Physics of Turbulence
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Lecture - 4
Basic Hydrodynamics: Conservation Laws

So, last 2 modules, we covered basic equations and vorticity right. Now I will cover conservation laws in the third module for hydrodynamics, okay.

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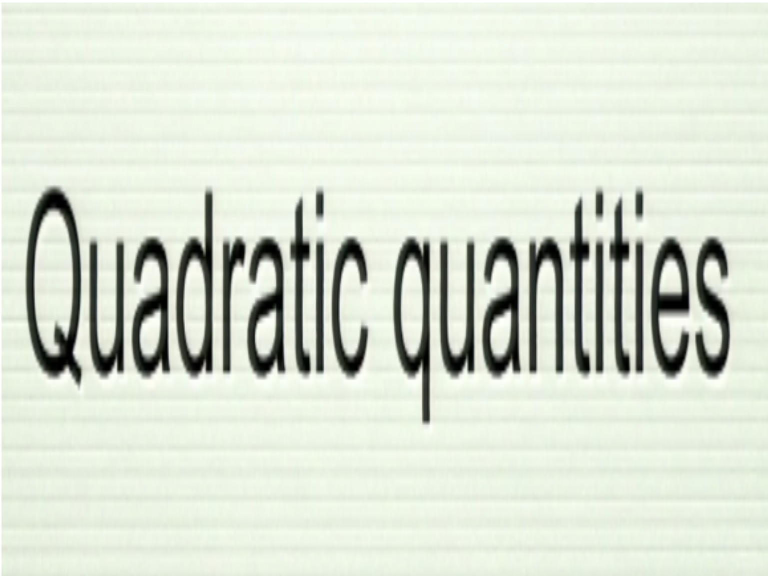

So first we will look at quadratic quantities which are very important for describing fluid flows. Quadratic means product of two fields and there, just with velocity, we can construct several quadratic quantities.

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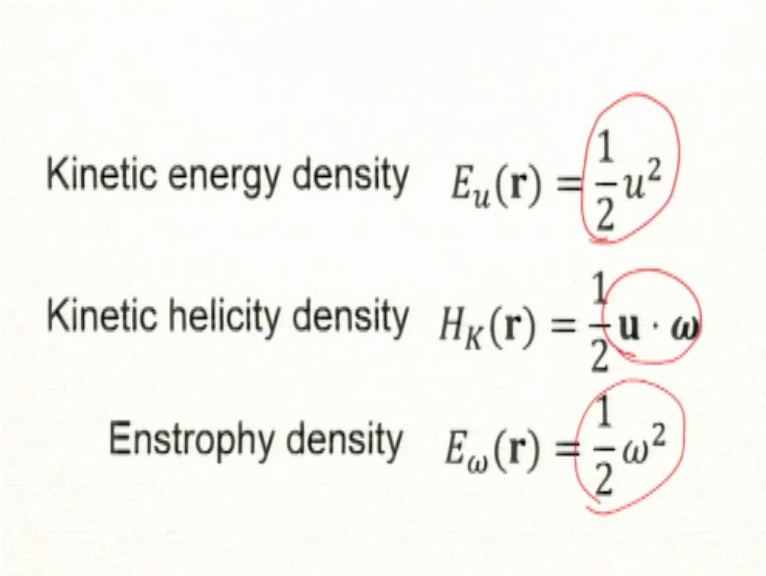

So first one is kinetic energy density. So I am going to be specific uh for the definition okay. So $\mathbf{u}$ is the velocity field, right. So we call 1/2, I put a factor 1/2 that by convention, no, so for particle of mass m, $\frac{1}{2}mv^2$. So similarly, we just say kinetic energy density is $\frac{1}{2}u^2$. Now density is 1, since density is constant, I don't carry density okay, so density is assumed to be 1 and this is in fact we can think of small, so since $\mathbf{u}$ is changing everywhere, so u^2 will change everywhere.

So you can think of even take a small volume dv, then energy content in that volume will be $\frac{1}{2}mv^2$ times z, so that is why it is called density per unit volume where the volume is small okay. So this is kinetic energy which everybody knows, but it is important to keep in mind this definition. Kinetic helicity. So you studied vorticity. So using vorticity and velocity field, you can define this helicity: $\mathbf{u} \cdot \boldsymbol{\omega}$, and factor 1/2, which is I, well, some books may not keep factor 1/2, but in my notation I always keep 1/2. So $\frac{1}{2}\mathbf{u} \cdot \boldsymbol{\omega}$.

So $\boldsymbol{\omega}$ is a vector, note please $\boldsymbol{\omega}$ is the vector. In 2D, it is only along z direction, but is a vector, is $\nabla \times \mathbf{u}$, so $\mathbf{u} \cdot \boldsymbol{\omega}$ is a scalar. So all these quantities are scalar which I am going to cover today and third one is called enstrophy and which is $\frac{1}{2}\omega^2$, so $|\nabla \times \mathbf{u}|^2$, okay. Now you may say why you have ω^2 and u^2separately, it turns out is it is useful to think in terms of $\boldsymbol{\omega}$ on many occasions which I will describe today okay a bit later. Now given that we have density, we can also construct total kinetic energy in a volume.

