Physics of Turbulence
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Lecture - 38
Stably Stratified Turbulence

We covered passive scalar, now we will cover simply stratified flows are very important for atmospheric applications.

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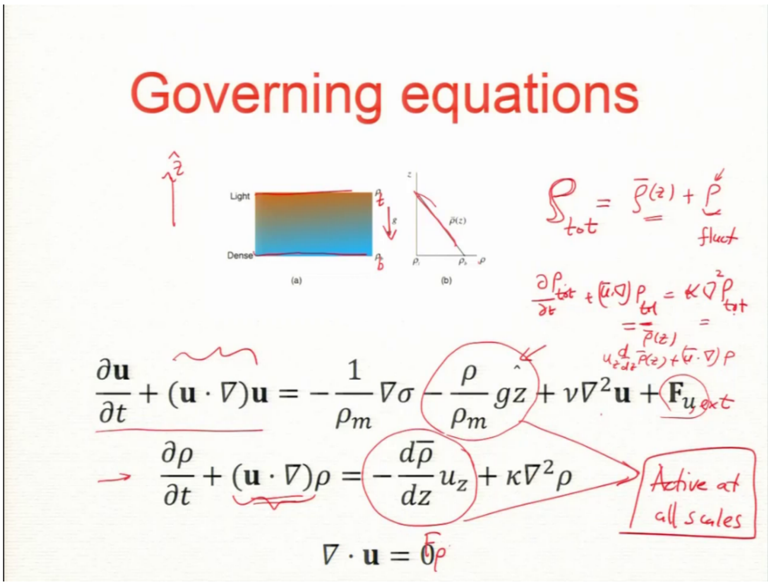

So, the configuration is the following. So, the dense fluid sitting at the bottom and light fluid above, of course there is a gravity, which will add force at every scale. So, in the atmosphere, fluid is heavy at the bottom and its density decreases when you go up.

So, we assume that density is decreasing linearly, the mean density with height. But it is true only for small patches of fluid, is not true for full 10 kilometer of the atmosphere. The density ρ_b at the bottom level and ρ_t at the top level. We take small patch, then apply this theory.

So, what is the equation for this? Ok. Equation is very similar to what we derived for our scalar field, now with slight modification. So, I am just going to show you the equation with the PPT.

So, the governing equations for stratified flows are coupled owing to buoyancy term. Gravity is in down direction and z direction is up. So, the minus sign is coming because of that, straight forward. The buoyancy force works at all scales because density is acting in all scales.

Now, we have situation when force is acting at all scale and that could affect the scaling of the spectrum or flux.

Total density $\rho_t = \bar{\rho}(z) + \rho$; where $\bar{\rho}(z)$ is the mean density and ρ is the density fluctuation about this mean. Substituting the density decomposition into the continuity equation, we get the governing equation for the density fluctuation.

So, $\frac{d\bar{\rho}(z)}{dz}$ is constant, but u z is there at all scales. So, this is these two forces are active at all scales and that is going to change our physics, fine. Now, all our formulas which we did in the last class, on the scalar flux, scalar mode to mode they will be applicable here is well. Because all the mode to mode and shell to shell energy transfer, and flux transfer, they all depend on the structure of the non-linear term, So, they remain the same. So, all the formulas for the flux will be the same, ok. So, there is no change; except how the flux change with k that will be coming from this k dependent stuff, ok.

