Physics of Turbulence

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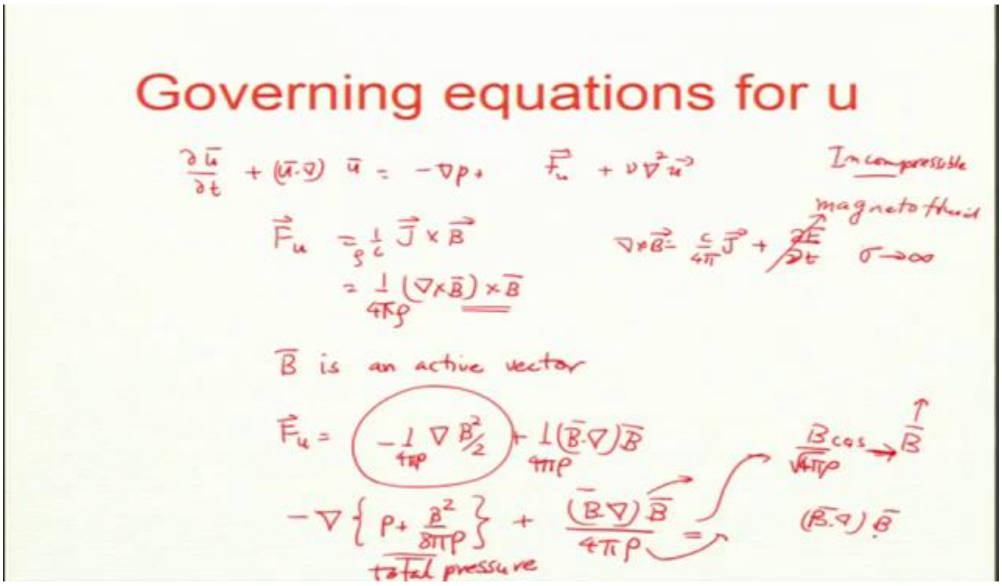
Lecture - 41
MHD Turbulence
Formalism

So I start the new topic MHD Turbulence, but this on the frame work of flow with the vector. Now magnetic field is the like a vector invaded in velocity in the flow; in the flow. So, the two vectors basically $\vec{u}$ and $\vec{w}$, but velocity field I found it is as if it is advecting the magnetic field ok. And, now we will use magnetic field as a $\vec{B}$ variable not $\vec{w}$ there is a standard notation in literature ok.

1. Formalism
2. Energy transfers and fluxes
3. MHD turbulence models
4. Dynamo

So, the topics will be formalism I will set the equation like energy transfer and fluxes. Then I will just describe few turbulence models there are many models, but I will describe what I considered to a considered important one and then dynamo ok. So, we will quickly go through these topics each of it is like one full course, but we will try to do in 1 hour and we will see.



So, formalism so let us first set equation for $\vec{u}$. So, we already have set equation for $\vec{u}$. We need to only worry about $\vec{F}_u$. So, that is there was formalism. In fact, I do like the way it has been written and it is very useful to think in a global way you know. So, let us rewrite that equation for $\vec{u}$, u ∇p plus $\vec{F}_u$ vector u plus $\nu\ \nabla^2\vec{u}$. So, what is $\vec{F}_u$? So, I am assuming constant density here. So, we will assume incompressible which is good approximation for many lessons, not so good for some extra physical applications. But it is quite good for sun or this they work quite well.

Sun, yeah sun is nearly incompressible roughly ok. Quite good I will not say best, but it is quite good. For galaxy also, but I will not discuss it privately we can discuss. So, there are ways to say that incompressible is reassembly ok may be I will make a remark. So, first no I think let us skip this because this will go into various approximations and so on $\vec{F}_u$ just think of liquid metal ok. So, inside that there is a lot of metal and there is a magnetic field. So, and it is like molten metal so it is like water you know. So, it is incompressible. So, that is what you should think in your head. So, what is the force density? So, this is the force density right and all of this is.

ρ equal to it is a force density force per unit volume ok. So, what is force density formula for in a electromagnetic fluid or it is called magneto fluid. You will find this is what fluid which is response to magnetic field water does not respond very well, but salt water will respond weakly. But metals will respond heavily free unit free electrons ok. So, $\vec{F}_u$ is $\vec{J}$

$\times \vec{B}$. Current density it is like it is a Lorentz force ok. Now, if you use S I or if you use CGS then there are different.

Student: Ok.

