Physics of Turbulence

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Lecture - 37

Passive Scalar Turbulence

So now, we will discuss Passive Scalar Turbulence, ok. So, we have done the formalism. So, includes definition of flux spectrum. So, 1D spectrum will be defined similarly you know. So, energy in a shell is 1D spectrum for θ. So, we look for spectrum E_θ, is it some power law or not power law what property.

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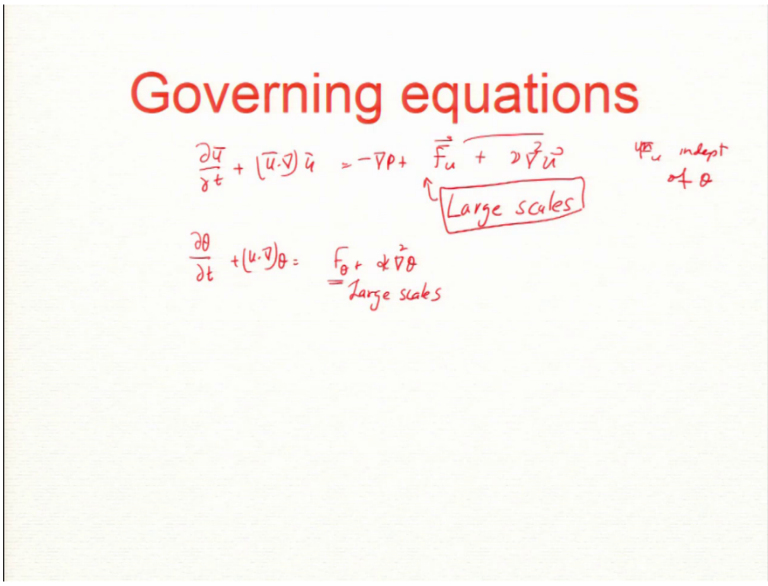
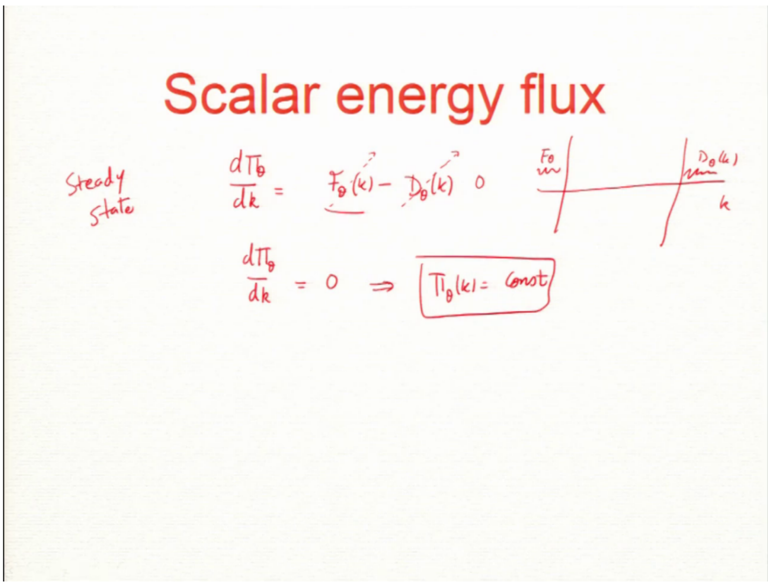


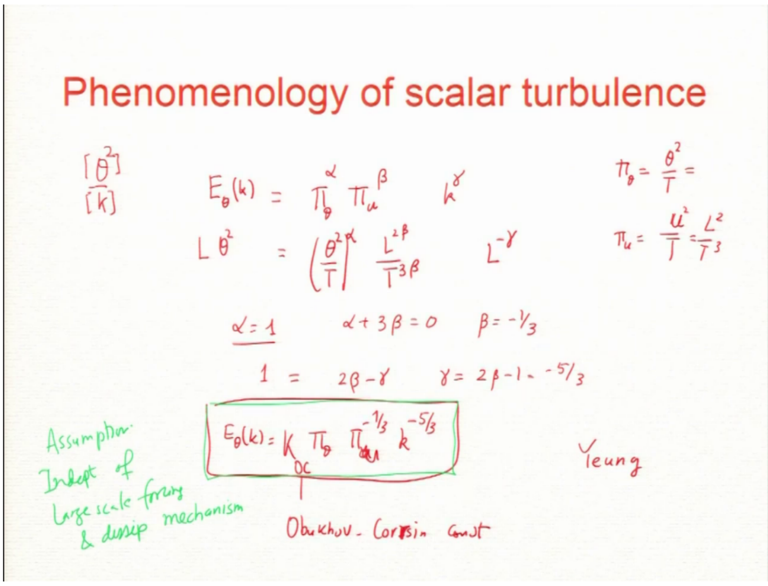
So, I rewrite the equations; velocity field is exactly same as before except that is $\mathbf{F}_u$ is not a function of θ and we will assume that it is forcing only the large scale, the activate large scales remember. So, we had said storing at large scale. Now what about θ equation? So, $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\theta + F_\theta$ I will also assume F_θ, but this is also at large scale.

If I do not put F_θ then it will decay by κ right, if this diffusivity and if you do not put F_θ, it is decaying theta squared. So, we will assume F_θ is at large scales and non-zero. So, it will induce a cascade of θ^2 theta squared which we discussed. Now, so, it is exactly same set of equation I mean, but I made an assumption that is $\mathbf{F}_u$ is independent of θ. So, that is why it is called passive.



So, scalar energy flux with which we already have; so, vector is a flux or velocity flux is same as before, a scalar energy flux will be so, I am assuming steady state. So, $\frac{d\Pi_\theta}{dk} = F_\theta(k) - D_\theta(k)$ and the flux is usual definition, theta is where coming out of the sphere, rate of θ^2 coming out of the sphere by non-linear terms, ok. Now, in the inertia range we let this go to 0 and also this goes to 0 because, this is the active at large scale. So, if you look at k space k so, dissipation is active here and F_θ is active here.

So, the inertia range, here both are absent like Kolmogorov theory. So, I will say that $\frac{d\Pi_\theta}{dk} = 0$; implies $\prod_\theta(k) = const$ which was easy to derive. So, $\prod_\theta$ must be constant of course, would be 0, but I assume that these have sink κ is non-zero supplied large scale that generates a flux and that is non-zero. If you force here in Fourier space and sink here and there is a channel of energy transfer, ok. So, $\prod_\theta = const$ I already derived from formula of variable energy flux there is this formula, ok.



So now, what about spectrum? What can you say about this spectrum? tSo, $E_\theta(k)$ this is what I am looking at, this dimension is $\frac{\theta^2}{k}$ you agree. So, $\int E_\theta(k)dk \approx \theta^2$. So, it was dimension of $\frac{\theta^2}{k}$, but this can depend on what all? I will do dimensional analysis. The Kolmogorov derivation, a Fourier space again; there are two derivations we are done so far. Kolmogorov 4/5 law which was lot of mathematics real space, but a Fourier space it was proving further $\prod_u = const$ then apply dimensional analysis.

So, now $E_\theta(k)$ can depend on various things. So, it can depend on k local wavenumber and flux rate. So, how many flux rates are there now?