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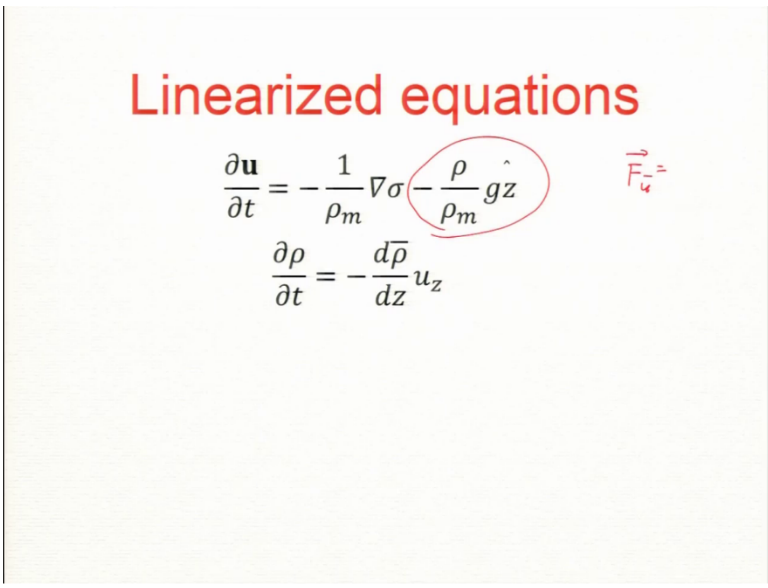

So, before we go on, so we need to see what this buoyancy force does. There are two forces now, $\mathbf{F}_u$ and F_ρ. Now, you will call this mathematical. So, F u is if I F u is the buoyancy which has a physical interpretation as ρg. Now, that easily you can do, but I will just say when a time in I will not tell you exactly. So, my buoyancy is written as ρg, where ρ is the fluctuation and $\bar{\rho}(z)g$ is absorbed in the pressure term with some small algebra.

So, I want to see whether this buoyancy is giving energy to kinetic or taking energy from kinetic that is a key part, ok. So, this is the force, can we say something about sign of injection of kinetic energy, ok. So, to understand whether energy kinetic energy gains from buoyancy or not we need to analyze the linear energy equation. Now, I will tell you the result. So, linearized here is that I drop the non-linear term and the diffusion and dissipation term.

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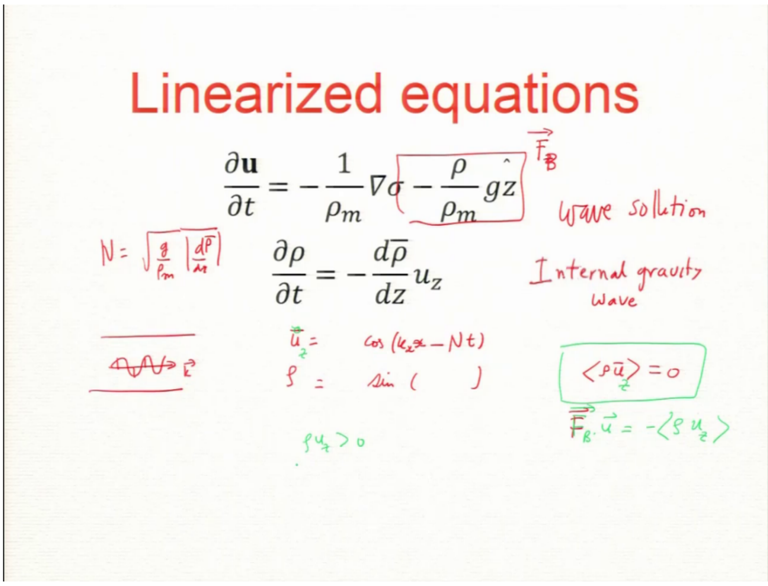

So, this equation give you wave solution, ok. And it is the very important name is call and this you can see in my book on buoyancy, ok. So, this is derived in Fourier space, in fact, using (Refer Time: 08:52) herring basis. So, what is gravity wave? Gravity wave is the ocean surface or water surface we have fluid going up and down, and it moves, right, I mean the wave is if you just prove the drop of is not one stone then wave this gravity wave on the surface it moves. So, that is called surface gravity wave. What is internal gravity wave? It is inside the fluid, so inside the like you go inside and do some disturbance inside the water this is the wave which propagate, that is why it is called internal gravity waves, ok.

And these waves travel and they travel with the frequency, and that is called Brunt Vaisala frequency (N), and that depends on this $\frac{d\bar{\rho}(z)}{dz}$, so it depends on this property. So, I can write down what is Brunt Vaisala frequency. So, $N = \sqrt{\frac{g}{\rho_m}\left|\frac{d\bar{\rho}(z)}{dz}\right|}$, it is very similar property of gravity wave, but it is inside the fluid, ok.