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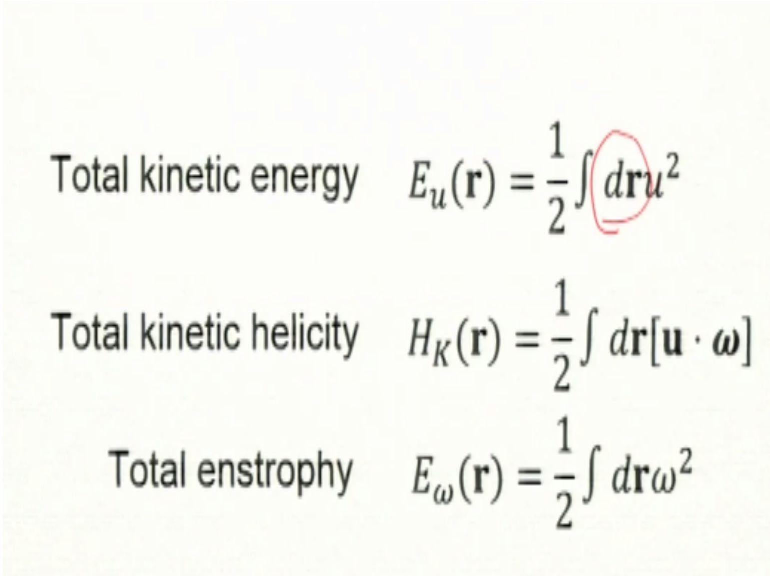

So given volume, you can construct total kinetic energy okay, we should just integrate our volume. So this means integrate over whole volume. Now volume could be periodic box or box with walls. So this is, the volume I have not specified, but total means over the whole volume. So similarly, you can construct total kinetic helicity which is again integral over the whole volume of $\boldsymbol{u}\cdot\boldsymbol{\omega}$ factor 1/2 including and similarly to total enstrophy, it is this. So density in totaled you must differentiate and when I say conservation means for the total.

So $\mathbf{u}^2$ will change, but if I sum over whole volume, I will show that $\mathbf{u}^2$ total kinetic energy for some conditions will be conserved okay. So now we have this quantity, so we can write down evolution equation for this. How does $\mathbf{u}^2$ change with time okay, so that is my next objective.

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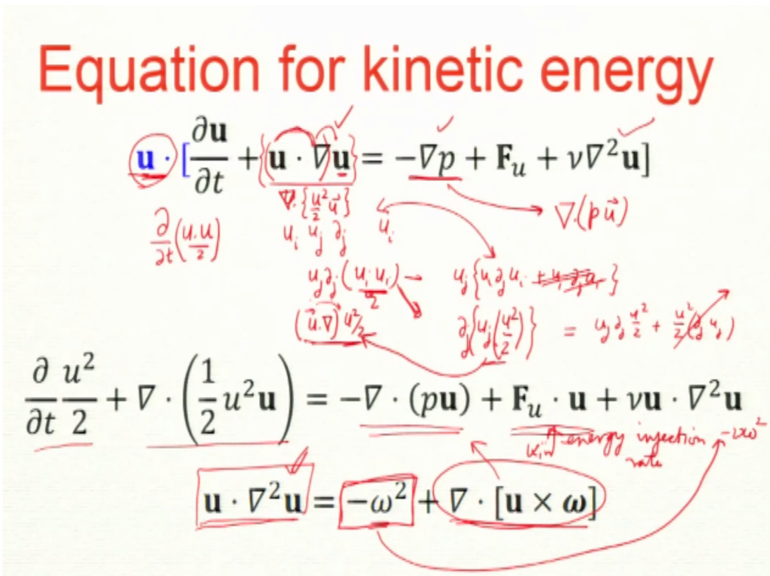

Equation for kinetic energy density okay, though I didn't write here, but kinetic energy density. So how do I derive this? So all these can be derived from Navier-Stokes equation, you can derive from the first principle. So I have the Navier-Stokes equation, this one, right, I dot this with $\mathbf{u}$. So what is the first term by the way, so this is $\frac{\partial}{\partial t}\left(\mathbf{u}\cdot\frac{\mathbf{u}^2}{2}\right)$ by product rule. it is easy to derive for incompressible flows. So, I am - let me use tensor notation okay. The tensor notation is very useful and should just do some exercises. So in fact for some of earlier derivation, no, I said curl of $\nabla \times (\mathbf{A} \times \mathbf{B})$, now those big expression.

Now you can derive this from first principles or using tensor analysis. So that will be one of the exercise I am going to give you okay. So, I will show you how to do this for with tensor analysis $u_i u_j$. So I want to construct write this in tensor notation, *ij* notation. So u_i and so this is *i*th component and this will dot with this, so u_i here. Now but then no index should repeat more than twice, if you do it, then you are making mistake. Two means sum, it already has been contracted, now you can't put another *i* because that will be wrong.