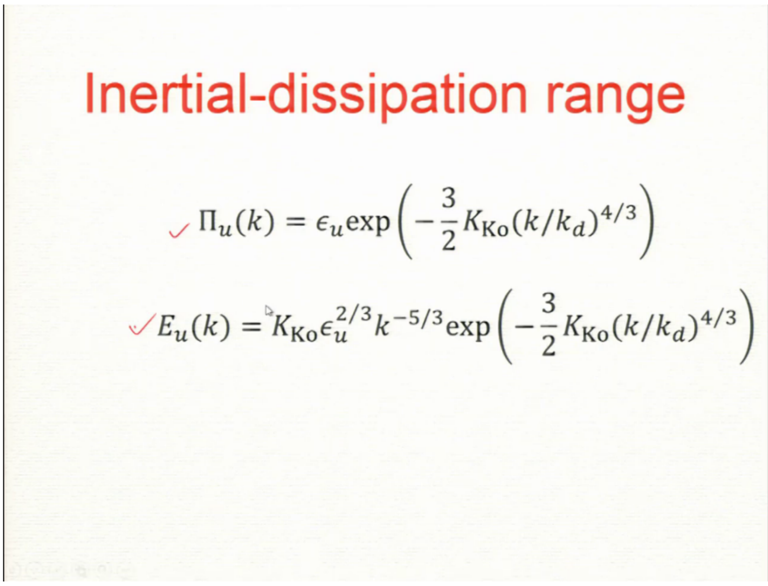
The two fluxes rate is right. So, there $\prod_u$ and $\prod_\theta$ so, the three of them. So, we have instead of $\prod_u$ and k for hydrodynamics you have three of them, now $\prod_\theta$ for. So, I am going to call it α, β and γ ok. Now, we need to determine α, β and γ So, these dimensions; so, what is dimension of $\prod_\theta$ by them? It is theta squared by time injection of rate of injection of θ^2. So, this will be $\frac{\theta^2}{t}$. so, we need to put all the dimensions properly. So, the $\frac{\theta^2}{k}$ means θ^2L. This is $\left(\frac{\theta^2}{T}\right)^\alpha$, k^γ means $L^{-\gamma}$ $\prod_u$ is.

$\frac{a^2}{t}$. So, there is a $\frac{u^2}{t}$ sorry $\frac{u^2}{t}$. So,$\prod_u = \frac{L^2}{T^3}$. So, this is $\frac{L^{2\beta}}{T^{3\beta}}$. So, now, we can figure out what all the three quantities by the θ has coming the new quantity again. So, that is why again determine all three. So, with the L and T would to would not have been possible, but θ has come as new. So now, we can easily see that α=1 right. From here $2 = 2\alpha, \alpha = 1$. β is from here.

So, $\alpha + 3\beta = 0$. So, $\beta = -\frac{1}{3}$ and now γ you can figure out. So, 1 plus, $1 = 2\beta - \gamma, \gamma = -\frac{5}{3}$, right. So, my $E_\theta(k) = \prod_\theta {\prod_u}^{-1/3} k^{-5/3}$. So, this is a formula for E_θ. So, it is 5/3 again, but now my level of the spectrum depends on both $\prod_u$ and $\prod_\theta$ and one next one it is 1 other one is -1/3.

Together gives you 2/3. So, that is the interesting, but $\prod_\theta$ and pi were different quantities, ok. Now, this is spectrum for passive scalar. It turns out it works quite nicely, people have verified it in experiments and simulations and a I will not and there are lots of literature, ok. So, you can look at some papers by P K Yeung and there is a constant in front is called Obukhov Corrsin constant. So, this constant; now so, we have constant so, if you force it large scale then$\prod_\theta$=const and spectrum is -5/3. So, this is the phenomenology for scalar turbulence. We get this $k^{-5/3}\prod_\theta {\prod_u}^{-1/3}$, but there is one catch is that we assume that the spectrum is independent of forcing at large scale and dissipation mechanism. So, this assumption which you should remember assumption independent of large scale forcing and dissipation mechanism or diffusion mechanism, ok.

So, mu and kappa do not come into it and also how you force it do not come it come into it, ok. So, this is another assumption which we assumed here to derive 5/3. If you do not force θ^2 then it will be decaying θ^2 and decaying turbulence also is expected to be; if your energy is large and initially five-third then it will go into some different power law. But it will decay in time ok, but that I will not discuss; decaying part I will not discuss. Is there any question on this?



So, we can also define a quickly what is the inertial dissipation range so; that means, this formula is what I described in the last slide is only for the inertia range. Now can we combine both? Is there a formula which can do both? Now, velocity field will be what? What is a good candidate for you for the velocity field? So, power formula should be good know written. So, since it is saying know when passive scalar velocity field is exactly same is hydro, no change.

So, Paul's formula is good and it seems to work reasonably well ok, this one. So, is it $\prod_u$ and $\prod_\theta$ where $\prod_u$ and **u.** What about pi theta and, E theta what do you want to make some guess?

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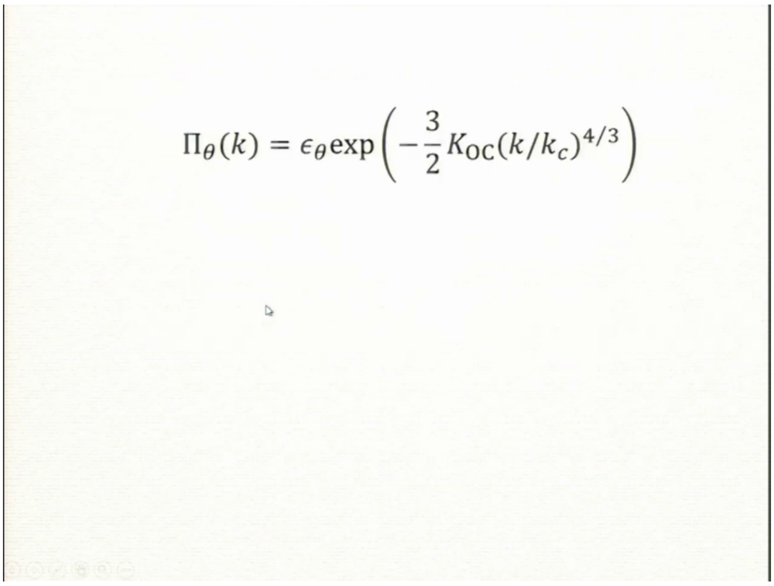
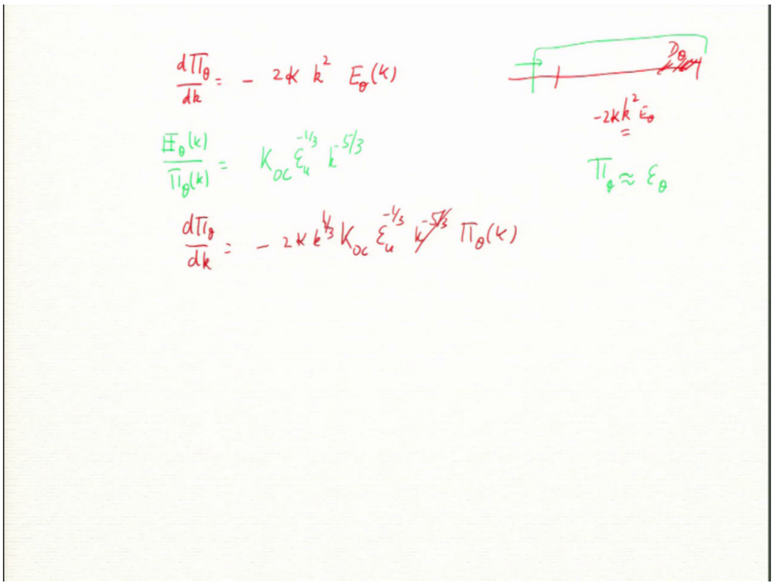


So, I still want to make Π_θ. So, I will give you the how I should do it. So, in the inertial dissipation range $F_\theta = 0$. So, we in this wave number I am in this range.



