Physics of Turbulence
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Lecture - 5
Basic Hydrodynamics: Examples

So, this is the fourth module for basic hydrodynamics. So, we will cover some examples. So I will take some vector fields, in fact very simple examples, no complicated pipes, channels and all, just simple velocity field and we will compute energy, vorticity, and so on. So these are velocity field.

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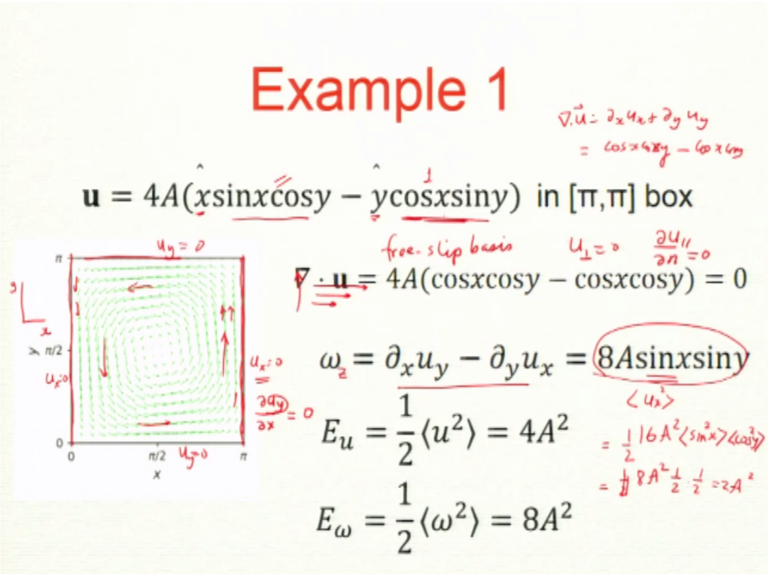

So $\hat{x}$ means x-component and this $\hat{y}$ means y-component and it is functional dependent, so u_x is sin$x$cos$y$ and u_y is cos$x$sin$y$. Now this is a very important velocity field at least for me. So there are some interesting properties. So we are going to do it for a box, let me also say this, I want to do for this a 2D field, so $\pi \times \pi$ box, so box size is $\pi \times \pi$. So let me first show you the vector fields. So you can plot the vector field, even easily do this in π term okay. So, this will be some exercises you will be asked to do.

So are the arrows visible, so this velocity field is going like that okay, you can see is like that. Now at x = 0, this x = 0 line, now sin$x$ is 0, so the first term is 0 altogether, it has only u_y component and u_y component is in fact it varies, cos$x$ is 1, this 1, at x will be 0, it varies the siny. In fact, you can see that velocity is u_y sign so negative sign, no, so it is negative sign, so is always 0 to π is negative, you just say is negative here, is going like this, is a steady flow, it doesn't change with time.

Now one important point is that the u_x is 0 at these 2 walls x=0 and π, but u_y is not 0 at these walls, the wall with red line, you know this wall. So this is not, in fact it is not true for rigid walls, rigid wall I said the velocity both the components must be 0. So this wall has a peculiarity that the velocity is not 0, that tangential component is not 0 at the wall and this is called free-slip basis okay, this is free-slip basis. So my perpendicular component toward the wall is zero 0, but the tangential component is not 0, same thing here.

For here u_y is zero 0. In these walls, u_y is 0, u_y is vertical for them because is y direction, no, this y, this is x axis, this is y axis, and here u_x is 0, but you can easily see for this wall, for the vertical walls, partial $\frac{\partial u_y}{\partial x}$ is 0. So if I take this u_y, the u_y, this one, if I take derivative with x, what will I get, sin$x$ that becomes 0. So this is called stress free, so stress is if you know well stress is proportional to the tangential derivative, so you know is the velocity field, how does it change in that direction.

If the velocity does not change perpendicular to the wall, then is called stress free because there is no stress, stress when it comes, when your partner is not going along with the same velocity that is stress. You know there is a race you know people are not running together, then there will be stress. So here if they move with the same velocity, then there is no stress. In fact, you see that this u_y are roughly you can see, well is not very visible but u_y are approximately equal near the wall, so stress is 0 at the wall, therefore stress free okay.

