Physics of Turbulence
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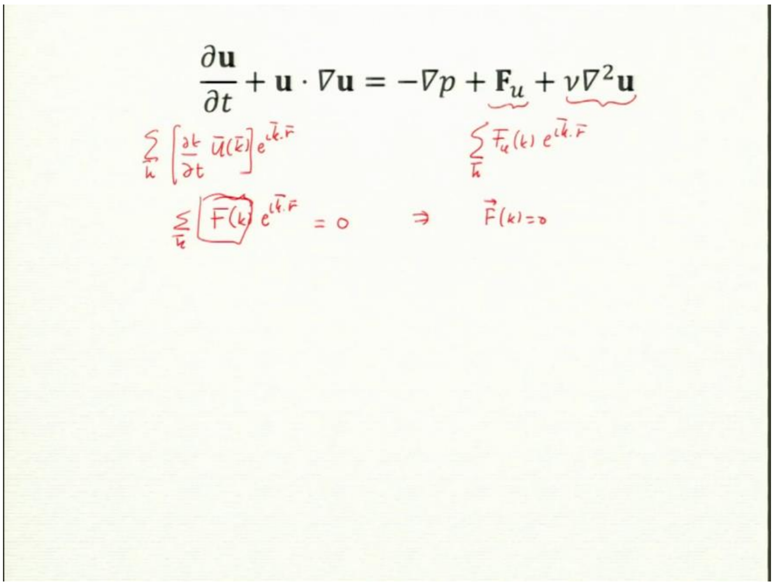
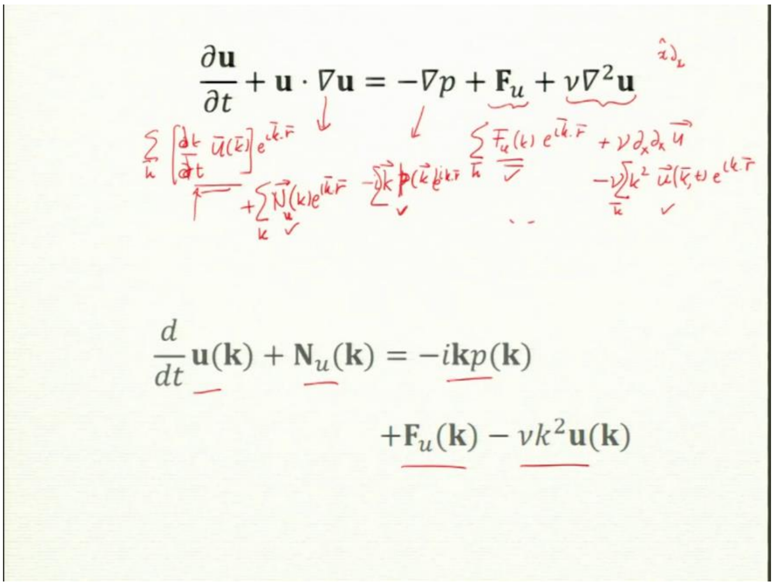
Lecture - 07
Fourier Space Representation Flow Equations

This part we are going to do the Flow Equation. I will do incompressible Navier-Stokes.



The above figure shows Navier-Stokes equation, which we did it for in real space. Now, I want to convert this in Fourier space. We can do d/dt first and sum later and vice versa. So, I am going to take that d by dt inside Please remember $\boldsymbol{u}(\boldsymbol{k})$ is function of time; so, Fourier mode is changing with time. I am going to write every expression in terms of sum.



What about $\nu\nabla^2\boldsymbol{u}$; it is double derivative along x, double derivative along y, and so on. See the above figure.