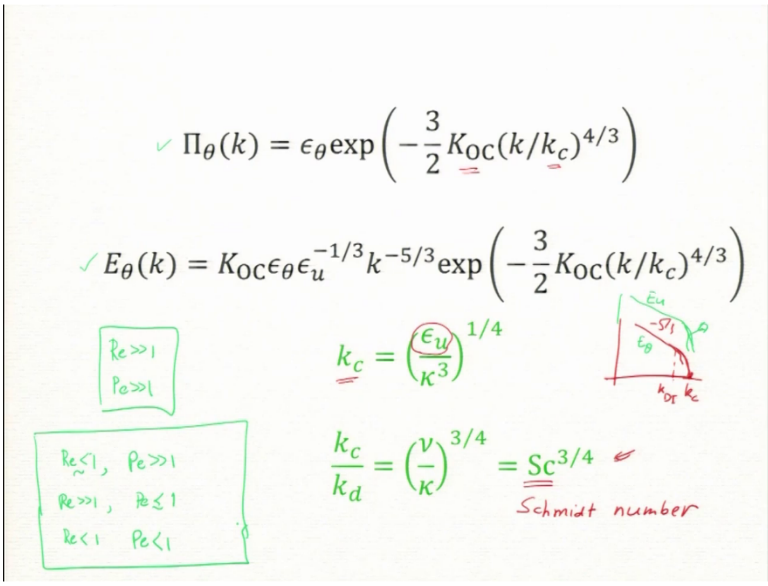
So, the dissipation is strong here, but is everywhere is strong into the large k ok, $-2\kappa k^2 E_\theta$. So, $E_\theta = 0$; that means, dissipation is there even for lower numbers, but as we get it, it becomes stronger at k^2 ok. So now, we can write $\frac{d\Pi_\theta}{dk} = -2\kappa k^2 E_\theta(k)$, these are the definition. $F_\theta = 0$ so; it comes follow some variable in g flux.

Now, I make the $\prod_{zs}$, what is $\prod_{zs}$? $\frac{E_\theta(k)}{\Pi_\theta(k)}$ is independent of ν and κ. So, I say there now there is no diffusion coefficient so, independent of both ν and κ. So, it is function only of k $\boldsymbol{\varepsilon}_u$ and $\boldsymbol{\varepsilon}_\theta$ and $\boldsymbol{\varepsilon}_\theta$ must be same as $\prod_\theta$. Yes Because, if I we can derive it exactly like what it is if it integrates from here to there then energy injection rate must be equal to the total dissipation rate.

So, $\prod_\theta$ by that this derivation you should do yourself you know, this you can do it from first principles; $\prod_\theta$ approximately $\boldsymbol{\varepsilon}_\theta$. Now, $\boldsymbol{\varepsilon}_\theta$ is always well I cannot $\boldsymbol{\varepsilon}_\theta$ approximately equal not quite equal ok; now the function of these three. So, by dimensionalize what should I can derive it, but I already know the formula. So, we derived in the last slide.

So, this k Obukhov Corrsin constant then $\boldsymbol{\varepsilon}$ so, $\prod_\theta$ is going to come in $\boldsymbol{\varepsilon}_\theta$ here. So, that should go away -1/3 and $k^{-5/3}$ right, now $E_\theta(k) = \prod_\theta \prod_u{}^{-1/3} k^{-5/3}$. So, this is what should be right over, I mean you could do it with division analysis, but this is what is going to come substituted there. So, what will I get? $\frac{d\Pi_\theta}{dk} = -2\kappa k^2 E_\theta(k)$ Obukhov Corrsin $\varepsilon_u{}^{-1/3} k^{-5/3}$ and $\prod_\theta$. So, I get a function 1D ODE first ordered ODE of $\prod_\theta$, I can solve it.

Now, this -5/3 and k^2 gives you 1/3, if you integrated I will get $k^{4/3}$. If I derive exactly the same way as what we did for hydro, ok. So, this is how we can derive.



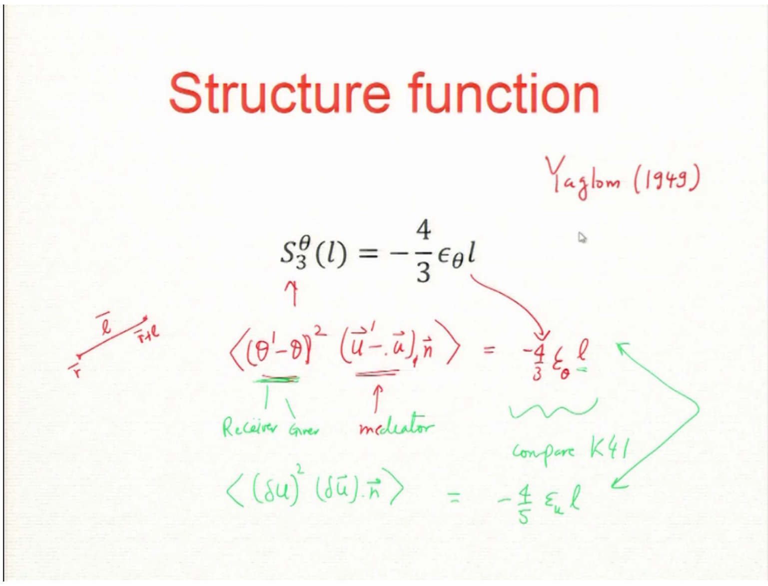
And so, I will expect this instead of Kolmogorov I will get this and instead of k_d I will get k_c it depends on κ rather than u and E_θ will be this and k_c is the Kolmogorov constant; Kolmogorov wave number for θ. So, these were my θ^2 spectrum is like this now -5/3, then it should steepen and at some point θ^2 is no more fluid. So, this k_c; k_c is not here and so, this please remember k_c is not here.

I call it k_d. So, this is ratio is factor 2, but k_d and k_c are different. So, k_c is where fluid velocity of fluid the θ^2 is losing the fluid meaning and we can now this is $\boldsymbol{\varepsilon}_u$. Interestingly epsilon theta does not come fall over k_c $\boldsymbol{\varepsilon}_u$ comes, the derivation is interesting $\boldsymbol{\varepsilon}_\theta$ does not come. So, what is k_c/k_d? Yes, ν/κ, now I forgot to say this ratio is called Sc for (Refer Time: 16:29). For once did even flows is called Pa, but for all the (Refer Time: 16:40) literature you will find this is called Sc and is 3/4.