"Professor - student conversation starts" Ya. We will be evaluating the stress after… So given the velocity field, I can compute stress okay. Now, this is not I am not solving for you from the Navier-Stokes equation, I am just giving the field and we will analyze few properties..
"Professor - student conversation ends"

So this wall has in fact **u** perpendicular is zero and **u** parallel u_n, $\hat{n}$ is normal okay this is 0, so this is called stress free. Now, I can show that the divergence of this is also 0. So, let us compute the divergence. So first term if, so what is divergence by the way, so I am going to write it here

$\nabla \cdot \mathbf{u}$ is $\partial_x u_x + \partial_y u_y$. So when I take the derivative of this with respect to x, what will I get, $\cos x \cos y$ right $\sin x$ will be $\cos y$.

So first term will give you $\cos x \cos y$, what about second term $\cos x \cos y$ right because $\sin y$ becomes $\cos y$, so $\cos x \cos y$, so that is why everything is 0, so it is divergence free, in fact is nice for lot of calculations, is divergence free and is very easy to write, it is stress free and is nice circulation, in fact in convection we get nice this connection role you know, so we use this as an interesting basis function or basic mode for representing thermal convection. What about vorticity, does it have vorticity? Can you make a guess it has vorticity?

It looks like there is some cyclone, now, this hurricane, so you can compute vorticity, yes indeed. So I just, I will leave this as an exercise okay, I will not do this, but you can just do this partial x for u_y, it is a 2D, so you don't need to compute all of the 3 components, just compute this one and it is $8A \sin x \sin y$.

"Professor - student conversation starts" this is 0, 0 no. No not this.. **"Professor - student conversation ends"**

So u_y - If I take the derivative of this with x, we will get minus $-\cos x$ cos will give plus sign $\sin x$ here and if I take derivative of this, so so that with y will be minus, so minus minus becomes plus, so it is $8A \sin x$ okay, so both give you plus contribution, minus minus becomes plus, well if I take *dy dy dy* I will get this minus $-\sin y$ and this is minus sign again and that becomes plus okay. It is nonzero, you can see this is hurricane, now vorticity is 0, what about energy, how do you compute kinetic energy, $\frac{u^2}{2}$.

Now, I want to compute the average energy. So this is important quantity which we want to compute average because if I make a bigger box, then my energy will be more, no, so in my notes at least I can also well let us compute average kinetic energy, means energy by volume, now it will be area, it is 2D space area. So, let us do one of the terms. So $\sin x \cos y$, square it, so the first term I am going to do. So $\frac{16A^2}{2}$, I am doing u_x squared, so this is $\langle u_x^2 \rangle$, $\langle 16A^2 \sin^2 x \cos^2 y \rangle$, they are independent no, x and y, no coupling.

So it can be written as $\sin^2 x \cos^2 y$ and what is the average of $\sin^2 x$, $8A^2$, what is average of $\sin^2 1/2$, so 1/2 and $\cos^2 y$ also is 1/2, you can do over a periodic box, average is 1/2, so that gives you $2A^2$. So u_x^2 is A^2, u_y^2 is also $2A^2$. So total is $4A^2$ okay. So that is kinetic energy. Helicity is zero, $\mathbf{u} \cdot \boldsymbol{\omega}$ is 0, no problem. By the way, this $\boldsymbol{\omega}$ is there, no, so this this $\boldsymbol{\omega}$ is there.

You can also compute enstrophy E_ω. Now, this I will again ask you to do it, so you need to compute as square of this by 2 okay, so you can easily do it. It is nonzero. This enstrophy is a 2D system. This example 1 is free slip, is nice convection role, nice role sitting there and we will use it in future okay. So my examples will be like this. In this course, we will consider these kinds of examples. We will work this in Fourier's space, we will put some more modes you know, this is only single mode. I will discuss the Fourier space, then you say well this is basically one Fourier mode.