So now I have another dot product between $\mathbf{u}$ and ∇, so what do I do with the $\mathbf{u}$ and ∇? Should I use same *i*th index, no I should not use the same index, different index because two *i*'s are already covered, so use j, $u_j\partial_j$, so these are second index. You can use as many indeed you want, but dot means that those index should be repeated okay. Now this is what I got. Now so $u_j\partial_j$, now I can use product rule. So this u^2, so u I can go inside like this because if I expand this what will I get. You can expand this one okay.

So this expansion will be u_j is silently sitting outside, $\partial_j u_i u_i$ + same thing again, so twice okay, so you get again $u_j u_i u_i$, so I should put factor 1/2 that two of them, so I should put 1/2 factor to get, no not 1/2, ya ya 1/2 here. So, I want to get this one, so I put a factor 1/2, so this will be same, so this basically half - half of this will go okay, this is same as what I have written here, these two are equal. So what is this object $\mathbf{u}\cdot\mathbf{u}\cdot\nabla\mathbf{u}$, no, $u_j\partial_j$ is $\mathbf{u}\cdot\nabla$, right and this $u_i u_i/2$ is $\frac{u^2}{2}$.

So this is how you can derive quite easily using tensor notation. Now what about the next term $-\nabla p\cdot\mathbf{u}$? So since, $\nabla\cdot\mathbf{u} = 0$, so this is what you should keep in mind, $\nabla\cdot\mathbf{u} = 0$, so using $\nabla\cdot\mathbf{u}$ I can simplify this further. This one can be simplified further, so I think before I go to $\nabla p\cdot\mathbf{u}$

or $\mathbf{u} \cdot \nabla p$, let me simplify further. Can I push **u** inside, **u** vector? For incompressible flows, can I push **u** inside, yes you can push inside. So in fact, you start form here okay, we start from here. So this can be done as $\partial_j u_j$ and this $\frac{u^2}{2}$.

In fact, these are scalar quantity, this one. Now apply a product rule again. So one term is equal to that and second term will be, if I expand this one, it is $\frac{u_j \partial_j u^2}{2} + \frac{u^2 \partial_j u_j}{2}$. What is $\partial_j u_j$? 0, since $\nabla \cdot \mathbf{u} = 0$, so this goes away. So these 2, this one and this one, these 2 are equal. So in fact, you can write this object **u** dot this thing as divergence of **u** vector $\frac{u^2}{2}$ dot, this is scalar okay, is that fine? Now what about this minus $\mathbf{u} \cdot \nabla p$, can you go inside, so I am interested in this versus this, dot product.

In fact, yes, it can go inside, where again you apply the product rule, you will find that $\nabla \cdot \mathbf{u} = 0$, so this will be there. This term gives you this. Next term is $\mathbf{u} \cdot \mathbf{F}_u$, keep same, and the last term is $\mathbf{u} \cdot \nu \nabla^2 \mathbf{u}$, is a viscous term, which can be further simplified which I will show you below. So, I am going to erase this, this stuff because I think some equation is going to appear below. So this term is divergence of $\frac{u^2}{2}\mathbf{u}$, this is what will appear okay.

So these are 2 very important products, this one and this one will play very important role in conservation law okay and I am not putting sum, but sum is implicit. When I write this one, this means $u_x \partial_x + u_y \partial_y + u_z \partial_z$. So this is Einstein notation, if I have repeated sum then is added, so this is what I have written. So this is this one and this is divergence stuff, so which I have written on the top, this is coming as the pressure term, this is in fact this is force field. Now what is force times velocity? for particles? It is work done, no, work done for unit time is power input.

So is a power energy supply per unit time by force to unit volume okay. So this is called energy injection rate, kinetic energy. So you can write kinetic energy injection rate okay. The last term is dissipation because viscous times velocity is viscous force, so dissipation by viscosity. So we can simplify further which this identify I will not prove it, but is there in the notes. So it is nice to rewrite this one $\mathbf{u} \cdot \nabla^2 \mathbf{u}$ as this $-\omega^2$ and that, okay, and many times you see this appearing in the dissipation term.

But this $\boldsymbol{u} \cdot \nabla^2$ is not equal to $-\omega^2$ okay, so this derivation in notes, it is quite simple to derive, you will find thus two case, so replace this by $-\nu\omega^2$ and this divergence is one of the term which is absorbed here, divergence here. So this will become you can write either in terms of $\mathbf{u} \cdot \nabla^2\mathbf{u}$ or in term of $-\nu\omega^2$ okay. So you will find that there. This term, divergence term will become 0 if integrated for whole volume okay. So I am going to come to that in next slide.