Constant in front normalization factor; so, I will use SI you know I will use CGS, I will use CGS unlike you had other forces, but I will use CGS ok. So, plasma I think lot of you will use CGS so you will use c 1 by c; c is speed of light. Now under MHD approximation, so this is another Maxwell equation which is d by d t no d$\vec{B}$ by dt is J minus so 4 π by c J minus no $\nabla \times \vec{E}$.

Plus or minus I am I should not.

Student: Sir, $\nabla \times \vec{B}$ I think.

Oh yeah it $\nabla \times \vec{B}$ $\nabla \times \vec{B}$ and these are d$\vec{E}$ by dt right.

Plus.

Student: plus.

Ok, but in MHD approximation we drop this term. So, we assume that electric field is much smaller than $\vec{B}$. Now I will not describe these approximation justification and so on with these what is assumed. So, electric field is much smaller is highly conducting fluid. So, conductivity is tending to infinity and for that this is a good approximation. So, we will focus on energy transfers, fluxes, models. We will not discuss too much into this details ok.

So, if you put that in we will get c c. So, 4 π by c; you know c by I mean too many mistake c by 4 π so c c cancels. So, you get a $\vec{J}$ is $\nabla \times \vec{B}$ curl B by 4 π and then the density is I am assuming one right now, but the density also comes. So in fact, forcefully there is a density should be sitting here first force density, so we divide with ρ ok. So, this formula for $\vec{F}_u$, so force on u is depends on B so; that means, B must be passive scalar or active scalar.

Student: Active.

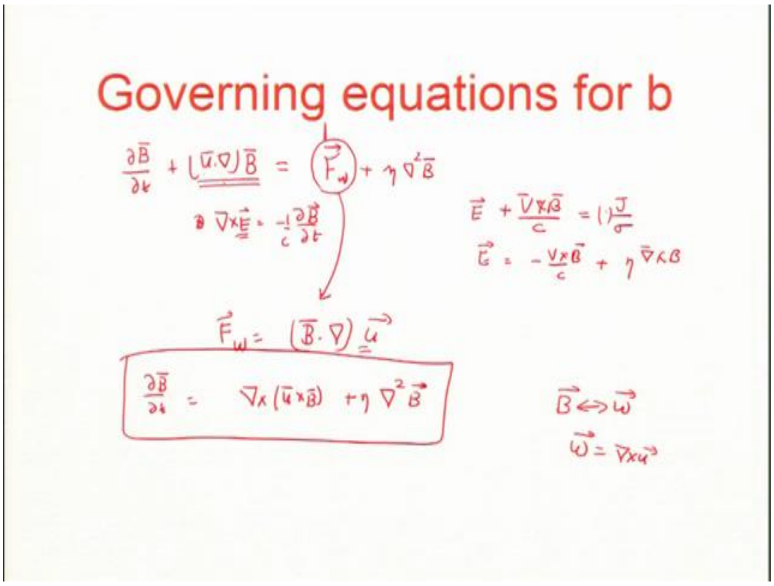
Active scalar, so B affects u. So, B is a active scalar an active scalar you know active vector sorry not active, $\vec{B}$ is an active vector ok. So, our formula which I derived for passive vector will not work ok. So, and moreover the force is quite non-linear know.

Is B B which is a non-linear force unlike buoyancy force which is linear buoyancy was theta you know or density. So, this is non-linear force. So, that makes life reassembly complicated for MHD ok. So, this is what we have to keep in mind that a force is curl of $\vec{B}$ **X** $\vec{B}$. In fact, we can simplify this force let me write it here. So, we can write curl of $\vec{B}$ **X** $\vec{B}$ is minus.

Curl of B^2 by 2 this is 1 by 4 π ρ is outside, plus $(\vec{B} \cdot \nabla)\vec{B}$. So, this is the interesting form. Why it is interesting? Because this part can be observed with pressure right, so we can take this part. So, I say minus grad p plus B^2 by 8 $\pi\rho$ and the other part is $(\vec{B} \cdot \nabla)\vec{B}$ by 4 π ρ interesting know so this is called total pressure.

So, it is a hydro pressure p plus B squared by 4 pi rho which is 8 pi rho which is coming from magnetic pressure. So, it is p is called thermodynamic pressure plus magnetic pressure ok. So, these are total pressure and this is the so I am missing out some vector signs, but these are all vectors. So, this is our force which is non-linear where this is also non-linear know, but I have you remember the pressure does not participate in energy transfers.

