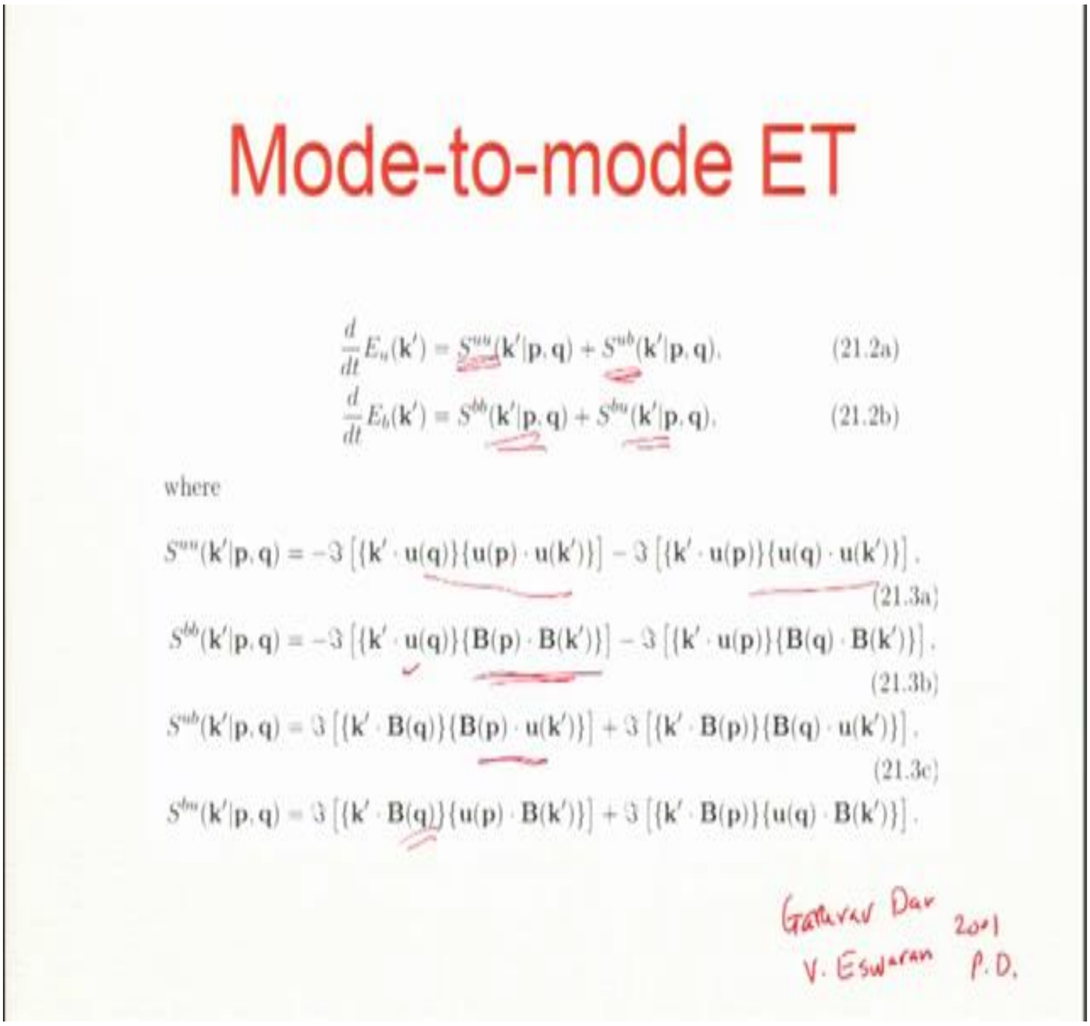
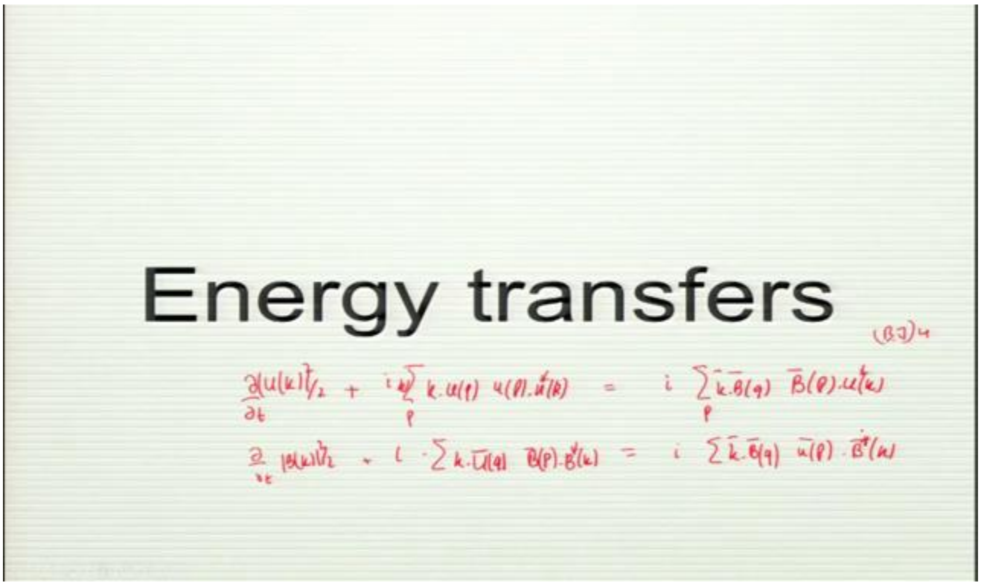
Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture – 42
MHD Turbulence
Energy Transfers