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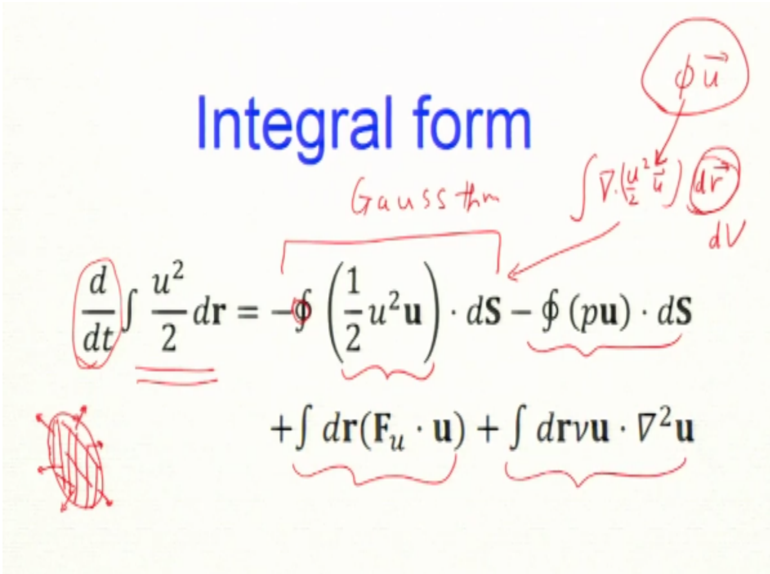

So the integral form, so what I showed you in the differential form, derivatives, but integrate this - the old form - to any volume, any volume okay, not the total volume, integrate to any volume. So the first term is *d/dt*. So this is coming from the old slide. Now by the way, I am not using partial, why I am not using partial, I am using total because this is not function of *x* anymore, this is energy of a given volume okay. So, I use total of a given volume. Now, what about the next term?

So remember it was divergence of u squared u, so now what is the theorem, it is called Gauss theorem. In the Gauss theorem, I have replaced the volume integral to surface integral okay. So if you recall that was divergence of $\frac{u^2}{2}\mathbf{u}$. Now I integrate over volume. So by the way $d\mathbf{r}$ means volume, this is the notation which you follow, various people follow various notation, it is in physics, this is quite common that where $d\mathbf{r}$ that means is a volume integral. So it is $dxdydz$.

Some people write *dv*, but I don't want to use it because the *v* some people will use for velocity, no, so I just follow this $d\mathbf{r}$, this means volume. This is coming from Gauss theorem, so this is divergence of a vector, $d\mathbf{r}$ is vector dot ds and this is over the close, full closed surface, this this this means closed surface. So, I do this flux over full surface. I had to integrate over the full surface. So by the way, this is flux of kinetic energy.

This is important which I am not going to emphasize in present course due to lack of time, but for any scalar quantity or any vector quantity, you can define flux by $\phi\hat{u}$, this for fluid mechanics, now in electrodynamics is there the different definition. For fluid, this flux is the scalar quantity or the vector quantity carried by velocity field. So it will be the quantity crossing per unit area per unit time okay. So this is flux of the quantity in real space, I will do some other flux which is energy transfers will be in Fourier space but that will be different okay.

So this is flux. So instead of ϕ this $\frac{u^2}{2}$, you have put this kinetic energy flux okay, density is one. Again, I reemphasize density is one, so $\frac{u^2}{2}$ is kinetic energy okay, that clear. Now this one is $\oint(p\mathbf{u})\cdot d\mathbf{S}$, so is a pressure flux. Pressure is convective. So pressure has in the $\rho = 1$ language. It has same dimension as energy. Pressure is ρu^2- ρ is 1 okay. Now, this is energy supply rate by external force and this is viscous dissipation. So, these all integral form, the earlier one is called differential form.

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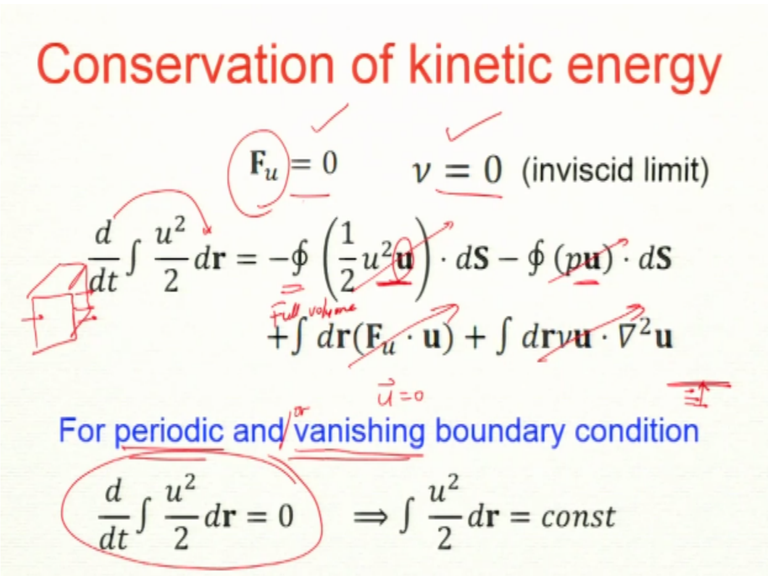

Now, we can go to conservation laws. So this integral I am going to do the whole box, kinetic energy integral. So let us take a special case when the external force is 0. External, I particularly mean one which is not pressure, other than pressure gradient and viscous okay. They are also a force on a volume, a force on the fluid element, but we mean this $\mathbf{F}_u$ means some gravitational force or magnetic force. So this force, I mean $\mathbf{F}_u$ okay. So this, any external force in engineering they call body force okay. This force is 0 let us assume that and we assume that viscosity is 0, so that means dissipation time is gone.