Now, so we can write down the solution which I have in my notes. So, $\overline{u_z} = \cos{(kx - Nt)}$. So, if I look at in the fluid, so if my $\mathbf{k}$ is along that direction k_x direction. So, the fluid will be going up and down. So, the fluctuation is perpendicular to $\mathbf{k}$ because it must be, right $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0$ and just going up and down. So, the wave will move in one direction.

And these also write down what is rho fluctuation, ok. So, because of this relation ρ will be Sin function and this is the amplitude. So, what about $< \rho u_z >$? Average of Cos and Sin will be, if a average were the full either in time on space, zero in both cases. So, what does ρu_z give you? What is the meaning of this? So, our force is of this form, right or this is F_z or this is the force by buoyancy, I call it $\mathbf{F}_B$, B for buoyancy.

So, $\mathbf{F}_B \cdot \mathbf{u}$ is the force supply rate by buoyancy which there is $-\rho u_z$. If I take the average it will be that. So, $< \rho u_z >$ gives you the direction. So, $-\rho u_z$ is the injection rate of kinetic energy to the system.

The injection rate is zero for waves. What does it mean? That on the average it is sometimes taking something to losing its like potential energy you know, kinetic potential, but if you average over time it has 0 net transfer, ok. Now, what will happen if I put non-linearity? When the sign of this guy change and what should it be? Should it be positive, or it should be negative? Now, is the question clear. So, it is 0 for linear solution, but if I put non-linearity can you argue that what should be the sign of $< \rho u_z >$. So, I give my argument.

Now, the system is stable, the wave solution the system is stable. If I put viscosity the energy of the system will die out, it is the reason why the system is called stably stratified system. By experienced in the earth atmosphere you do any perturbation it will die by viscosity. If we have viscosity turned off it will just begin. So, in this system if this is ρu_z is positive then, so you need to write down equation for the kinetic energy. So, let us write down the equation for the full, full non-linear term. Please refer to the slide on next page.

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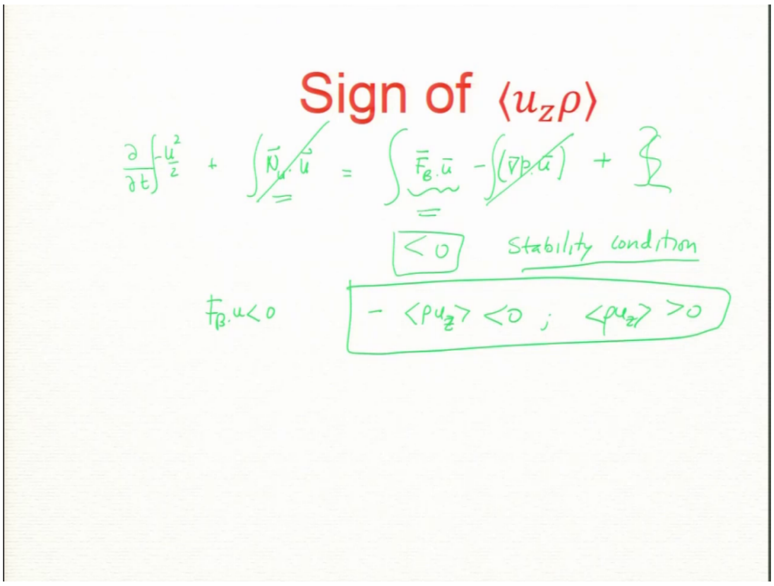

Now, if that is the case then we have very good idea about the flux. Now, I am building static versus some direction will the flux increase with wave number or will it decrease with wave number that is what I am after, ok.