So, we will do energy transfers in MHD now ok. So, similar well, generalization of hydro where is quite complicated, but if you are got a good hang of hydro, then you can easily do all of it.



So, the equation for energy; now, maybe I should write down equation of energy. So, we wrote the equation for the non-linear term. So, I think let me write down and go we go back. You know, I do not have a slide.



So, let us right down here, equation for the energy.

So, what is equation for kinetic energy? So, it is $|u(k)|^2$ by 2. Modal energy you know this one. So, the first non-linear term we will give you u dot grad u. So, that will give you k dot u(q) u(p) dot. This in fact, have you done it $u^*(k)$ right. Sum over ps right, Is that correct? I mean this will come.

So, non-linear dot u star the next term is right hand side is minus ∇p. So, ∇p gives you 0. Right? Because, pressure is in the k direction and I am taking dot product u(k) is 0. So, you get only the term for B dot grad B, which is, this is i sitting here ok; i k dot B(q), B(p) now dot with what?

Student: u.

Dot $u^*(k)$ sum over p's, q is k minus p. So, this is I drop the viscous term and this equation for energy. Now, we write down equation for magnetic field. $B(k)^2$ by 2 and you get i k dot, k you know sorry k is inside, in the above k dot u of q.

B of.

Student: p.

p dot.

Student: B.

B star of k, and the right hand side is i k dot.