With this, what do I get. So this is called inviscid approximation, viscid is viscous, inviscid means without viscous okay, inviscid, this is the technical name. Now integrate this for the whole volume. Now this volume is full volume, not arbitrary volume. This could be box, cylinder, sphere, whatever you may like, this full volume okay. Now these are the 4 terms which you had, but these anyway are 0 because I make the assumption, this $\mathbf{F}_u$ and ν will just, this is copied from the old slide. Now you have only 3 terms.

"Professor – student conversation starts" Yes? Are we assuming that volume we are considering is not changing the time. Yes, it is a total volume, volume doesn't change the time, yes. So that is why the *d/dt* does not act on *d***r** indeed. **"Professor – student conversation ends."**

So now we will make another condition. So one condition, these 2 conditions already there, so in addition to these 2, we make another condition on the boundary condition. So this QR cylinder, we assume that periodic boundary condition. A cube is easy to see, for cylinder it may not be easy for like cylindrical wall, but vanishing means the velocity is 0, vanishing is **u**=0 at the wall, at all walls.

Periodic means for cube is easy to see this wall and this wall has same velocity at a given point, right, periodic function, say if I just go along x direction by box length, I get the same value along y, along *z* or you could also have a combination. So periodic along *x*, but vanishing along *y* or vanishing around *z*, so any combination of this. So if I have this, then what happens, it already got vanished. So what happens to this term. Any vanishing is easy to see. This term will be 0 because **u** is 0 at all the point, right.

This term also will be 0 because both *p* and **u** are vanishing, well only **u** is vanishing sorry, **u** is vanishing, so both the terms become 0. So vanishing will easily see these 2 sums gone. What about periodic, periodic whatever contribution I get from one wall gets canceled by the other wall. For one wall, velocity is going out, another wall velocity is coming in. So this integral **u** one has positive sign, other one has negative sign okay. So this cancels, so that is how period wall also gives zero contribution.

So the surface integrals vanish for periodic or and vanishing boundary condition. Equal also have combination, but combination is periodic along pair, so it is periodic along x, so for that means periodic has to have periodic in which direction, x direction or y direction or vanishing along z direction. So combination is also used. So we use that combination.

"Professor - student conversation starts." Sir vanishing means it is like some confinement or. So in this room, velocity at the surface is 0, so the fluid flow inside this room is confined okay. You know velocity must be 0 at a viscous wall. Viscosity makes the fluid flow having a 0 velocity at the wall okay. In fact it is natural way and vanishing boundary condition is natural for solid walls. But velocity may be tangential to the wall may be.

No for viscous boundary layers and for real solid walls all the components of velocity must be 0, all 3 components. The vertical velocity, okay let us take a wall, vertical velocity must be anyway 0 because it cannot penetrate. The horizontal velocity here is because of viscosity, so viscous stress makes the velocity 0 at the wall okay. Of course, away from the wall, it becomes finite, but at the wall viscosity makes it 0. **"Professor - student conversation ends."**

So as a result, what do I get for so these 3 conditions, force external force 0, viscosity 0, and we have this boundary condition, then we get u squared/2 dr $\int \frac{u^2}{2} d\mathbf{r}$ is the integral over the full volume is 0. What does it mean, $\frac{u^2}{2}$ integrated over the whole volume is a constant okay, so that means kinetic energy in a box is conserved, means it doesn't change with time, okay, it will remain the same.

So this is conservation of kinetic energy for fluid flows which are inviscid that means viscosity 0 and there is no external force and of course is vanishing and/or periodic boundary condition. Actually it should be, it can't have both at the same wall but ya it should be and/or okay. So, this we should, I can't have periodic as well as vanishing at the same time. So this should be clear, no, but anyway this is a technical English part, okay clear this derivation, it is important derivation, but these are the steps involved.

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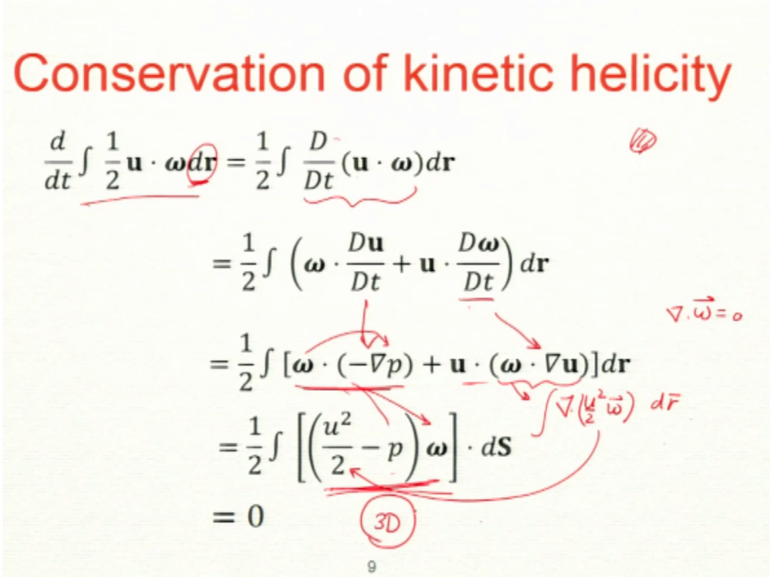

Let us look at kinetic helicity. So I gave the definition of helicity, it is $\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{\omega}$, I have to integrate for the whole volume, this one. Now, I need to take the derivative. Since my $d\mathbf{r}$ is not changing, this dr volume, so I take a small element. Since $d\mathbf{r}$ is not changing, I do not take derivative with relative to $d\mathbf{r}$, but I take total derivative with related to the function $\boldsymbol{u}\cdot\boldsymbol{\omega}$ okay, but this is total derivative *D*, capital D okay.