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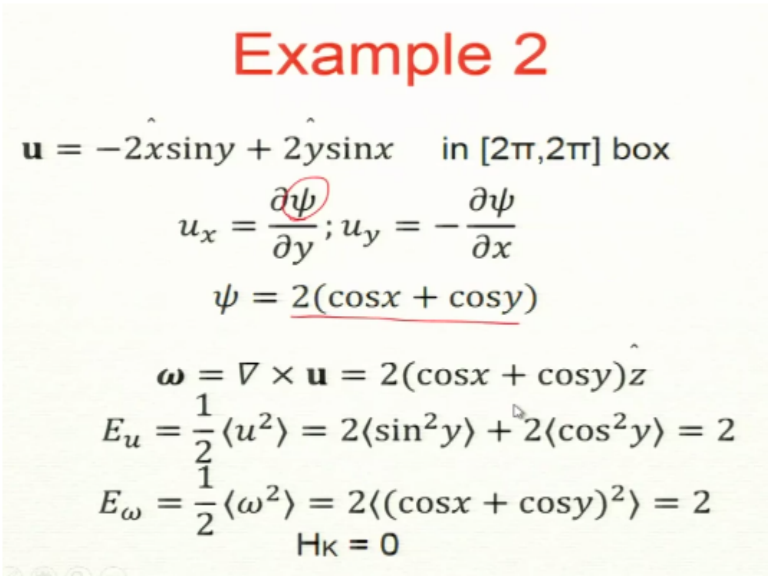

Example 2. So my velocity field is this $-2x \sin y + 2y \sin x$. The box is I made it a bigger box $2\pi \times 2\pi$ box, 2D. It is periodic with periodicity 2π. Now we can again compute the same stuff. So this is standard exercise. So well before I do, so stream function okay, this stream function is useful, so you can compute stream function and the differential of this psi is stream function. So you can compute u_x and u_y given stream function or I can put ψ given u_x and u_y okay.

This is simple mathematics and you should be able to do it. I can compute ψ okay. So, this I will leave it, this part, fine or not fine. Okay so, compute ψ is $2\cos x + 2\cos y$. You can also compute vorticity. So vorticity is partial, so that $\partial_x u_y - \partial_y u_x$, it is this. So vorticity is along z

direction, it is $2(\cos x + \cos y)$, of course $\boldsymbol{\omega}$ and $\mathbf{u}$ are perpendicular because $\boldsymbol{\omega}$ is along z direction. You can compute kinetic energy. So please make sure that is $u_x^2 + u_y^2$ okay, u^2 is sum of 2 for 2D and some of 3 quantities in 3D, $u_x^2 + u_y^2 + u_z^2$.

Now what is enstrophy, it is 2D, so there is only one $\boldsymbol{\omega}$, ω_z, so is 2. How does flow look like. It is an interesting flow, I will show you the picture. This picture you know, I mean in our python course in the past you have done it, so please use those scripts to make this plots. Kinetic helicity zero naturally because of $\boldsymbol{\omega}$.

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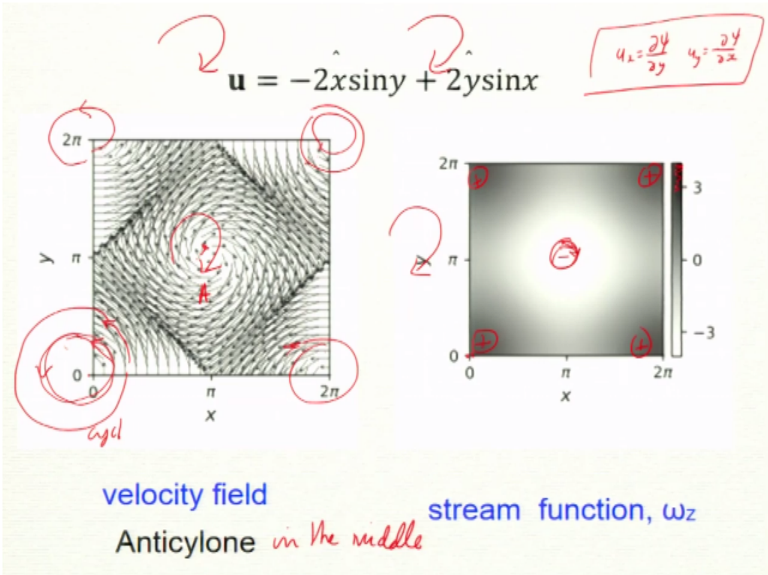

So this is the picture of the velocity field, it is not coming out very well but you can see that it is going in that direction, is clockwise. Now, here it is anticlockwise like that, here it is whether it is the same is periodic. So this guy, this guy here is repeated here. So there is a pattern here. This counterclockwise, so this is cyclone, this is anticyclone. What about this, this is again, in fact this is periodic, so it should be here, cyclone, this is also, so in fact the 4 cyclones here and 1 anticyclone, but why not, I mean these looks are symmetric, no, the four 4 cyclones and 1 anticyclone.