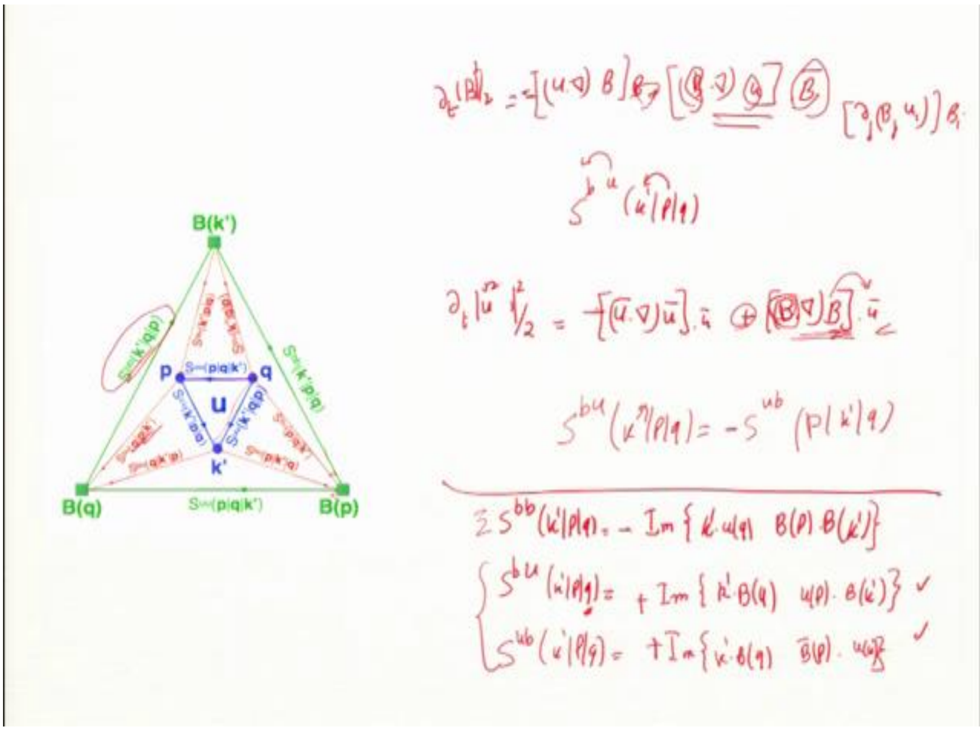
No, it is remember is B dot grad u.

Student: B.

So, k dot B, B of q, u of p. So, the difference here is who is the mediator? u of p dot B star of k, just straight forward ok; so, this is what I wrote in the next slide. So, you have two terms, now I just rearrange them. You take left hand side, right hand side. There is a u u transfer, which is coming because of u u terms. This is like hydro. There is no difference between hydro and formula and this formula. There are u B term which is this term and at this term u B term. So, this is coming off $(\vec{B} \cdot \nabla)\vec{B}$. Now you got, B B term which is here. Here u(q) advector. Remember if it is in the left, $(\vec{u} \cdot \nabla)\vec{B}$. Now that last one is.

Student: B dot.

$(\vec{B} \cdot \nabla)\vec{u}$, now which is here. Now, B I keep as total B. But, today's lecture I will not really emphasize, but it is very convenient to keep total B to see the effect of B naught, ok. But we will not delay with your on it, ok. Now from this, you can easily see that how to get to mode to mode energy transfer, right? That is what we did long time back in 2001 with my student Gaurav Dar. Sorry Gaurav Dar. His full name is Gaurav Dar and Eshwaran ok, 2001 Physica D, ok.



So, let us see the how the modes look like. So, let us focus on a triad, right. We focus on a triad only mode to mode is in a triad. So, we have three magnetic fields. Modes magnetic modes; B(k′), B(p), B(q), and three velocity modes u(p), u(q), and u(k′). Blue is velocity and green is magnetic term. Now, these are exchange of velocity among themselves, right. u to u transfer. Now, this is the B to B transfer, which is coming from which term?

The u dot grad B term; there has to be in B^2, mod B square by 2. So, this term will be the term responsible for B to B. It does not talk to you for energy transfer, its talking to u for mediation. So, here u is the mediator. So, for these transfers u is the mediator, but there is no exchange from u to B. Now, this red lines are energy going from u to B.

So, energy magnetic energy can grow or can decrease by this red lines, and that is coming from which term.