So again, I apply product rule, so product will be $\boldsymbol{\omega}$ then this is *d/dt* of the first function, this is product rule right, when $\boldsymbol{\omega}\cdot\frac{D}{Dt} + \mathbf{u}\cdot\frac{D\omega}{Dt}$. Now what is *D/Dt*, now again inviscid approximation. So assume force 0 and viscosity 0. So what is *D/Dt*? So again look at your notes, is ∇p. So this is $\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u}$. So this $-\nabla p$, the viscous term is 0, $\mathbf{F}$ is 0 right and ρ is 1. So this one is $-\nabla p$, first one, and what about second term.

So it is, we have done this before, is no no it is not 0 for 3D, stretching term, $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$, this one, $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$, this is the term okay. So, this is $-\nabla p$ and this $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$. Now what about this first term, this one. Since $\nabla\cdot\boldsymbol{\omega} = 0$, right, because divergence of curl of a vector is 0, so if any of these condition is there and divergence free vector can be pushed inside okay, a simple algebra that taken also push both inside.

So this can be written by following the similar steps which I did for the last derivation, these divergence of $\frac{u^2}{2}\boldsymbol{\omega}$ okay. So this I will leave it for you to derive, it is following pretty simple steps, but please cover this, do it yourself. So this is what we will get. So this is the first term is here, second term is here. Now, I apply Gauss theorem again. So divergence of this $d\mathbf{r}$, what will that be, by Gauss theorem it is $\frac{u^2}{2}\boldsymbol{\omega}\cdot d\mathbf{S}$ okay and the first term will give you, this one will give you this, it is Gauss theorem.

Now for vanishing boundary condition or periodic boundary condition, what this will be. So **u** is 0 or $\boldsymbol{\omega}$ is 0, periodic boundary condition will give you okay, and so this thing is 0, this term is 0 for periodic or vanishing boundary condition okay. So, we get 0. So, kinetic helicity 2 is conserved for 3D flows okay. So, this is for 3D, what about 2D, in fact 2D, what is $\mathbf{u}\cdot\boldsymbol{\omega}$ is 0. So, kinetic helicity itself is 0 trivially okay, so 0 remains 0, it doesn't change, so that part is obvious okay.

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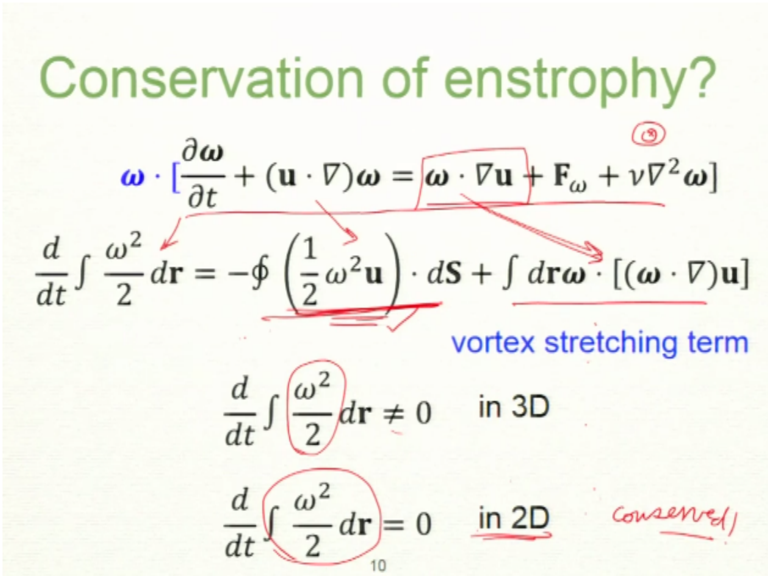

So go to the next slide. So left go get enstrophy, is it conserved or not. So we will go step by step. So we start with enstrophy equation, well we start with first vorticity equation. This I derived it in the last lecture. I dot product this one with omega. So the first term is straightforward. So, first term is added okay. Now, in fact I have to save steps, so I already written in the integral form. So this is this term. This term will give you this one, now I am not deriving it, but you can easily derive, and the third term is this.

Now, this term can be written as divergence of vector and that is how it converted using Gauss theorem two of suffice term, but this term cannot be written divergence of that form okay. There was **u**, it would have gone in, but this $\boldsymbol{\omega}\cdot\boldsymbol{\omega}$, you try it, you cannot convert it to divergence of a vector okay. So for periodic and vanishing boundary conditions similar way, this term will go to 0 okay. This term conserves, but for conservation you want both the terms to go to 0 for the whole volume.

