is a injecting energy and this is the losing energy by viscosity and is each Fourier mode is doing it. I am able to identify this Fourier mode, how much it loses a viscosity, how much it is getting for external force and how much it is getting for non-linear interactions.

$\boldsymbol{k}$ vector is a vector in Fourier space. Which direction $\boldsymbol{u}(\boldsymbol{k})$ cannot have any component along? Since due to $\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{k}) = \boldsymbol{0}$, it has to be perpendicular. It lies in this plane perpendicular to $\boldsymbol{k}$. So, $\boldsymbol{u}(\boldsymbol{k})$ has no component along $\boldsymbol{k}$. In 3D I need to take two basis functions. Now, let us compute NLIN: non-linear term. See the below figure.

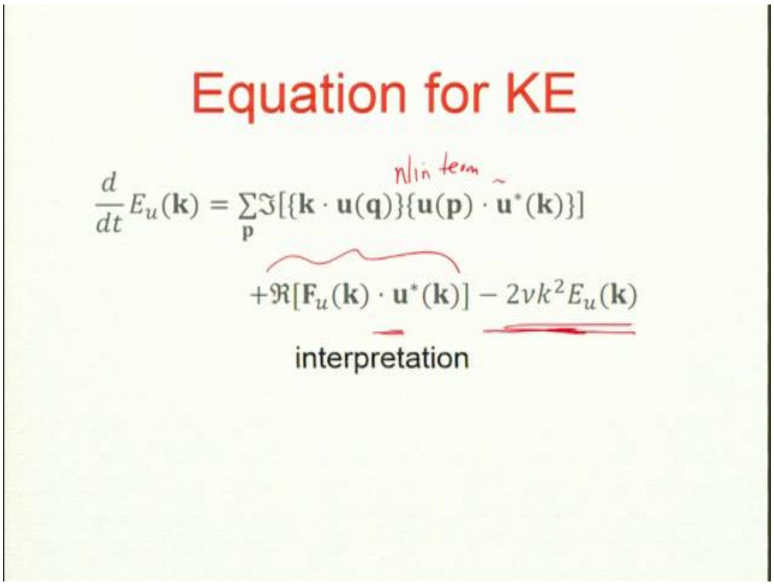


I have to take dot product as u star k real part. I will spend lot of time on this term a bit later. Now if product of three, one of them is complex conjugate.

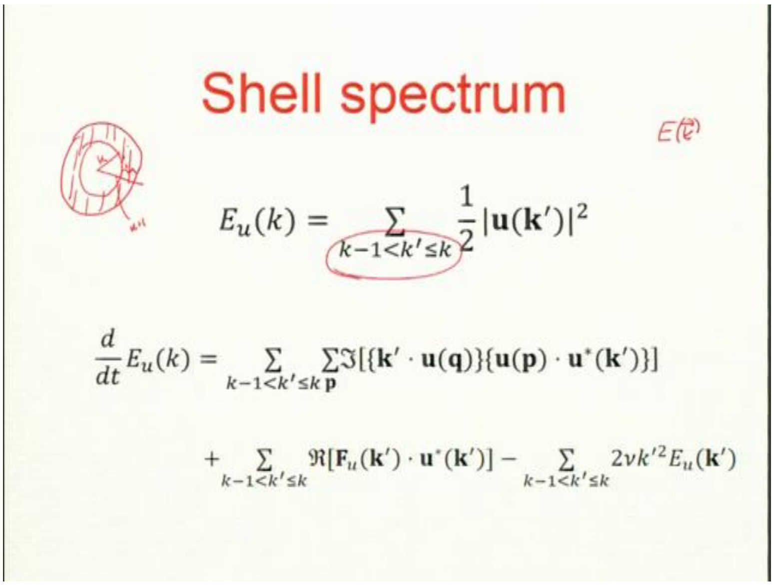
I need to of course, take the real part of this. So, we take the real part of this. So, now, this is straightforward. We will simplify further and further but for next set of lectures I am going to use this and compute them. Wewill do bit of hard work for given flow field what is energy transfer by non-linear interaction.

You must also look at to understand this. A Fourier mode gains energy from other modes via non-linear interactions. This term is telling you that is gaining energy from other modes equal to this number. Now what are the other modes; $\boldsymbol{u}(\boldsymbol{p})$ and $\boldsymbol{u}(\boldsymbol{q})$ are giving energy to $\boldsymbol{u}(\boldsymbol{k})$. So, in non-linear systems u will be receiving or losing energy to other modes by non-linear interactions.