So, this term will be silent on energy transfers and we make another change of variable. So, I will make a remark here in CGS you make this variable $\vec{B}$ by $\sqrt{4\pi\rho}$ this as dimension of velocity ok. Now this $\vec{B}_{CGS}$ this is dimension of velocity, so we make a change of variable B CGS to this. So, this has very convenient know now $\vec{B}$ has dimension of velocity. So, we can write this as this part is B dot $(\vec{B} \cdot \nabla)\vec{B}$ and 4 $\pi\rho$ is disappeared ok.



So, now let us go to velocity field magnetic field. So, magnetic field equation you will again borrow the equation for active vector or for the vector. So, now instead of w I am just writing as B. So, $d\vec{B}$ by dt plus $(\vec{u} \cdot \nabla)\vec{B}$. Now the right hand side is F w plus η grad square $\vec{B}$. So, I need to derive this some Maxwell equation. So which Maxwell equation I should use? So, we have one equation is curl of $\vec{E}$ is minus $d\vec{B}$ by dt.

And in CGS E and B have same dimension. So, there must be 1 by c here know ok. Now these are nitpicking on dimensions, but I will not I will basically ignore units I rather I will ignore this constraints, but this there must be c. Now E in I mean that MHD approximation is minus V. So, in the moving frame of the fluid if we go with the fluid the electric field inside is 0 for ideal flow.

Now, electric field inside a conductor is 0. So, $\vec{E}$ plus so by Maxwell not by Maxwell by the relative c transformation electric field inside the fluid is this, under u by c must less than 1 ok. This is a transformation of $\vec{E}$ and $\vec{B}$ under relativity so you will find this is my E prime ok. So, this will assume to be 0, but if the conductivity is large then this is J by σ. Now, if you want to want to get this proper units no I do not derive it here ok.

So, there is a $\vec{J}$ by σ, but there is a some appropriate factor ok. So, I can find E know, $\vec{E}$ is minus $\vec{V} \times \vec{B}$ by c plus $\vec{J}$ by σ which is curl of $\vec{B}$, in fact η comes here made it diffusivity here ok. So, plug this in if we plug this in for electrical field here, you plug this in then I will get a term for F_w. So, this is you define this is how you derive equation for $\vec{B}$. So, I write down the equation now so F_w under my new units, my units were B is in the terms of velocity F_w is B dot.

Student: Ok.

Grad $\vec{u}$ ok, this is just simple right it is nothing complicate. So, you have to use the formula curl of $\vec{A}\times\vec{B}$ or curl of $\vec{u} \times \vec{B}$ and you will get this. In fact, you can also write in other way $d\vec{B}$ by dt is curl of $\vec{u} \times \vec{B}$. Now these derivations I am not emphasizing because lack of time, but these are interesting derivations. Now these equation looks very similar to another equation we will just seen in the course.

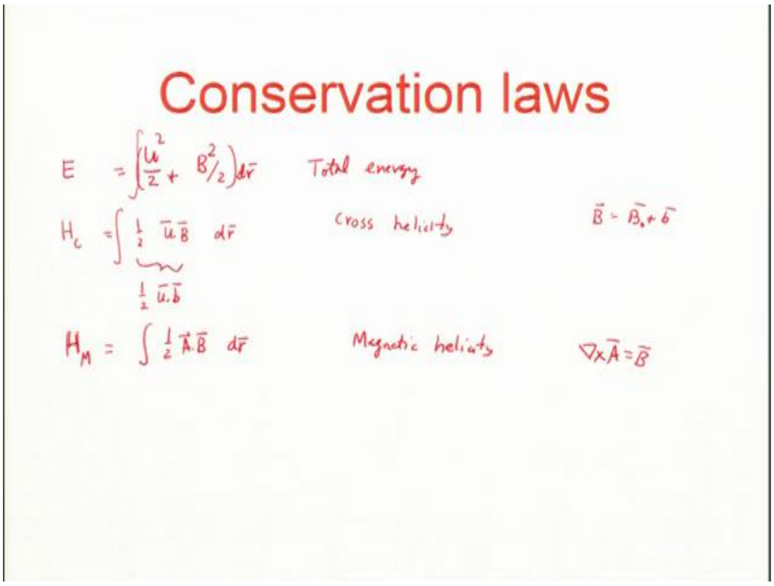
Student: Vorticity.

So, equation was the vortictiy know. So, B and w are similar in fact, they are not identical, but the equation is equation is identical. But they are not same because vorticity is.

Student: Curl.