The first term will go to 0 for periodic and vanishing boundary condition, but the second term does not. In fact, second term is the vorticity stretching term. As I said, you know, the vorticity if the velocity field that will stretch the vortex, so that way vorticity becomes thinner and omega speed increases, the vortex tube becomes thinner. So this, the second term does not become 0 for the whole volume and that is called vortex stretching term. So that is why in 3D, this $\frac{\omega^2}{2}$ integral is not 0. So enstrophy is not conserved in 3D okay.

It is straightforward this one, but is it conserved in 2D. So what happens, what is this term, $\boldsymbol{\omega} \cdot \nabla\mathbf{u}$? 0, because $\boldsymbol{\omega}$ is along z direction and grad is along x direction no, so this $\boldsymbol{\omega} \cdot \nabla$ is 0, so in fact this vortex stretching does not happen in 2D, these are infinitely long vortex tubes which can only move, there is no use to stretch it. So for 2D, the vortex stretching is 0. As a result, $\frac{\omega^2}{2}$ is 0 integral. So $\frac{\omega^2}{2}$, this integral is constant in 2D.

So enstrophy is conserved in 2D okay, in 2D conserved, but not in 3D. So summary is kinetic energy is conserved for both 2D and 3D, kinetic helicity is conserved in 3D, of course it is 0 trivially in 2D, but enstrophy is conserved only in 2D, not in 3D okay for pure hydrodynamics, inviscid hydrodynamics. Now, these are 3 very important quadratic conserved quantities. So, you saw all of them are product of 2 vectors.

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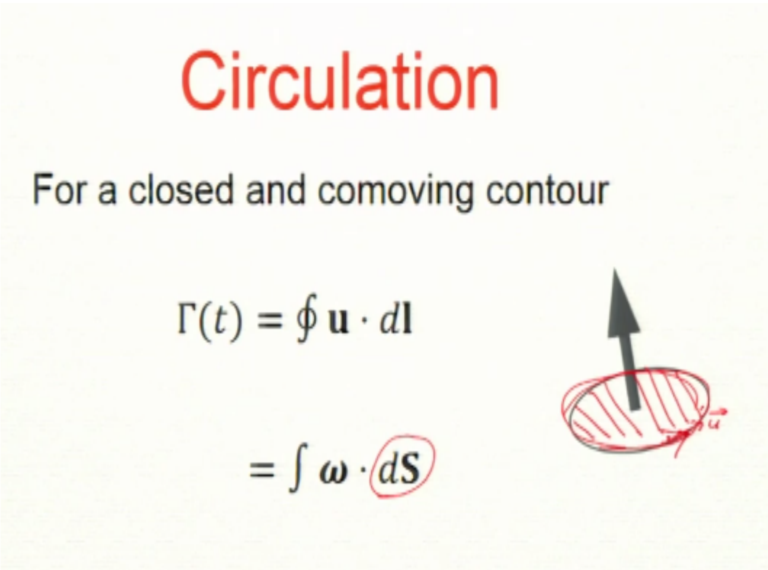
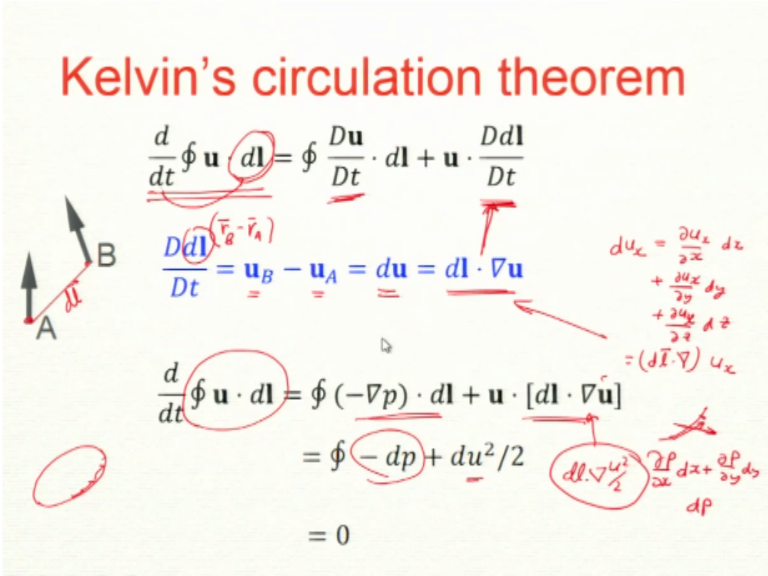


Now there is a quantity called circulation which is not product of 2 vectors. It is a very important quantity and so it is defined like this. For closed and co-moving contour, co-moving means it moves along with the fluid. So just think of you know this cigarette smoke and you just follow that smoke, one fluid parcel, so make a contour, so I am just I have a picture here. So this contour your particles are sitting here of the fluid elements and when the fluid element moves, your contour also move with it.