We have large eddies and that breaks into smaller eddies that means large eddies are giving energy to smaller eddies or losing some energy to smaller eddies and this is by non-linear process. Some eddy is growing like hurricane is growing; how does hurricane grow? Of course, some of it is by buoyancy but some of it is by non-linear interaction where small eddy is giving energy to large eddies; it is something like poor people give money in bank and there is some interaction so, via transfers. So, these kinds of interactions happen in fluid flows. We will be working on these interaction quite a lot.



Equation for kinetic energy in summary is this. See above figure. So, these are coming from a non-linear term and this is energy injection rate and this is dissipation. The injection rate by force for example, buoyancy can give energy to a Fourier mode that is why it is rising going up and it loses by viscosity. The viscosity is taking energy from each Fourier mode this is always negative with a minus sign but others could have both positive and negative signs. Positive means is gaining and negative means losing.



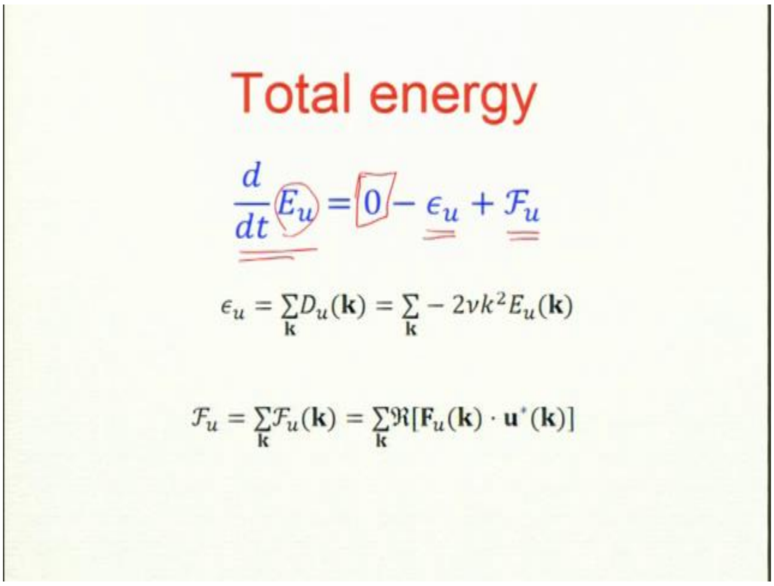
Now, we also need something called shell spectrum. There are just too many modes; like in a fluid system, in 3D there are N^3 Fourier mode and for each direction is there N modes.

So, it is a good idea to say for isotropic systems; isotropic means it is there is no direction. So, in this room if you ignore the wall effects; if the people have scattered uniformly in the room, x direction is equivalent to y direction equivalent to z direction.

For those kinds of flows I can draw a shell, 3D sphere and within, one sphere of radius k, other radius sphere of k+1. I am simplifying my life and not making dk. So, this shell has unit length. this in fact, motivated by simulation. some people will say we will use dk its fine, but I will make dk 1. There are many modes inside shell.

You have to think of 3D lattice and there are some lattice points lie within the shell. Now I sum over all the points in the shell. So, instead of $E(k)$ which is called modal energy, I am defining shell energy, shell spectrum which is sum over all the modes in a shell. Kolmogorov theory which we will discuss bit later is for a shell not for a single mode.

So, $E_u(k)$ is sum over all the modes within the shell. since we know the equation for each mode, I can also write equation for the whole shell and; you just basically have to sum. This you need to perform a sum of over all the modes. This is called shell spectrum.



We can also write down for total energy. The equation for total energy is this, see above figure. total means the whole box and whole box of course, corresponds to sum over all the Fourier modes. Note that the non-linear term only exchanges energy among themselves which I will prove it. Non-linear interaction does not increase energy of the whole box; is only exchanging among themselves in hydrodynamics.

The energy injection rate which could be positive or negative and other one is coming from dissipation. dissipation is always negative; if I do not force it my total energy will decrease right; if I F_u is 0 then dissipation is negative and energy will decrease. If you can help it to sustain, it can be positive. These are the three terms: non-linear term, dissipation term and external force injection by the external force term.

Pressure does not get involved in the energy transfer because this $\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k})$ becomes 0. So, pressure does not get involved, pressure does something else which we will do bit later.

Thank you.