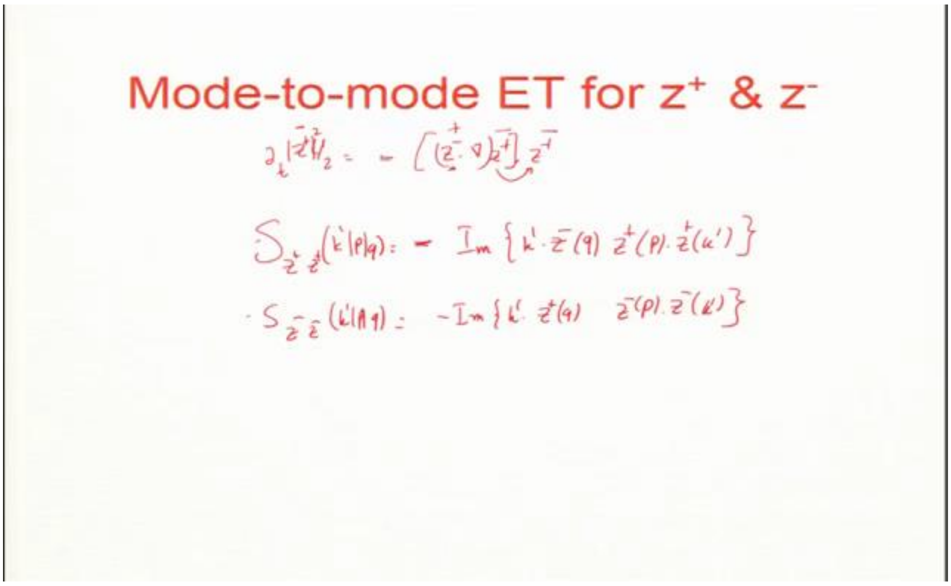
The other term B dot grad u dot with B, these are responsible for this. So, we write this S_{bu}; so, our notation is right to left. The giver is in the right, receiver in the left, ok. That is a notation we need to keep. And we write k′ p q. So, p is a giver k′ is a receiver. And we have red lines which are u to B. Is the energy transfer from B to u?

Student: B to u.

Yes, because if we write equation for u squared u(k) mod square by 2. Then we have minus u dot grad u dot this with u plus B dot grad B dot this with u. Well, I should not really write k, this is this one. This is in real space. So, this is the energy transfer from B to u. B this is a giver and u is a receiver. It is from green to blue. In fact, that should be equal and opposite know? So, you can see that, you can easily see from here if for mode to mode by property, k′, p, q. Here, u is giving to B. You can write this as u b.

Student: p.

p is the receiver, k′ q with a minus sign straightforward know. So, this is better with, this is like energy is here like a quantity, it is like probably money or any this is a property of transaction, any transaction; transaction by scalar quantity. That is what it will happen, you should follow this laws. So, you can follow similar derivation. So, what I have in the previous slide is combine energy transfer. Previous slide was, u(k) was getting from both p and q.



Now, we can derive in the same lines which I will not derive it here, but we can I will just give with the formula, no. So, I will just write down the formulas, ok. So, what are those formulas in fact, you can easily write them yourself.

So, let us write down the formulas. S_{uu} and S_{bb}, I do not need to write. Correct? Because you done it in the past. S_{uu} you have done before. S_{bb} will be like what?

Like passive, the vector which I did because that is advection, coming from u dot grad omega w. So, let us write for completeness i write, S(k′|p|q) is minus imaginary, k prime. So, is between B and B; So, B(p) dot B(k′), and u is a mid-vector or is a mediator so, u q; so, this as B(p), fine. Now this, we can write down how much goes this guy S_{ub}, S_{bu}. This is the S_{bu} term, this one. So, who is a mediator?

Student: This B.

This B is a mediator, this B right. So, to the left of grad is a mediator ok. That is what i, so.thing term coming to the left of grad. So, this as if you write this ∂_j B_j u_i dot with B_i. Correct? So, who is the mediator? Bj.

Student: Bj

Bj is the mediator. So, you will write k′; B is a mediator so, B of q. Mediator we write as q. That is our standard notation. q is the mediator here, fine and we get.

Student: B.

Who is the receiver?

Student: B(k).

B(k); B(k′). So, B(k′) and who is the giver?

Student: u(p).

u(p). So, u(p) is the giver; u(p). So, this is intuitive advector is to the left of grad and giver is to the right of this is the giver and this is the receiver, ok it is straightforward. What about S_{ub}? k′ p q minus imaginary. So, this coming from here right, S_{ub}, u is getting from B. So, who is the mediator?