It is useful formula which comes anyway, ok. So, this is how we can generalize and we have not tested it and I am not sure, now this needs to be tested whether it is lower for (Refer Time: 17:01) is that clear I mean. So, these are conjecture whether it works or not we can figure out from experiments, nominal experiment is easiest. So, these are the equations for $\prod_\theta$, $\prod_u$ and $\prod_\theta$ E_θ and we did in the past u and $\prod_u$, but these are applicable when both of them are 5/3 u and E_θ, right. So, there is there will be u as well on this u and this is E_θ.

So, both are five-third and then this part we are focusing on the dissipative range. So, this formula is both dissipation and inertia, but the assumption that both are 5/3 assumes that both Re and Pe are much bigger than 1. So, their regimes when Re may be order 1 small and Pe may be large. So, laminar flow, but which for with the θ part is turbulent, right.

So, the regimes which I am not covering here, but you can look at it in the notes Re much less; Re this 1 and Pe much bigger than 1 or other way around. Re much bigger than 1, Pe less than equal to 1 or both less than 1. These regimes I have not covered in these lectures ok.



Now, structure function; so, in the exactly same thing like Kolmogorov and this was done by Yaglom first in 1949. So, instead of 4/5, it is 4/3 and this proof is simpler. So, you can have a look at the proof, it is $-\frac{4}{3}|\theta|$ l, but is following the same lines, but it is simpler because scalar. So, we do not have third rank so, these tensors become simpler with θ, we define S_3^θ.

For which there is exact relation. So, S theta I have not written here the definition is so, the 2 points **r** and **r+l** so, difference is **l**. So, $(\theta' - \theta)^2$. So, scalar value at **r+l** =-r$^2\delta$u; so, u prime minus u parallel$(\boldsymbol{u'} - \boldsymbol{u})_{/\!/}$, parallel means I take the component along **l** .dot $\hat{l}$ or we had notation of **n**.

So, this is the component of the velocity fields along the **l** direction so, $(u' - u)_{/\!/}$ take the projection so, I will say **n**. So, this quantity is S_3^θ and this goes as this object $-\frac{4}{3}\varepsilon_\theta$, a dissipation rate of scalar quantity, ok. So, now, what is giver and what is a receiver for this? So, from this you can make interpretation know.

So, here this is the mediator right because **u** is the mediator. So, and both receiver and giver are here; one of them is receiver one of them is giver. But it is a cumulative so in fact, this is receiver as well as giver; one of them is giver one of them is receiver, ok. If I take the diverges of this, now is supposed to be θ that is equal to flux, ok. So, derivation you will find that we have to take the divergence to equity to the flux with some pre-factors.

So, the divergence will kill n, ok. So, the point is even in real space we can interpret is a something is mediator, something is a giver and something is a receiver. So, these are beautiful way to interpret these formulas, and also works for MHD which I will try to explain when I do MHD. So, this is $-\frac{4}{3}\varepsilon_\theta l$, now when you compare with Kolmogorov k-41, k-41 therefore, -4/5. So, which is $\langle(\boldsymbol{\delta u})^2(\boldsymbol{\delta u})\cdot\boldsymbol{n}\rangle$, right.

So, it was all three of them and there is -4/5$\boldsymbol{\varepsilon}_u$l. So, this l is proportional to the distance between the two points, ok. So, this formula is a very similar, the pre-factor is only different -4/3 and -4/5. One of them is dissipation of scalar, other one is dissipation of u^2, ok. This has been verified by P K Yeung, there is a paper in 2005 at least one paper I know, but there could be more papers. So, let us stop.

Thank you.

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Lecture - 38
Stably Stratified Turbulence

We covered passive scalar, now we will cover simply stratified flows are very important for atmospheric applications.

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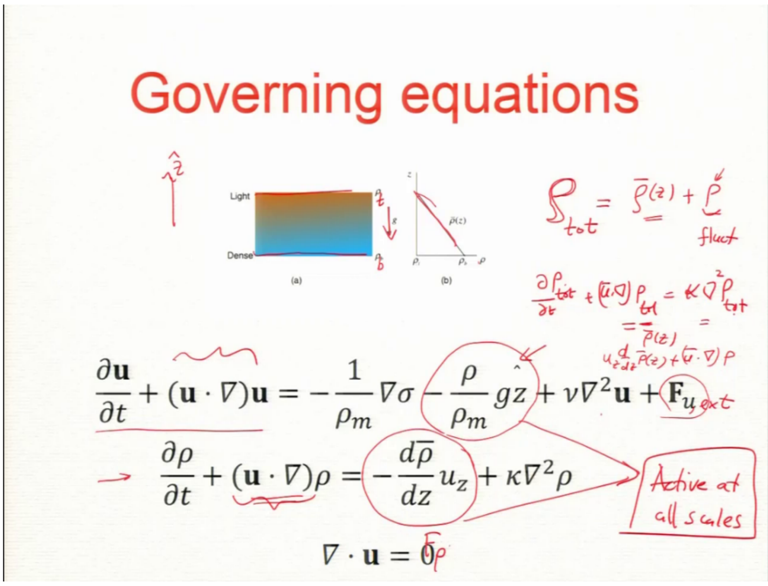

So, the configuration is the following. So, the dense fluid sitting at the bottom and light fluid above, of course there is a gravity, which will add force at every scale. So, in the atmosphere, fluid is heavy at the bottom and its density decreases when you go up.

So, we assume that density is decreasing linearly, the mean density with height. But it is true only for small patches of fluid, is not true for full 10 kilometer of the atmosphere. The density ρ_b at the bottom level and ρ_t at the top level. We take small patch, then apply this theory.

So, what is the equation for this? Ok. Equation is very similar to what we derived for our scalar field, now with slight modification. So, I am just going to show you the equation with the PPT.

So, the governing equations for stratified flows are coupled owing to buoyancy term. Gravity is in down direction and z direction is up. So, the minus sign is coming because of that, straight forward. The buoyancy force works at all scales because density is acting in all scales.

Now, we have situation when force is acting at all scale and that could affect the scaling of the spectrum or flux.

Total density $\rho_t = \bar{\rho}(z) + \rho$; where $\bar{\rho}(z)$ is the mean density and ρ is the density fluctuation about this mean. Substituting the density decomposition into the continuity equation, we get the governing equation for the density fluctuation.

So, $\frac{d\bar{\rho}(z)}{dz}$ is constant, but u z is there at all scales. So, this is these two forces are active at all scales and that is going to change our physics, fine. Now, all our formulas which we did in the last class, on the scalar flux, scalar mode to mode they will be applicable here is well. Because all the mode to mode and shell to shell energy transfer, and flux transfer, they all depend on the structure of the non-linear term, So, they remain the same. So, all the formulas for the flux will be the same, ok. So, there is no change; except how the flux change with k that will be coming from this k dependent stuff, ok.