So, it should be periodic, so where are other anticyclones, is periodic, so there is anticyclone sitting here, here, there is anticyclone sitting here, anticyclone sitting here. So this 1 cyclone in the middle and 4 anticyclones around. So that in fact is a periodic lattice of cyclones and anticyclones. So cyclone sitting in between 4 anticyclones and 1 anticyclone sitting in between 4 cyclones okay. So, this is nice example for rotating flows or even 2D, this is what we tend towards.

Now since I already computed stream function, in fact I am more interested in well right now in stream function, but I show you later, so this anticyclone in the middle I should say not everywhere, in the middle okay, it is at (π, π) okay, this place in the middle, this coordinate is (π, π). Now you can also compute this is my stream function okay and ω_z in stream function are the same functions which is there in the slide before. So ω_z in stream function, a very similar property that it is here, it is positive, ω_z is positive no, here?

This is positive and it is negative here. So I defined in the last slide, so you can see that u_x is $\frac{\partial \psi}{\partial y}$ with plus sign no and u_y is $-\frac{\partial \psi}{\partial x}$. So stream function is by definition contours with constant, so velocity field is tangential to the stream lines okay. So velocity field is, so if you just join all the lines, basically ya you just draw the lines and velocity field is tangential to the stream line okay. It is useful for lot of application, but if you don't to be understand it or not fully -

"Professor - student conversation starts" If we find any scalar potential associated with **u** that could. No, no, no, no that scalar is different. So stream function has a very specific meaning, it doesn't satisfy this property. So ψ is here. Stream function, ψ is stream function, not potential function. Okay. Okay. **"Professor - student conversation ends."**

So you can see that it is negative vorticity, a positive vorticity here, this dark area, density plot. So dark is positive because this is cyclone and in between is bright white, so it is anticyclone, anticyclone is clockwise, this clockwise, so this clockwise. So minus plus plus plus, the sign of ω_z. This is an example where we have cyclones, but no helicity, there is no kinetic helicity, 0 okay.

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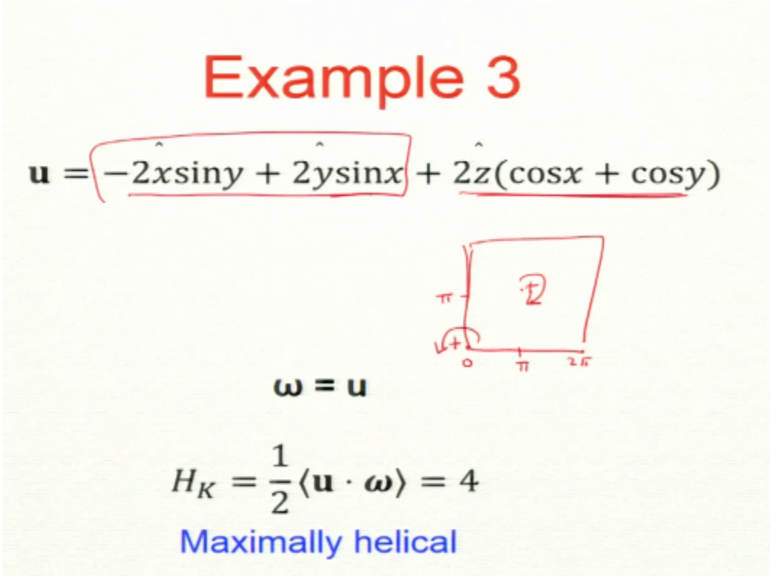

So, example three 3, this is helicity okay. So in fact, this u_x and u_y are exactly same, but I am putting u_z. So this has this in the middle anticyclone, but now this is a vertical velocity, so this is a chance of having $\mathbf{u} \cdot \boldsymbol{\omega}$ is nonzero. For this, you can compute $\boldsymbol{\omega}$. Now I will I think I have a space for doing it, but I will not do it, you can just do this algebra and you will find that omega is equal to u. Example is chosen such way omega is equal to $\mathbf{u}$. So certainly, this is an exercise for you, just compute and $\boldsymbol{\omega}$ is equal to $\mathbf{u}$ and picture I am going to show you I think is in the next slide.

So Hk is positive, kinetic helicity is maximum. Now the thing is here so is maximally helical. Now, if you look at the picture, so let us try to understand okay. So this was my vector field. In the middle, it was going like that right. So this field is exactly same as what I did in the last slide. Now, its vorticity is negative down, what about vertical velocity in the middle. Remember middle is (π, π) okay, this is 2π, this is 2π, 0 and this is also π. So $\cos\pi$ is negative, so my vertical velocity at the center is negative downward.