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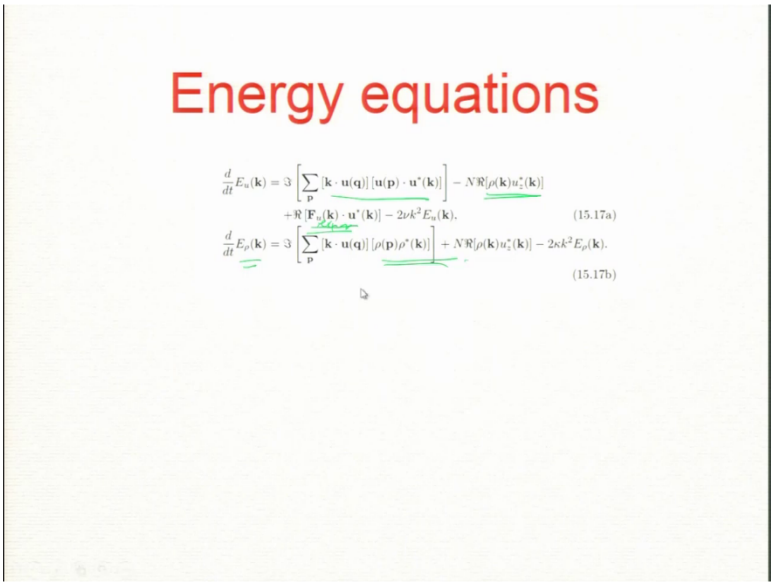

So, this is the energy equation, right. So, this is coming from non-linear term, this is the injection by buoyancy force. $\mathbf{F_u}$ is the external force. So, this is external force, in addition to buoyancy. So, if since buoyancy is decreasing the kinetic energy, so if you want to maintain a steady state then you need to supply with large scale. So, in our case if you do not supply something coming from the ocean or some shaking up it will just become quiet. In fact, wind also will not move. You cannot, this will be all quiet. So, this is the external force this one. Please refer to the above slide for equations for $\mathbf{E}_{\text{u}}(\mathbf{k})$ and $\mathbf{E}_{\rho}(\mathbf{k})$.

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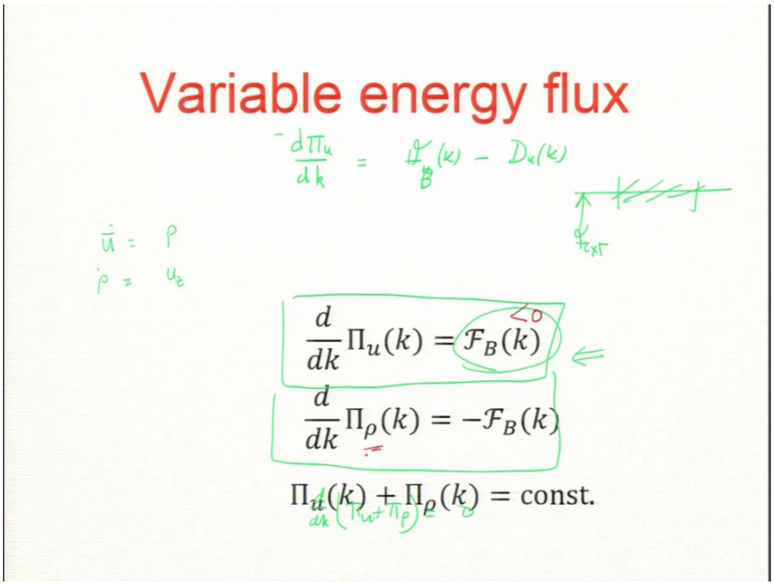

And it turns out we can also derive. So, I think we will focus on only one part then, I will. So, we have variable in the flux. So, let us consider steady state what did you derive this, we have derived on many occasions (please see above slide). We focus on the inertia range varies only with the buoyancy part. So, external part is 0 in inertia range and dissipation also not active in this range. So, this is the in the equation with the flux in the inertia range.

If you sum these two equations for the fluxes $\frac{d\Pi}{dk} = 0$, where $\Pi(\text{k}) = \Pi_u(k) + \Pi_\rho(k)$ is the total energy flux. that means, $\Pi_u(k) + \Pi_\rho(k) =$ constant. The key ingredient for stably stratified flows; so, these two fluxes give you constant.