Curl of $\vec{u}$, but $\vec{B}$ is a independent vector that is the difference, otherwise equation is exactly the same else is some of the dynamics. So, vorticity is stretched by the velocity fluid and this is called stretching of the vorticity you know same thing happens for the magnetic field. Magnetic field is stretched by the velocity field and that some of the magnetic field increases in dynamo. So, magnetic field is generated by this process of stretching of magnetic field ok.

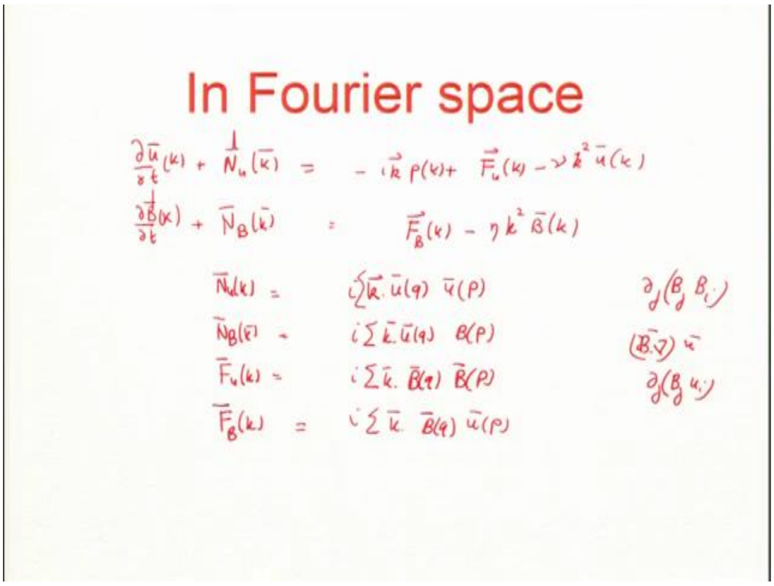
So, may be briefly I will describe it in the end ok, so this is the equation for $\vec{B}$ now. So, this is the advection term and this is u is acting on B of course, so this is like $(\vec{B} \cdot \nabla)\vec{u}$. But it has very similar form is $(u.\nabla B)$ you know except B has come before and u has gone later. So, these are similar non-linear terms.



So what is the conservation laws? Now because F_u and F_b; u squared is not conserved, F_u is not 0 right. I mean if you put in magnetic field I cannot set the force to be 0 and F_b is also not 0. So, they make the conservation law quite complicated.

So, u^2 is not conserved B^2 is also not conserved, but together they are conserved, this is called total energy. So, integral nah integral of this quantity over whole space is conserved. Another quantity called H_c it is called cross helicity which is defined as half $\vec{u} \cdot \vec{B}$. So, $\vec{u} \cdot \vec{B}$ right now I am keeping B as a total field it can have a some mean component. So, B field could be $\vec{B}_0$ plus $\vec{b}$, but when I put a $\vec{B}_0$ u is fluctuating. So, $\vec{B}_0$ will not contribute know such that we will get cancelled $\vec{u} \cdot \vec{B}_0$ will be 0.

So, this is same as half $\vec{u} \cdot \vec{b}$ ok. And there is another consecutive is constrained a which is constant is kinetic helicity, a magnetic helicity. Magnetic helicity which is half $\vec{A} \cdot \vec{B}$ is a vector potential ok. So, curl of A is B. So, proof I will not discuss, but these are three conservative quantities for MHD, u^2 and B^2 are not conserved individually the total is. This is for 3D; 2D has certain modification, but we will skip 2D discussion ok.



Now Fourier space how the equation look like? So, is I mean it is straightforward if you just have to look at your notes and pick up the threads. So, du dt plus Nu, so my notation is the in this book is ok. So, I write like $\vec{N}_u(\vec{k})$ and minus i $\vec{k}$ vector p(k) plus $\vec{F}_u$ of k is force. So, it is function of $\vec{k}$ and minus ν k^2 u(k). Let us something that coming here ok. I am going to write $\vec{N}_u$ and $\vec{F}_u$ soon and for $\vec{B}$ $\vec{N}_B$ of $\vec{k}$ non-linear term coming from u dot grad B and this is a $\vec{F}_B$ of $\vec{k}$ minus η k^2 $\vec{B}$ of $\vec{k}$.