So, my vorticity is pointing down, vertical velocity also pointing down. So here there is a kinetic helicity, this is like the helical flow. So think something is going like in a circle but is also being pushed, so it is like corkscrew, it is going down okay. So that leads to the helical structure. So this is what the picture which I was trying to show you is exactly this. Here what happens here, in this middle. It is cyclone, so the structure is like that, yes, I plotted last time okay. What is u_z here, u_z is 0; x is 0 and y is 0, u_z is positive.

So here, this is going like this. So this part is contributing positive, this part is also contributing positive. So each of these places we get positive contribution for $\mathbf{u} \cdot \boldsymbol{\omega}$. If $\mathbf{u}$ and $\boldsymbol{\omega}$ are say equal, this is an example where we have $\boldsymbol{\omega}$ is not perpendicular to $\mathbf{u}$, is 3 component, is not 1, for 2D of course it is true. For the omega is three component and if you just work it out, you will find they are equal, $\boldsymbol{\omega}$ and $\mathbf{u}$ are in the same direction okay.

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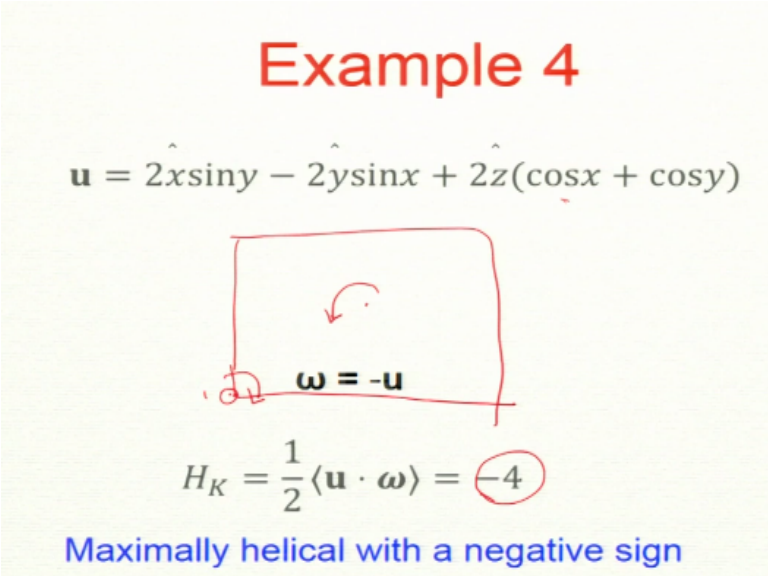

Example 4. I just make only one change that my 2D fields are just changed direction -- changed sign, sorry change sign. So they were minus here, they were minus here before and plus in the previous example, I just changed the sign. So wherever, there was a cyclone will become anticyclone, right because I changed the velocity sign. So cyclones will be anticyclone and anticyclones becomes cyclone. So this picture will be, so let us just draw this picture. So in the middle will be, it was anticyclone, so it becomes cyclone like that.

Here it was cyclone, so it will become anticyclone, yes okay. what about vertical velocity? This one here what is the vertical velocity? Positive. Vertical velocity is, I didn't change the vertical velocity, I only changed u_x and u_y. So it is positive. Vertical velocity positive. So my vorticity is down, my vertical velocity is up. So what happens to $\mathbf{u} \cdot \boldsymbol{\omega}$? Negative. In fact, this example $\boldsymbol{\omega}$ is $-\mathbf{u}$. You can just work it out, $\boldsymbol{\omega}$ is $-\mathbf{u}$. So, what happens to H_k is maximally negative because $\boldsymbol{\omega}$ and $\mathbf{u}$ is there totally opposite and $\mathbf{u} \cdot \boldsymbol{\omega}$ will be maximally negative okay.

So, I mean, for any configuration H_k cannot be lower than this okay. So we will do this constrain in Fourier, Fourier becomes easy, it does not show straightforward in real space, but I will show you what is called maximal helicity okay. So, this cannot go lower than -4 and this is called maximally helical with negative sign. So, earlier example was positive sign and this negative sign. So these are the 4 examples I will like you to keep in mind and we will revisit them in several times okay. So, I will stop. Thank you.