So, both change with k. So, because the buoyancy is the activity all scales it will change with k, but the sum must be constant. Now, how will they change can you make a guess? So, F_B is negative I proved the sign, right I mean I hope you are convinced. So, F_B is negative. So, Π_u should decrease with k naturally. So, flux will decrease with k. And $\Pi_\rho(k)$ will increase with k.

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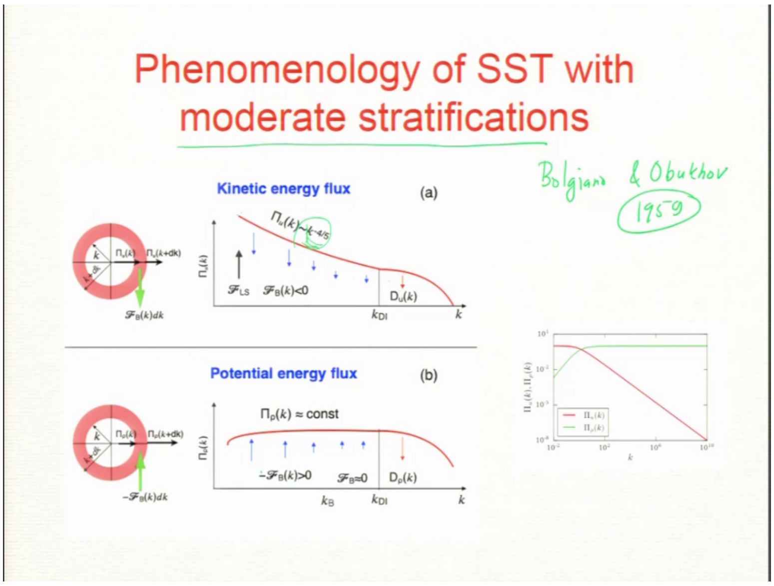

Stably stratified flows come in 3 types. So, I will just write down quickly maybe you should know this. So, I will just write down the regimes of SST.

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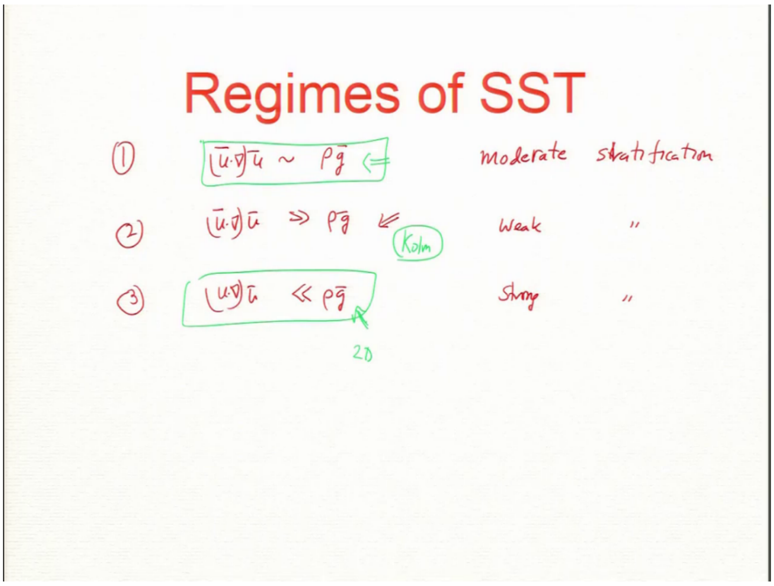

One regime is when gravity and non-linearity both are equally strong. So, this ρg term and $\mathbf{u} \cdot \nabla\mathbf{u}$ both are of the same order, roughly equal. And this is called moderately stratified flow, moderate stratification, ok. Second, if $\mathbf{u} \cdot \nabla\mathbf{u}$ is much bigger than ρg, this is called weak stratification or now because gravity is weak, so weak stratification. And third is other way round $\mathbf{u} \cdot \nabla\mathbf{u}$ is much less than ρg. So, it is a strong stratification.