Student: B.

This B is the mediator. So, k′ dot.

Student: B of.

B of q, and who is the receiver?

Student: u.

u of k′. This is the receiver. The u(k′). And who is the giver?

Student: B(p).

B(p), This is B is the giver, ok. So, these are the formulas which we can write down.

From this, we can get fluxes which I will describe in the soon, ok. So, this is the terms here and here they are responsible for energy going from one to other. Now, so you can see that there are u to u transfer, B to B transfer, u to B transfer and B to u transfer. But, if we look at z^+ z^-, the equations are simpler.

Now, without proof I am also telling you that if I look at I sum up over a triad of S_{uu}, this triads, they sum up to 0. Now, I am not proving it. These sum of, these S_{bb}, if I sum over all the modes in a triad they are 0 and if you sum these two together, then they are 0, ok. It has important consequence, but I will not repeat. I will not discuss, sorry, not repeat, I will not discuss, ok.

Let us look at energy transfer for z^+ z^-. So, here for z^+ mod square by 2. What is the form? It is z^- dot grad z^+ z^+. Correct? So, who is going to whom? z^+ is giving to z^+. This is giving to this. And who is the mediator?

Student: z^-.

z^-. So, it is straightforward. We can write down energy going from $S_{z+z}{}^-$ plus k′ p q is here. There is no minus sign, this is the plus sign. Where is the plus sign?

Student: Already minus sign.

This is the minus coming from, this minus or this minus here, but these are plus signs know?

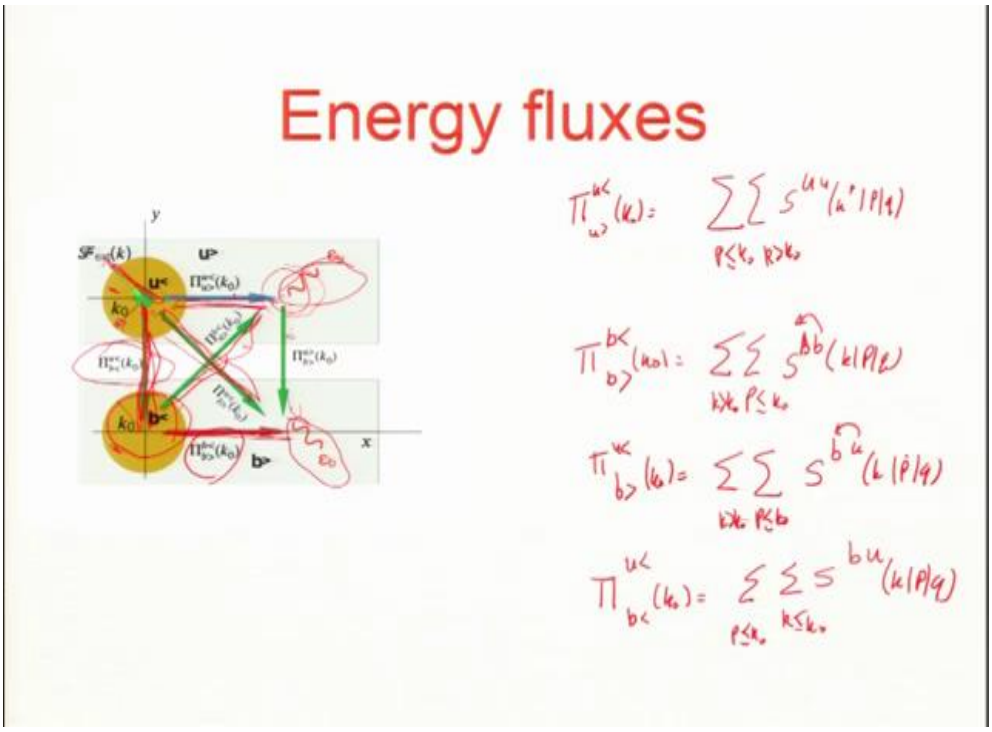
Student: Ok.