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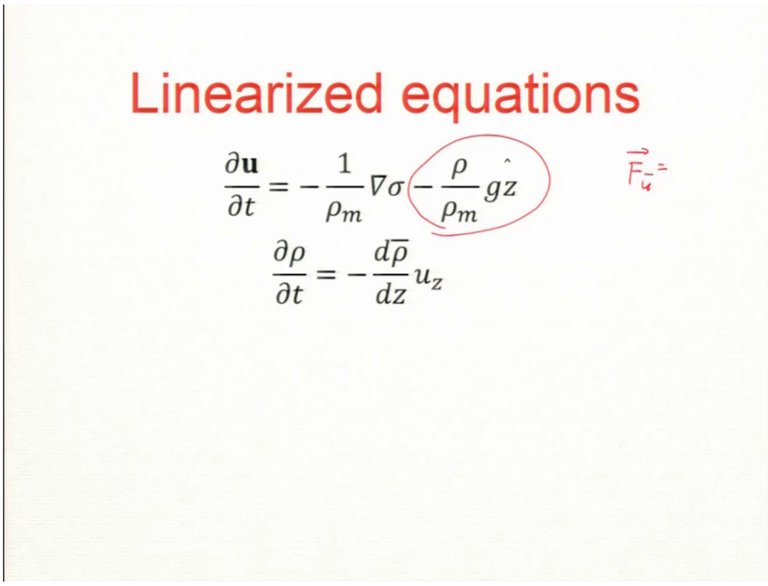

So, before we go on, so we need to see what this buoyancy force does. There are two forces now, $\mathbf{F}_u$ and F_ρ. Now, you will call this mathematical. So, F u is if I F u is the buoyancy which has a physical interpretation as ρg. Now, that easily you can do, but I will just say when a time in I will not tell you exactly. So, my buoyancy is written as ρg, where ρ is the fluctuation and $\bar{\rho}(z)g$ is absorbed in the pressure term with some small algebra.

So, I want to see whether this buoyancy is giving energy to kinetic or taking energy from kinetic that is a key part, ok. So, this is the force, can we say something about sign of injection of kinetic energy, ok. So, to understand whether energy kinetic energy gains from buoyancy or not we need to analyze the linear energy equation. Now, I will tell you the result. So, linearized here is that I drop the non-linear term and the diffusion and dissipation term.

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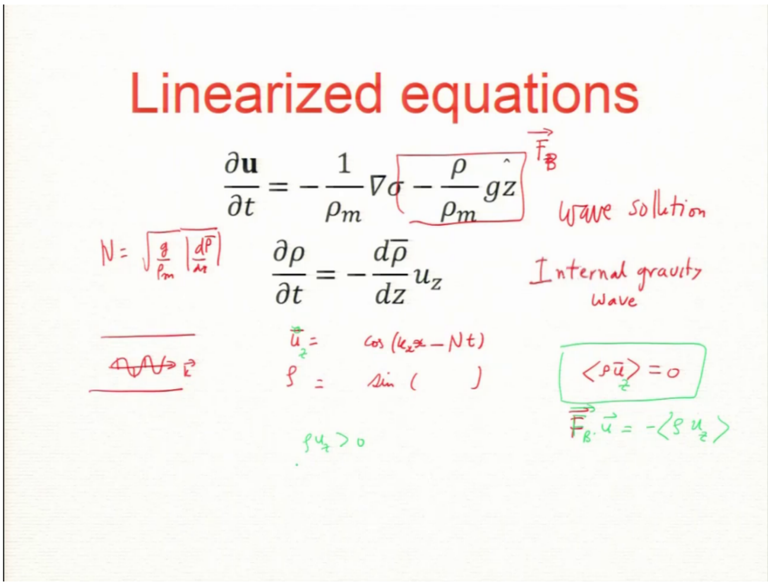

So, this equation give you wave solution, ok. And it is the very important name is call and this you can see in my book on buoyancy, ok. So, this is derived in Fourier space, in fact, using (Refer Time: 08:52) herring basis. So, what is gravity wave? Gravity wave is the ocean surface or water surface we have fluid going up and down, and it moves, right, I mean the wave is if you just prove the drop of is not one stone then wave this gravity wave on the surface it moves. So, that is called surface gravity wave. What is internal gravity wave? It is inside the fluid, so inside the like you go inside and do some disturbance inside the water this is the wave which propagate, that is why it is called internal gravity waves, ok.

And these waves travel and they travel with the frequency, and that is called Brunt Vaisala frequency (N), and that depends on this $\frac{d\bar{\rho}(z)}{dz}$, so it depends on this property. So, I can write down what is Brunt Vaisala frequency. So, $N = \sqrt{\frac{g}{\rho_m}\left|\frac{d\bar{\rho}(z)}{dz}\right|}$, it is very similar property of gravity wave, but it is inside the fluid, ok.

Now, so we can write down the solution which I have in my notes. So, $\overline{u_z} = \cos(kx - Nt)$. So, if I look at in the fluid, so if my $\mathbf{k}$ is along that direction k_x direction. So, the fluid will be going up and down. So, the fluctuation is perpendicular to $\mathbf{k}$ because it must be, right $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0$ and just going up and down. So, the wave will move in one direction.

And these also write down what is rho fluctuation, ok. So, because of this relation ρ will be Sin function and this is the amplitude. So, what about $< \rho u_z >$? Average of Cos and Sin will be, if a average were the full either in time on space, zero in both cases. So, what does ρu_z give you? What is the meaning of this? So, our force is of this form, right or this is F_z or this is the force by buoyancy, I call it $\mathbf{F}_B$, B for buoyancy.

So, $\mathbf{F}_B \cdot \mathbf{u}$ is the force supply rate by buoyancy which there is $-\rho u_z$. If I take the average it will be that. So, $< \rho u_z >$ gives you the direction. So, $-\rho u_z$ is the injection rate of kinetic energy to the system.

The injection rate is zero for waves. What does it mean? That on the average it is sometimes taking something to losing its like potential energy you know, kinetic potential, but if you average over time it has 0 net transfer, ok. Now, what will happen if I put non-linearity? When the sign of this guy change and what should it be? Should it be positive, or it should be negative? Now, is the question clear. So, it is 0 for linear solution, but if I put non-linearity can you argue that what should be the sign of$< \rho u_z >$. So, I give my argument.

Now, the system is stable, the wave solution the system is stable. If I put viscosity the energy of the system will die out, it is the reason why the system is called stably stratified system. By experienced in the earth atmosphere you do any perturbation it will die by viscosity. If we have viscosity turned off it will just begin. So, in this system if this is ρu_z is positive then, so you need to write down equation for the kinetic energy. So, let us write down the equation for the full, full non-linear term. Please refer to the slide on next page.