So this contour moves along with the fluid parcel. I will not call particle, we are not thinking of particle, but it moves with the fluid. You tag this fluid element, it is called co-moving, you move with the fluid okay. You hang onto the fluid and move. So for this contour, I have velocity field at every position, this is a velocity field, $\mathbf{u}$ field here. So I take a dot product. So $\oint \mathbf{u} \cdot d\boldsymbol{l}$ integral over the whole closed loop, it should be closed loop, this is a closed loop here. I integrate over the whole loop, $\oint \mathbf{u} \cdot d\boldsymbol{l}$ and this is called circulation.

So circulation of velocity you can think of. So velocity you have to integrate at every small line elements and you just sum it up. I can use Stoke's theorem, what does Stoke's theorem say? This line integral where the closed loop equal to surface integral for any surface whose edge is this contour, is not a closed surface, is an open surface. So you can think of a cap, so the cap's edge is what we have here, cap's edge, and the cap surface which can be moved anywhere you like, if this really is a topological result, is true for any surface as long as the surface edge is same as this closed curve.

So this is $\boldsymbol{\omega} \cdot d\mathbf{S}$, and this $d\mathbf{S}$, the surface, actually whole surface is the one whose edge is the contour, is not a closed surface okay. So, I am not writing a closed stuff here, it is open surface. Now this is circulation. It turns out for inviscid flows, viscosity is 0, this quantity is also conserved, but this is not a quadratic invariant, but this is very at least one of the important theorems and this conservation law goes to Kelvin okay, so this is called Kelvin circulation theorem and according to Kelvin circulation theorem, *d/dt* of $\mathbf{u} \cdot d\boldsymbol{l}$, this one, in fact shown to be 0, the proof is quite straightforward. So there is one difference between the earlier proof and the present proof, either this $d\boldsymbol{l}$, this $d\boldsymbol{l}$ line element is changing with time that is because my fluid element is also changing. So my time derivative will act on $d\boldsymbol{l}$ as well, earlier my $d\mathbf{r}$ was fixed when I was doing integral, my quantities were changing with time, but here now $d\boldsymbol{l}$ itself is changing. **(Refer Slide Time: 32:27)**



So we need to, we have 2 terms, first one is what used to be there in the first earlier 3 slides, the second term is coming from *d/dt* acting on ***dl***. So by product rule, I have to apply *d/dt* on both, whatever the changes, right, I mean this straightforward product rule. So now, we can just put in okay, but this is straightforward, no, *d/dt* of $-\nabla p$, it is easy, but what about this second term. Second term is a bit tricky, so the derivation I will show you one derivation.

This is ***dl*** okay, ***dl*** vector, so that 2 points at the two ends A and B, so what is the ***dl***, it is $\mathbf{r}_B - \mathbf{r}_A$. If I take the time derivative will be $\mathbf{u}_B - \mathbf{u}_A$. Now what is $\mathbf{u}_B - \mathbf{u}_A$ for small element, is $d\mathbf{u}$ and velocity changing at these 2 points which is $\boldsymbol{dl}\cdot\nabla\mathbf{u}$ okay.

So u is changing, how much does it change with ***dl***, this is ***dl*** dot, so you can look at du_x, how much du_x changes, it is $\frac{\partial u_x}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_z}{\partial z}dz$, right, this *dx* is changing is, this is okay. Now what is this one? this is $\boldsymbol{dl}\cdot\nabla u_x$, same thing for u_y and u_z. So this precisely what I have written here okay, so this is the stuff. So I have already got the answer for this one. So now let us combine these2 sums. So, I will get first one is $-\nabla p\cdot\boldsymbol{dl}$**,** second one is I just written it here.

Now, first one is taken, this guy, no, what is this one $-\nabla p\cdot\boldsymbol{dl}$, so this is a gradient and I go dismiss ***dl***, made into any direction, this gradient could be here. So $-\nabla p\cdot\boldsymbol{dl}$ will be *dp*, this become vector algebra, but no, this is fine or not fine okay. So this $\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$ ... like this, what is this, this is just *dp* change at *p* at two positions. So this is a straight vector stuff. So first one is *–dp*. Now, I have to sum up this *dp* over the whole close contour *dp dp dp dp* if I go over, how much will I get, what is the net change?

I go around, go around, I come back to the same point and these are vector sums, sorry well algebraic sum, not not, algebraic sum. So $p_B - p_A$, then $p_C - p_B$ and so on and finally I come back, so this term is 0 when integrated over the closed loop. This one is straightforward is $\boldsymbol{dl} \cdot \nabla \frac{u^2}{2}$, this one, so u can be pushed inside, now be careful, sometimes can be pushed, sometimes it can't be pushed, so here the **u** inside and **u** outside and can be pushed, you can check yourself and so this will be basically $d\frac{u^2}{2}$, same idea, go around, you come back to same point.

So this *df* at any scalar is 0 when I sum over the whole loop. So this sum is 0. So what is it mean, this circulation does not change with time, is conserved quantity okay. So this is also useful result, which we will use it later when I do more of MHD, in fact we will use some of this in MHD and so on. Okay, so these are the 4 conservation laws I discussed today, kinetic energy, kinetic helicity for 3D, then kinetic helicity and enstrophy for 2D, then circulation. So this ends, thank you.