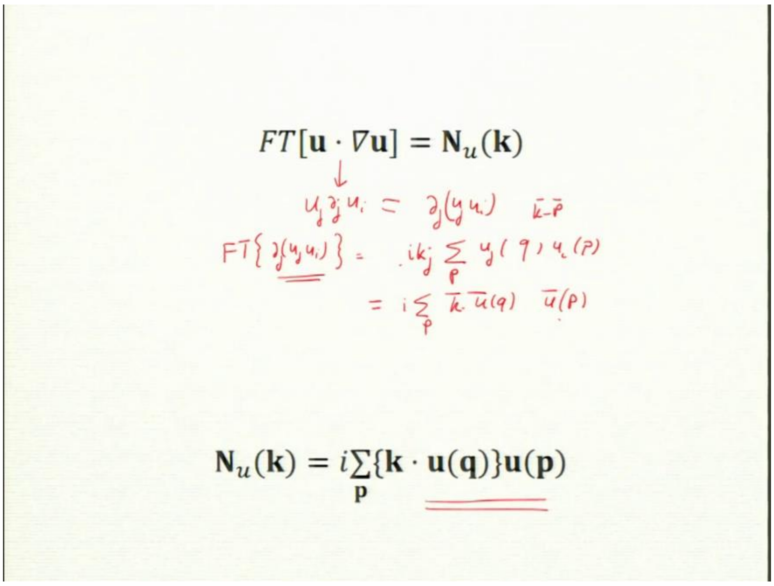
What about minus ∇p? It is a gradient. This p is pressure

Now $\boldsymbol{u} \cdot \boldsymbol{\nabla u}$ is a complicated; most complicated object is a product; it will become convolution. I am going to write this term as $\boldsymbol{N_u}(\boldsymbol{k})$, since this is a non-linear term corresponding to velocity field, it is a vector.

What are each terms of this equation giving you? The first term gives you derivative. So, let me remark that this is Fourier series, this is not Fourier transform. Fourier transform implies a box is infinite.

So, you take the box L_x going to infinite limit, L_y going to infinite limit and for that k_x, k_y, k_z become continuous. So, δ basically within two points in the lattice, it becomes 0 distances, tending to 0 distance. So, for Fourier transforms which corresponds to infinite box my k becomes real and then of course, derivation changes a bit not a lot. But, you have to be careful when we have infinite box, but I am going to consider in this course only finite box that is by design so, I just want to stick to it. So, this is derivative d/dt, there is only one variable time because k is fixed.

Now, the second term gives you $N_u(k)$ which we will do it a bit later and this is k vector times p, each term is a vector ok. And, fourth one is $F_u(k)$ and the last one is $-\nu k^2 u(k)$. Now, as I said I can take everything to the left hand side. So we have a non-linear term, pressure gradient term force in Fourier space and viscous term. This is my Navier-Stokes equation, I am not fully done I need to find what is $N_u(k)$. So, let us find what is $N_u(k)$ Let us work on this.

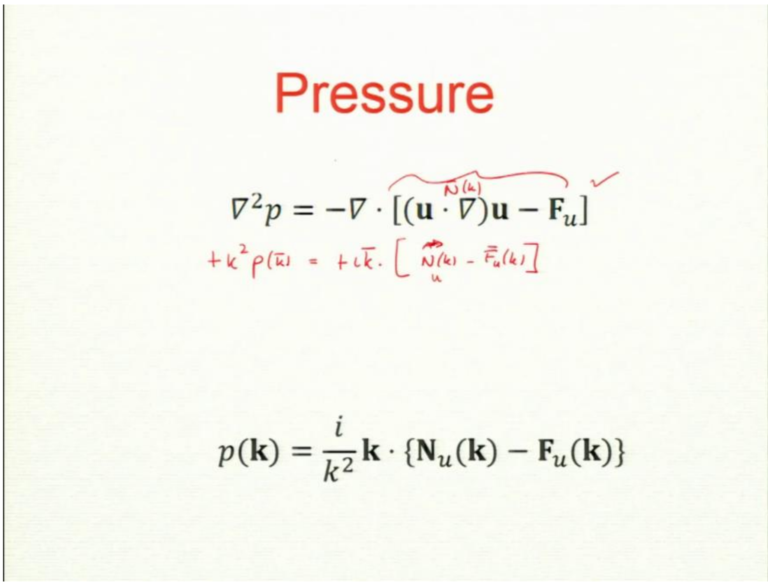


So, next slide is Fourier transform of $\boldsymbol{u} \cdot \boldsymbol{\nabla u}$. This is in real space is $i\partial_j u_j u_i$. Since u is incompressible, I write equal to $ik_j u_j u_i$?