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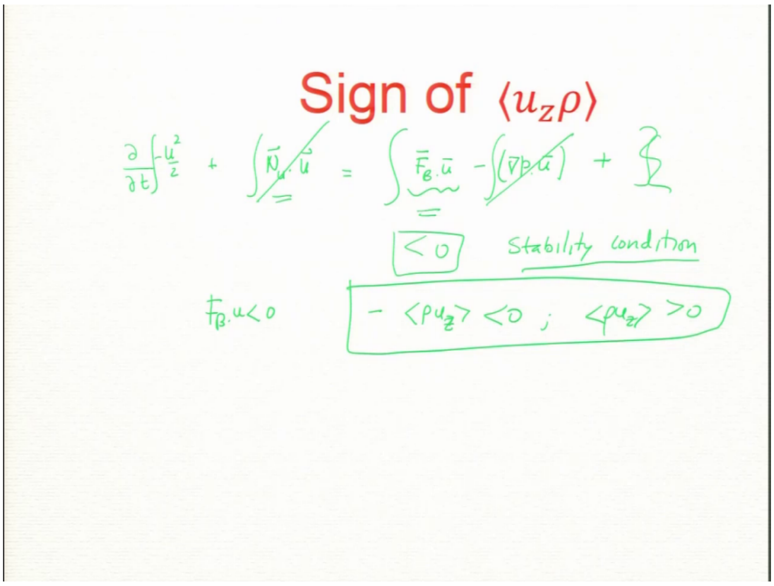

Now, if that is the case then we have very good idea about the flux. Now, I am building static versus some direction will the flux increase with wave number or will it decrease with wave number that is what I am after, ok.

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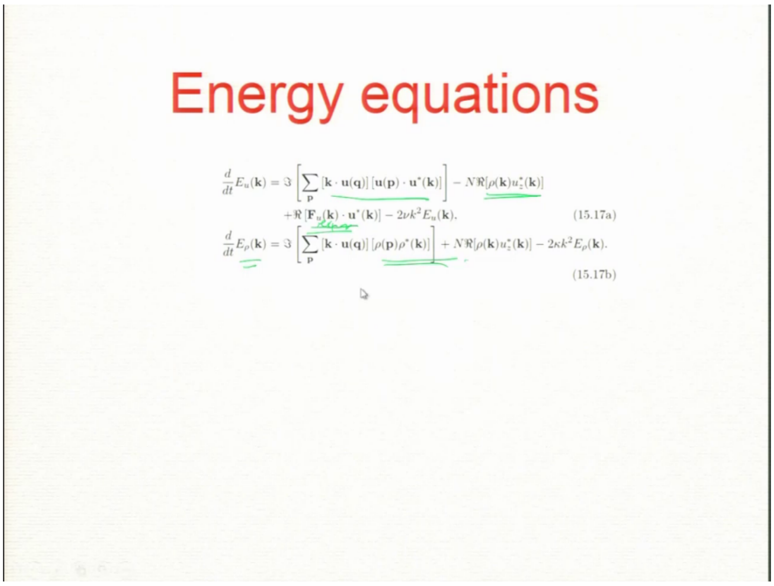

So, this is the energy equation, right. So, this is coming from non-linear term, this is the injection by buoyancy force. $\mathbf{F}_u$ is the external force. So, this is external force, in addition to buoyancy. So, if since buoyancy is decreasing the kinetic energy, so if you want to maintain a steady state then you need to supply with large scale. So, in our case if you do not supply something coming from the ocean or some shaking up it will just become quiet. In fact, wind also will not move. You cannot, this will be all quiet. So, this is the external force this one. Please refer to the above slide for equations for $\mathbf{E}_u(\mathbf{k})$ and $\mathbf{E}_\rho(\mathbf{k})$.

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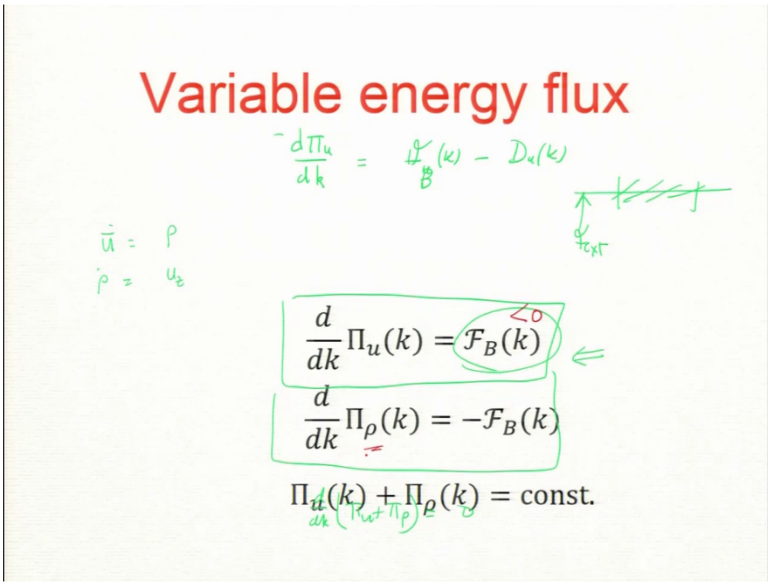

And it turns out we can also derive. So, I think we will focus on only one part then, I will. So, we have variable in the flux. So, let us consider steady state what did you derive this, we have derived on many occasions (please see above slide). We focus on the inertia range varies only with the buoyancy part. So, external part is 0 in inertia range and dissipation also not active in this range. So, this is the in the equation with the flux in the inertia range.

If you sum these two equations for the fluxes $\frac{d\Pi}{dk} = 0$, where $\Pi(\mathrm{k}) = \Pi_u(k) + \Pi_\rho(k)$ is the total energy flux. that means, $\Pi_u(k) + \Pi_\rho(k)$ = constant. The key ingredient for stably stratified flows; so, these two fluxes give you constant.

So, both change with k. So, because the buoyancy is the activity all scales it will change with k, but the sum must be constant. Now, how will they change can you make a guess? So, F_B is negative I proved the sign, right I mean I hope you are convinced. So, F_B is negative. So, Π_u should decrease with k naturally. So, flux will decrease with k. And $\Pi_\rho(k)$ will increase with k.

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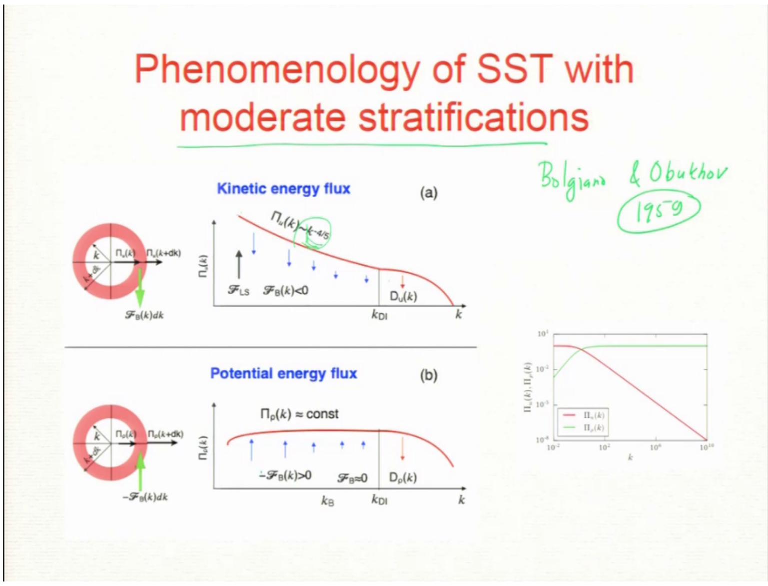

Stably stratified flows come in 3 types. So, I will just write down quickly maybe you should know this. So, I will just write down the regimes of SST.