So, it is like fluid solution Kolmogorov theory, if gravity is not doing much, some changes, but there is some inertia inertial gravity waves, but Kolmogorov theory works quite nicely. I will cover this one in this today's lecture, but this one is strong gravity it becomes 2D. So, gravity is too strong flow is moves easily in the horizontal direction north so easily in vertical direction and so this is called quasi 2D flow. So, in fact, earth atmosphere is in this regime, ok. So, this is the big topic which I will not discussed, but I will illustrate the dynamics of moderate stratification.

Now, this is the theory of for moderate stratification, so this is for moderate stratification. And this was theory was done by Bolgiano and Obukhov, 1959, ok, way back. So, according to their theory, the kinetic energy flux will decrease with k as $k^{-4/5}$. Why because this sink of, so this force is taking it away. It is like the analogy which I used to make is for financial system is corruption. So, there is some things I just taking your money flow, no. So, this buoyancy force is taking it away. And where is it pushing? It is pushing into potential energy and of course, finally, it goes into heat, ok. So, let us start to derive how to get $k^{-4/5}$from scaling arguments from dimensional analysis.

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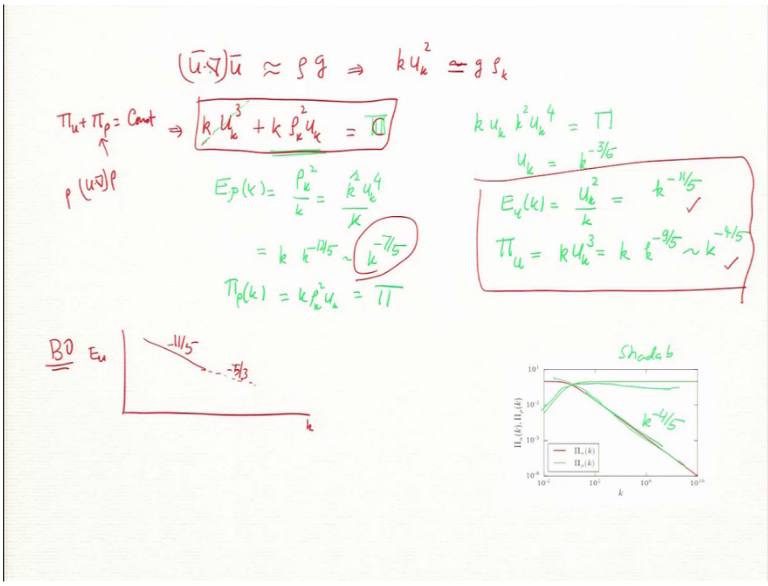

So, I will quickly derive it, ok. So, equate the non-linear term with gravity, but ρ_k is function of k, right, ρ is active at all the scales. So, it gives you $ku_k^2 = g\rho_k$. Now, you make another assumption, you take the flux argument t:$\Pi_u(k) + \Pi_\rho(k) =$ Constant.

So, what will that give you? $\Pi_u(k) = ku_k^3$ and $\Pi_\rho(k) = k\rho_k u_k^2$ in Fourier space.

I must make a remark that the Bolgiano and Obukhov dropped $\Pi_u(k)$ .So, they assumed Π_u will be small. So, Π_ρ equal to constant. So, that is an assumption. In fact, we looked at that assumption and we got some nice result, ok. Please refer to slide above for the derivation for the kinetic energy flux and the energy spectra.

So, Bolgiano and Obukhov say that kinetic energy decreases steeper. Steeper means faster than$k^{-5/3}$. And why is it steeper? Because some energy was supposed to flow which would have maintained five-third, but somebody is already taking things at everywhere like leakage in pipe you know there is the water is flowing in the pipe, but is you flowing in Fourier space.

It is not in real space water pipe is Fourier space work, but there holes everywhere, so water will keep decreasing when you go down and down. So, that is why flux, there is a flow is like water current and that is decreasing with k. So, that decreases as $k^{-4/5}$ and similarly the density spectrum is also affected, that is $k^{-7/5}$.

Thank you.