It is a convolution.

This is what I get for non-linear term. So, non-linear term because of convolution, it is nice simple, but now you have to deal with a infinite sum; that is the complication with Fourier.

Now, I want a vector, but if I want to write in terms of vector so, this is what it is. We are almost done, but what about pressure? As I said pressure is a dependent variable, not independent variable. So, I should write pressure in terms of velocity fields. See below figure.



So, remember this part which I did in real space. So, pressure comes from this equation, this I had derived it. I take the divergence of the Navier-Stokes equation and I will get this equation for pressure. Also, it also depends on F_u; so, let us keep force. Many times force is divergence free, but not necessary; so, I need to write this in Fourier space. But, easier is this is let us take Fourier transform of both sides. So, I say Fourier transform of $\nabla^2 p$.

It is twice derivative so $-k^2 p(k)$. So, I get $p(k)$.

This is equation for pressure. So, pressure I have to compute from the velocity field and the external force. So, we get a differential equation for every Fourier mode and so, instead of real space where$u(x)$, x is a continuum in real space; it is everywhere. But, now Fourier space $u(k)$ is defined only at discrete points. So, I do not have continuum, but the small k corresponds to large scale modes and you go down to smaller k, then it becomes smaller scale. But, if you do not do the largest wave number that is smallest scale in the system.

Now, we need to find out the amplitudes of all these modes. In fact, given initial condition I can solve for $u(k)$ at different times in future and that will tell you how the magnitudes of or how these modes are evolving in time. And, once I know the Fourier mode I can always compute real space fields. So, it is not that I only work in Fourier space, I will be coming back to real space and trying to interpret what is it in real space. I want to make one more remark. Many of you would have done finite difference method for solving differential equation; now we have we can solve this flow equation in Fourier space.

So, each Fourier mode is time advanced. There are lot of ordinary differential equations, they are not PDE's anymore. In fact, if you go back what I got is Ordinary Differential Equation: ODE's. Now, this I believe has inherent advantage in terms of physical terms. I am solving the flow patterns at different scales. In finite difference everything is considered at equal ground. I am to look at large scale and small scale. I need to basically look at averages at large scales, but with spectral I just look at small k, I get the large scale features.

It is like multiscale features are inbuilt in spectral method or with Fourier transform.

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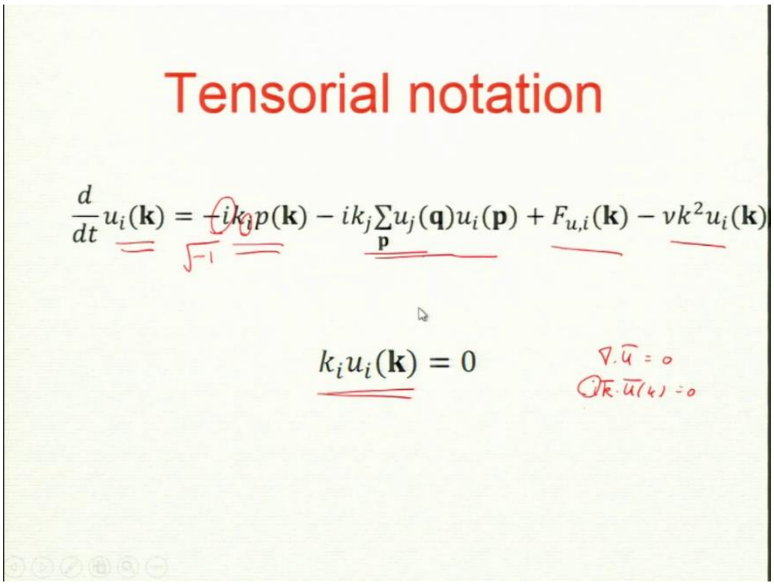

There is tensorial equation. We need to know something about tensors notation. So, it is very similar with what I was doing, but it is important to know. Incompressibility condition $\nabla \cdot u = 0$ is $ik_i u_i(k) = 0$. The tensorial notation which we will use it sometimes.

Thank you.