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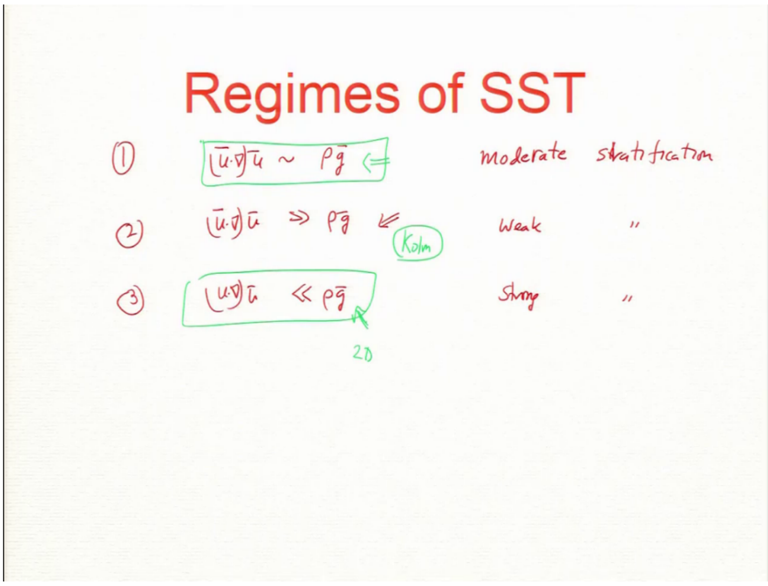

One regime is when gravity and non-linearity both are equally strong. So, this ρg term and $\mathbf{u} \cdot \nabla \mathbf{u}$ both are of the same order, roughly equal. And this is called moderately stratified flow, moderate stratification, ok. Second, if $\mathbf{u} \cdot \nabla \mathbf{u}$ is much bigger than ρg, this is called weak stratification or now because gravity is weak, so weak stratification. And third is other way round $\mathbf{u} \cdot \nabla \mathbf{u}$ is much less than ρg. So, it is a strong stratification.

So, it is like fluid solution Kolmogorov theory, if gravity is not doing much, some changes, but there is some inertia inertial gravity waves, but Kolmogorov theory works quite nicely. I will cover this one in this today's lecture, but this one is strong gravity it becomes 2D. So, gravity is too strong flow is moves easily in the horizontal direction north so easily in vertical direction and so this is called quasi 2D flow. So, in fact, earth atmosphere is in this regime, ok. So, this is the big topic which I will not discussed, but I will illustrate the dynamics of moderate stratification.

Now, this is the theory of for moderate stratification, so this is for moderate stratification. And this was theory was done by Bolgiano and Obukhov, 1959, ok, way back. So, according to their theory, the kinetic energy flux will decrease with k as $k^{-4/5}$. Why because this sink of, so this force is taking it away. It is like the analogy which I used to make is for financial system is corruption. So, there is some things I just taking your money flow, no. So, this buoyancy force is taking it away. And where is it pushing? It is pushing into potential energy and of course, finally, it goes into heat, ok. So, let us start to derive how to get $k^{-4/5}$from scaling arguments from dimensional analysis.

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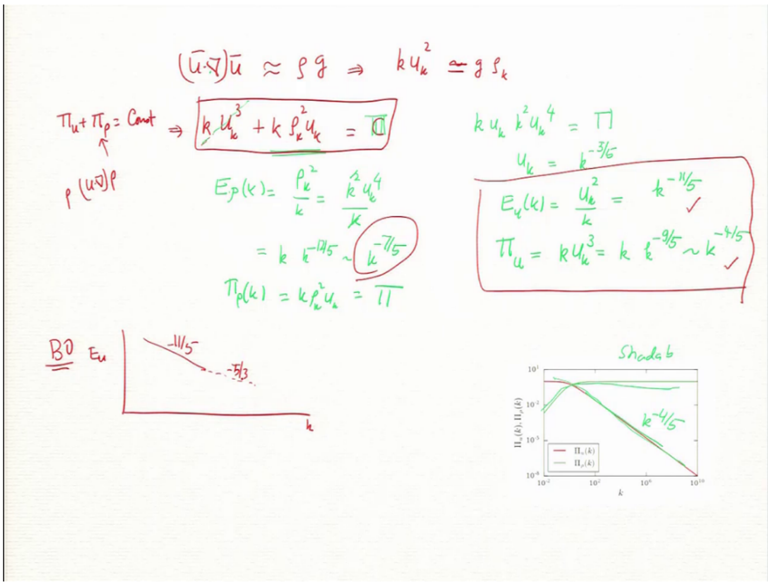

So, I will quickly derive it, ok. So, equate the non-linear term with gravity, but ρ_k is function of k, right, ρ is active at all the scales. So, it gives you $ku_k^2 = g\rho_k$. Now, you make another assumption, you take the flux argument t:$\Pi_u(k) + \Pi_\rho(k) =$ Constant.

So, what will that give you? $\Pi_u(k) = ku_k^3$ and $\Pi_\rho(k) = k\rho_k u_k^2$ in Fourier space.

I must make a remark that the Bolgiano and Obukhov dropped $\Pi_u(k)$ .So, they assumed Π_u will be small. So, Π_ρ equal to constant. So, that is an assumption. In fact, we looked at that assumption and we got some nice result, ok. Please refer to slide above for the derivation for the kinetic energy flux and the energy spectra.

So, Bolgiano and Obukhov say that kinetic energy decreases steeper. Steeper means faster than$k^{-5/3}$. And why is it steeper? Because some energy was supposed to flow which would have maintained five-third, but somebody is already taking things at everywhere like leakage in pipe you know there is the water is flowing in the pipe, but is you flowing in Fourier space.

It is not in real space water pipe is Fourier space work, but there holes everywhere, so water will keep decreasing when you go down and down. So, that is why flux, there is a flow is like water current and that is decreasing with k. So, that decreases as $k^{-4/5}$ and similarly the density spectrum is also affected, that is $k^{-7/5}$.

Thank you.

Physics of Turbulence
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Lecture - 39
Turbulent Thermal Convection

So now, we will go to the third system. So, we did passive scalar, stably stratified flows now we will go to thermal convection. So, this is another discussion of scalar; temperature being coupled with velocity field; so, Turbulent Thermal Convection.

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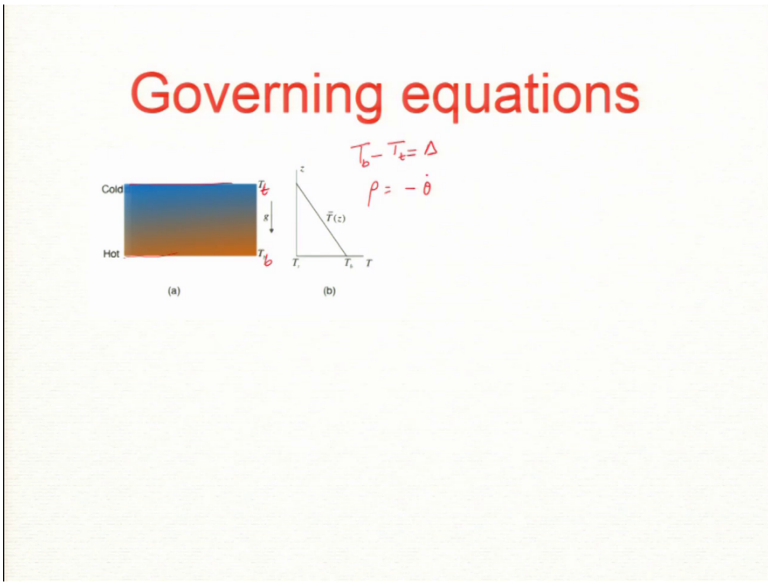

So, system is very similar except. So, let us look at the internal temperature. So, hot below know convection means you know heat from below, but cold at the top. So, this is slightly specialised is not like heating water in a container in a pan, but you also put a lid at the top.