These are plus signs. So, the sign difference between S_b and Su and S_{bb} and S_{uu}, ok. Here this is a minus sign here. So, this is a minus sign here as well, the proof is bit involved, but, you can see the signs we have to be careful. I am doing mode to mode know, so minus imaginary, k′. Who is the giver? z^+ is the giver; z^+ is the mediator, z^- is the mediator.

Student: z^+ is the giver.

z^+ is the giver, p and z^+ k prime is the receiver. What happens with the minus sign, z^- z^-? Exactly same with minus flip to z reverse. So, let me just write this equation. So, these becomes minus, these becomes plus, these becomes minus, these becomes minus. This is total symmetry, and whether the B naught term does not do any non-linear transfer. These are Alfven term know, B naught dot k. That term has no non-linear, because there is only is a linear term. So, there is no non-linear transfer coming from that. This is interesting linear transfer, but we will not discuss today.

So, you can write down equation for z^- as well, k′ p q is minus imaginary, k′ z^+ of q, $z^-(p)$ $z^-(k^{'})$, ok. So, there is energy transfer from z^+ to z^+ and z^- to z^-, there is no transferred from plus to minus. So, this is convenient. So, in fact, these are better variables for turbulence modeling, if you are looking at energy transfers and flux then ub. Though u b are measurable, that is what you measure, magnetic energy kinetic energy, but they are very useful variables.



Now, what about the flux? So, these are the fluxes of ub, ok. So, we have two spheres. You can see there is u less sphere this one and b less sphere. So, these are, so basically there is only one Fourier space, but I am writing, I am giving you on b sphere and u sphere or the same radius ok, same radius. The both of them are radius k naught. Clear?

Now, so we can define many many 6 fluxes. So, what are those fluxes? So, modes inside this sphere for u to modes u modes outside this sphere. That is Kolmogorov. So, $\prod_u$ less to u greater. $\prod_u$ less to u greater of k naught that will be S_{uu}, right. That is going from u to u, correct, Suu k prime. Let us not use k prime k p q. Where is p and where is q? Where is k? p is less than k naught.

Student: k is greater than.

And k is greater than k naught. So, this is usual Kolmogorov. This is velocity stuff and the formula is exactly same as what we did for hydro. But, now we have 6 arrows. You can see they are red and green, the 6 arrows. So, this is b to b transfer as well, this is b to b transfer. So, what is b to b transfer? So, we write Π b less to b greater, this one, k naught. What is the definition? So, b less to b greater which is S_{bb}; so, b to b know. So, giver and giver, this is giver and this is a receiver.

So, k p q; giver is within this sphere, receiver is outside this sphere, fine. You can define from, let us say this one, this guy u less to b greater. So, what is that? ∏u less to b greater. So, now, what should I use now please tell me?

Student: u.