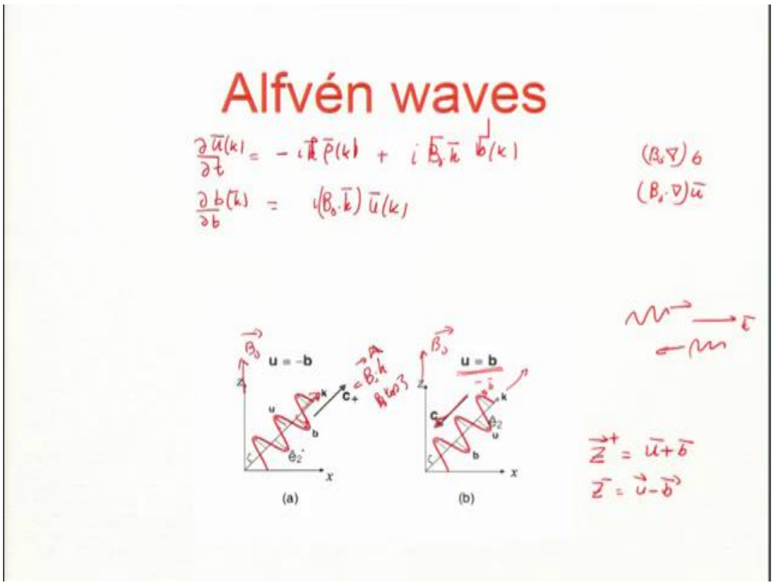
Now, $\vec{N}_u$ of $\vec{k}$ you know already right. So, let us for completeness I write it here $\vec{k}$ dot $\vec{u}$ of $\vec{q}$; $\vec{u}$ of $\vec{p}$, $\vec{q}$ is $\vec{k}$ minus $\vec{p}$ correct. What about $\vec{N}_B$ of $\vec{k}$? All vectors know no minus this is plus i $\vec{k}$ dot $\vec{u}$ (q) $\vec{B}$ of $\vec{p}$; $\vec{F}_u$ it is $(\vec{B} \cdot \nabla)\vec{B}$.

$\vec{B}$ is also divergence free right. So, you can write them as ∂_j B_j B_i.

Yes. So, we know what is ∂_j u_i u_j so it is very similar. So, i $\vec{k}$ dot $\vec{B}$ of $\vec{q}$ $\vec{B}$ of $\vec{p}$ right it is straight forward and F B of E sorry F B of k. So, this comes from $(\vec{B} \cdot \nabla)\vec{u}$ remember I just derived it in the last slide. So, this will be del j Bj u_i. So which is advected?

Student: Bj.

Bj is one with cons with thing. So, it is going to i $\vec{k}$ dot $\vec{B}$ ($\vec{q}$) $\vec{u}$ of $\vec{p}$. So, it is straightforward and these are the terms which will exchange energy among modes, they transfer energy from one to other ok. So, you see that very nice way to interpret these transfers and derive flux formulas.



So, I will make one more discussion is Alfven waves. So, these are the waves so in hydrodynamic flows they no really wave. Though some of you have got ODE's which are not like wave solutions, but there are oscillations right. Like we found that three more model has oscillations, but there is no wave really, but MHD has waves.

If you turn off the non-linear term we get wave solution and what are the wave solution. So, let me just make it linear equation. So, B gives you one term know so the pressure part is minus i $\vec{k}$ p of $\vec{k}$. So, this p is total pressure ok, but this is a term comes linear term is $(\vec{B} \cdot \nabla)\vec{B}$, so B is a total B. So, I can write as B naught dot grad B naught dot grad B right; so, I can write this as in Fourier space i B naught dot k b of k.

So, this is the term coming from $\vec{B}_0$ and the equation for B also has the term which is coming from $\vec{B}_0$. In the right hand side remember you have $(\vec{B}_0 \cdot \nabla)\vec{u}$ right. So, these are linear terms because $\vec{B}_0$ is a constant. So, the term which comes from is i B naught dot u of k ok. So, these are two linear equations I turn off viscosity and diffusivity. So, we can solve for them now is there in the notes you solve then you get two solutions.

So, we apply magnetic field along z direction. Then if the wave vector is along any direction wave vector can be along any direction. So, this is a direction k, then you get a solution where u and B are in the opposite direction. Can you see that, u is here and B is there.

Student: Yes sir.

And they will oscillate and also the wave will move wave moves with c plus Alfven's field and the c plus is $\vec{B}_0 \cdot \vec{k}$.

Student: k cap.