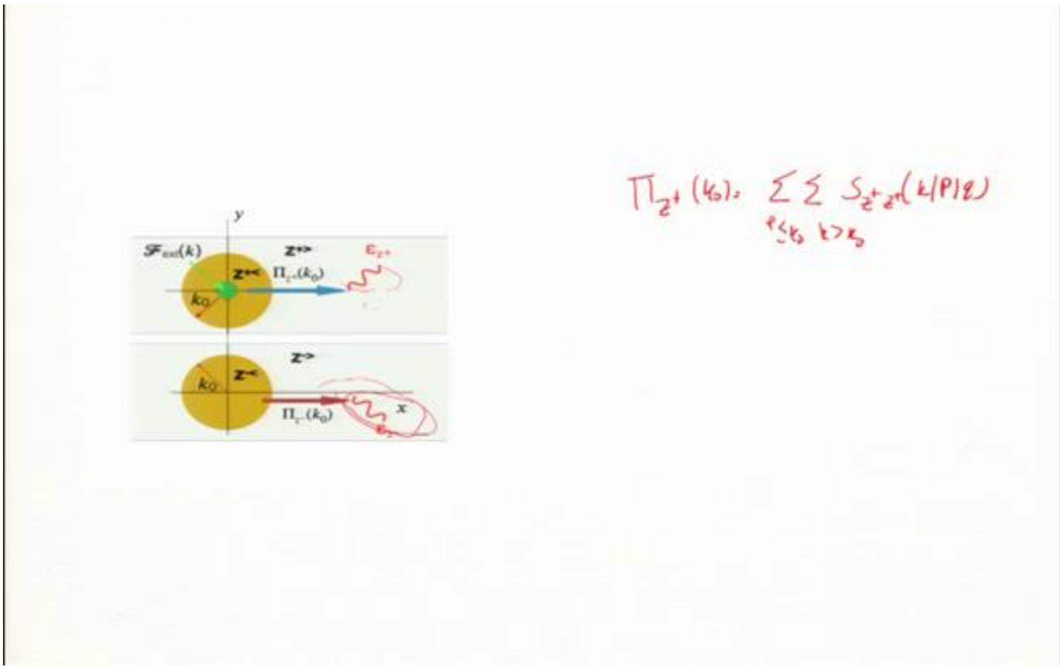
Yeah S_{bu}, u is the giver, so b receiver k p q. p less than k naught, k greater than k naught. Similarly you can define these guys, b less to u greater so, b is the giver u is the receiver. Now all these seems I mean why are you doing it, but I will tell you why we are doing it in 10 minutes, but this is interesting transfer this one, this one.

So, who is the giver and who is the receiver? ∏b less to b, no sorry, ∏u less to b less. Remember this one, you can see this in the screen is this small, this one, pi u less to b less. So, what should I write here? Is u is giver; b is the receiver, k p q. So, k is a less than k naught, because receiver is also in this sphere, but it is a different field. This is a b field in the same house. We have English, let us go we have some x people and y people and they are exchanging ok.

So, they are two different so, k less than k naught and p less than k naught, less than equal to I put, ok. So, just to and now you also define energy going from larger to larger. u greater to b greater, than the most probably outside this sphere. So, you just change the sign to greater, ok. And they are useful for dynamo ok, which we will discuss in 10 minutes, fine. Now, so these are six fluxes actually for u b variable, the 6 fluxes. What about z^+ z^-? How many fluxes do you expect?

Student: 2.

2, great, very good.



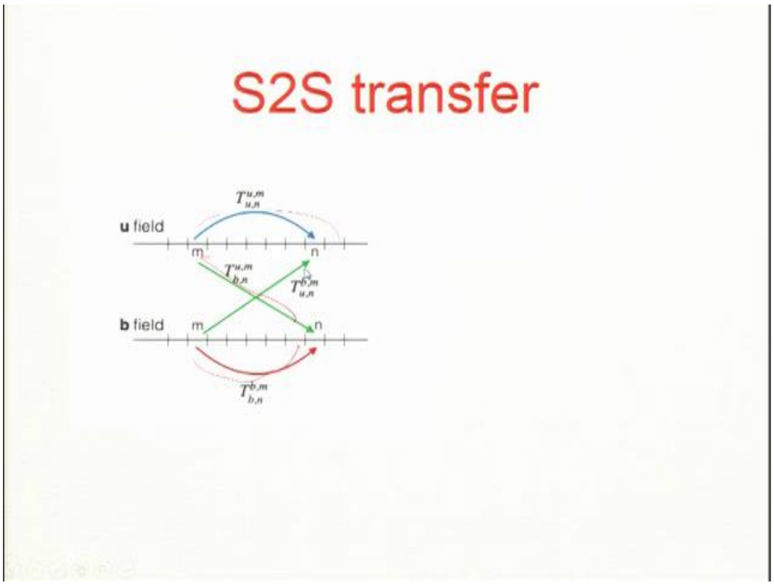
So, for z^+ z^-, there is only 2 fluxes. Now, this is z^+ channel and this is z^- channel. There is nothing across, so there is across ones are not there. So, we have $\prod z^+(k_0)$ which is Sz^+z^+ (k|p|q), k less than k naught. And now sorry p less than k naught you know, p is a giver and k greater than k naught. So, this is for z^+ and z^- will be just change z^+ with z^-. This is convenient this is only two variables.

Now, by the way these variables get dissipated. So, the dissipation here in fact, I am quite incorrect. Yeah, so there is a it is ok. So, these are dissipation here of z^+ and this is dissipation here z^-, ok. So, in the earlier slide, there is a dissipation of magnetic energy and kinetic energy.

So, let me just make a remark, because actually we need this. Here you find and all the magnetic energy here is dissipated at large wave number, right. So, that is why they are effective dissipation. A kinetic energy is dissipated here. So, how much energy is dissipated? Kinetic energy, is a just ∏ u less to u greater? This one or more?

Is more, because energy coming in is under steady state. You force at large scale here and here. The energy is going from this one, but then also this is another source of energy. B is giving to u. So, under steady state all the energy coming at large wave numbers must be dissipated. So, energy coming in here will be this plus this minus this, but the according to this notation is going away. So, whatever coming here net must be zero under steady state. So, there are lots of formulas we can derive on the properties of this fluxes, ok. For example, under steady state whatever energy coming in here, you know b is getting some energy; b sphere under steady state, this sphere should not get any net energy.

So, means whatever comes in is going into this and this channel. Understand know? So, if I have my income does not change, then whatever I get must be given away or dissipated. So, right now it is not getting dissipated because this small k and just distributed. So, you can define these laws for the fluxes.

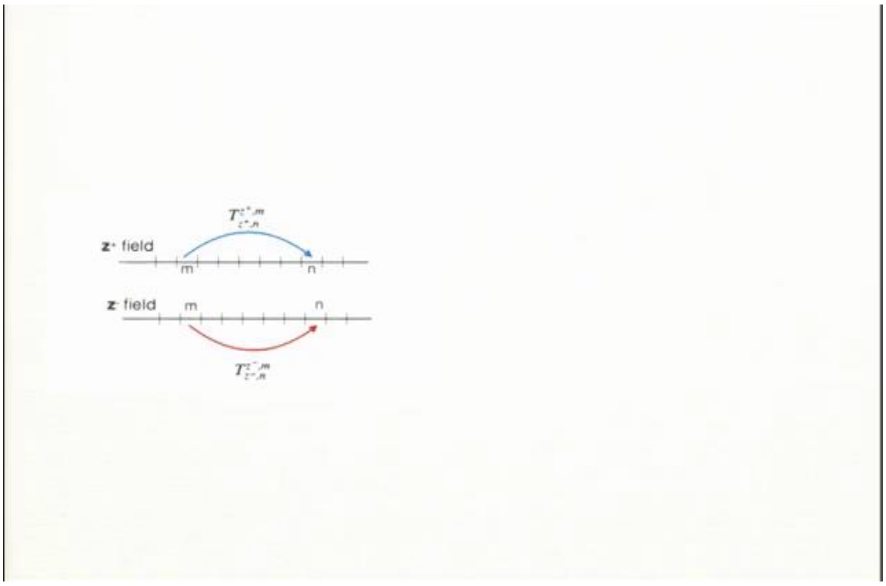


Ok, now we can also define mode to mode transfer, sorry shell to shell transfer. So, this is S2S shell to shell transfer. So, there is u to u transfer from shell m to shell n. There is a b to b transfer shell m to n shell, also we are transfer from u to b and b to u.

So, the 4 shell tran; shell to shell transfer into the ub variables, ok. We can define the, we already defined it, but the four of them, you should keep in mind. And for z^+ z^-,

Student: two.

Only two of them so, there is nothing cross.



And z^+ z^- is only two of them, ok. So, this is a nice way to compute these quantities, ok. So, we stop.

Thank you.