So, the Alfven field is B naught dot k cap so thanks to Naren. So, this is because it has dimension of velocity only if it is k k cap. So this angle is ζ, so B naught $B_0 \cos \zeta$ is the magnitude of the Alfven's field which is the different for different zetas. So, that is why it become 0 for ζ equal to 90 ok. This derivation I am skipping lack of time is nice derivation you can do it yourself. But is so c plus moves with B naught k, a solution is u and B are opposite and they are equal. So, there is a beauty of this unit that u and B have same dimension and in fact, they are equal so the equipartition know.

So, if you look at kinetic energy and magnetic energy on the average they are equal. So, this is another solution. If B naught is in that direction same thing, but k is going rightward, but the wave moves in the minus k direction ok. Wave is moving in c minus in that direction with the same magnitude B naught dot k, but it goes in the opposite direction. So, this is the wave which is going in that direction the wave going in that direction for given k and for this u and B are in the same direction.

So, if u is here and B will be same ok, so these are called Alfven waves and these are fluid mode. But these are wave modes in a MHD, and they have very important role to play in the dynamics ok. Now this I am very briefly describing it. Now we also have variable was z^+ and z^-.

Any questions?

Student: Small b dot k naught k.

Small.

Student: Small b dot k it is also 0 or not?

Yeah that is 0 because incompressibility.

Student: Same.

So, small b is perpendicular to the k right. So, fluctuations are perpendicular to k always. There is one subtle point what happens when the wave is along x axis.

Student: k.

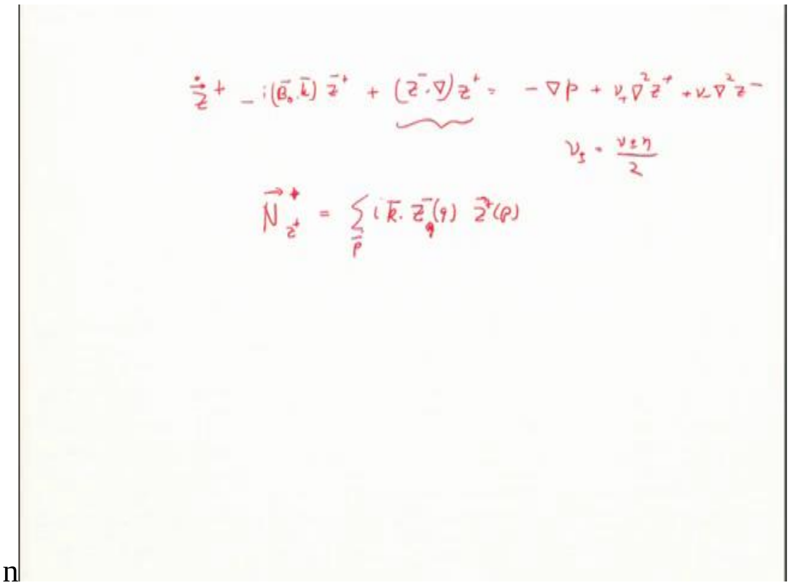
or k is along x axis. What happens to B naught dot k?

Student: 0.

Is 0 know.

Cos 90 is 0; so there is no wave on the plane on the equatorial plane. So, theta equal to 90^o has no Alfven waves ok. And, so they are non-linear term plays a big role on the wave on that on the plane. Now, we have two variables z^+ and z^- we define it for convenience u plus B and u minus B ok. So, given B you can write always z^+ z^-

Sum in difference and the equation is a much nicer in z^+ and z^-. Ok so let me write z equation in fact, this is just for, in fact we do lot of work with that.



n

So, let me write the equation for z^+ z^+ dot vector. So, this is a mean magnetic field ok.

It is a minus i B naught dot k z^+ plus z^- dot grad z^+ is minus grad pressure plus viscous terms ok, but I am not. So, let me write this I will not really discuss this terms $\nu_{\pm}$ r ν plus minus eta by 2. Now, equations are nice know because now we do not have 4 non-linear terms. We have one z^- dot grad z^+ this is for z^+.

What happens to z^-? We just flip plus to minus is symmetric and plus minus and except this sign becomes plus, plus wave moves in opposite direction ok. So, these are convenient non-linear terms which I will discuss in the next set of discussion. So, what will be the non-linear term by the way N in Fourier space N^+?

In fact, I write this is the N z^+ to differentiate between helical plus and helical minus, but too many plusses if you write for everything. So, the helical plus and helical minus know. So, that so I take N z^+ which is going to be i $\vec{k}$ dot z^- of q and z^+ of p ok. So, this is my equation sum over p ok. So, I think this is the last line.

Thank you.