So, hot plate below and cold plate above. So, there are two plates. So, this is somewhat simpler because you do not want to deal with the surface at the top. So, it is to simplify our life in analysis is assume that this is a plate at the top as well. So, this is the temperature difference. So, T_b, and T_t are the temperature of bottom and above plate respectively. $\Delta = T_b - T_t$ is the temperature difference. So, T_b is bigger than T_t what about the density, which is denser? Above is denser right because there the temperature is less. So, it is a system which is opposite of stably stratification that is why it is unstable. So, the top guys come down and the bottom guys go up. So, the hot looms they go up why you guys like it, which is wants to go up and the top one which is colder it is comes down and that is creates this roll and some of you already done in your project and like.

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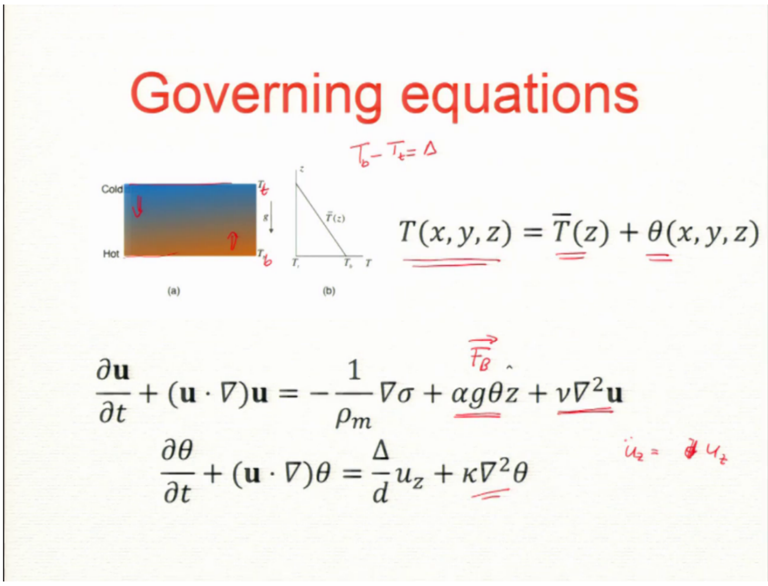

So, total temperature I write as this linear profile of temperature plus fluctuation ($\bar{T}(z) + \theta$) ok. Same idea like for density we had $\bar{\rho}(z) + \rho$ now we do the same thing here as well ok. So, θ is 0 at the top and bottom. So, θ is a fluctuation over a linear profile, now equation will be very similar to what we wrote for density. So, $\mathbf{F}_B = \alpha g\theta\hat{z}$ is the Force by Buoyancy, where α is the expansion thermal expansion coefficient.

Now, if I do the linearization and stability analysis what will I get? So, some of it I we already done know linear stability, it is a growing solution. So, this is unstable, but stability is hampered by the viscous term right. So, this we done in the pass. So, you agree that this is unstable and naturally precisely because hot thing is going up and cold is going down. So, it will make it unstable ok.

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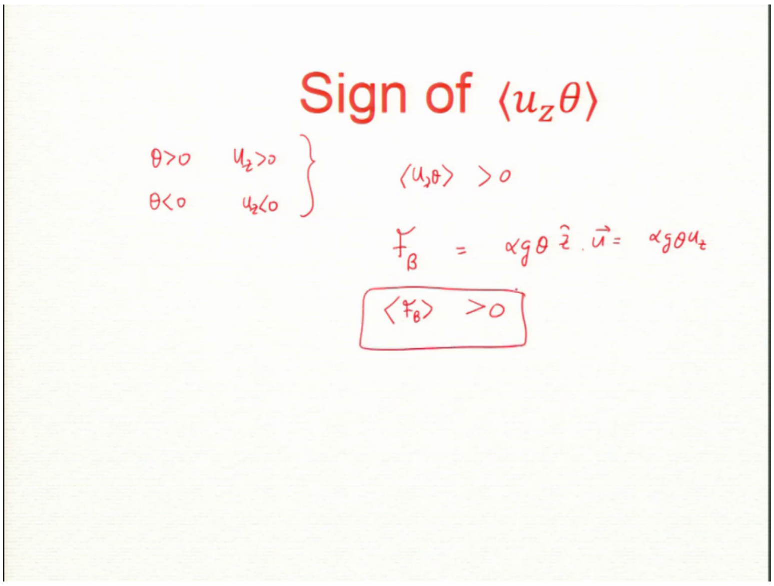

So, what is sign of u_z? I think some of you correctly said. So, u_z so, hot fluid θ positive u_z is positive right. So, θ here positive means is hotter than the ambient., if it is hotter then it will go up. So, θ and u_z have the same sign. So, $< u_z\theta >$ must be positive.

If theta is less than 0 means temperature is colder than the mean at that level; so, it will come down. So, u_z is negative. So, this implies $< u_z\theta >$ is greater than 0. So, what is sign of energy kinetic energy injection rate ($\mathcal{F}_B$) = $\alpha g\theta u_z$? If I do the average it will come out to be positive. It would have situation when it becomes negative for short while, but overall average is positive ok.

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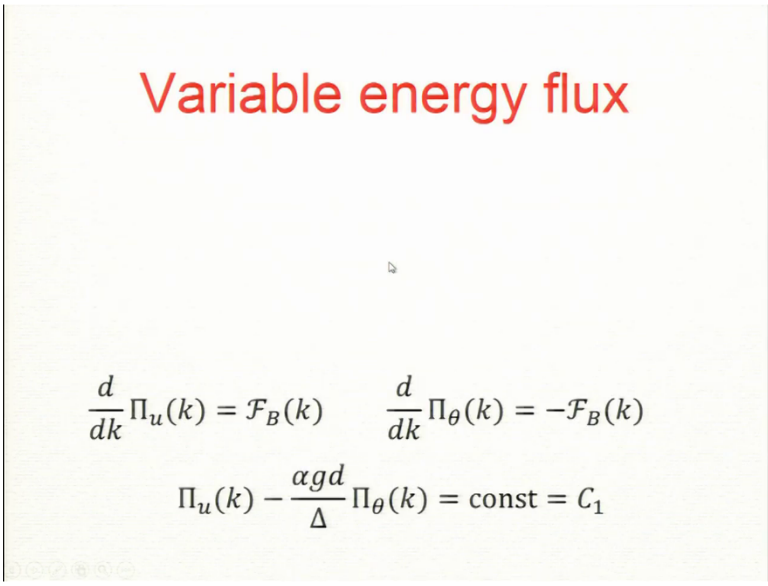

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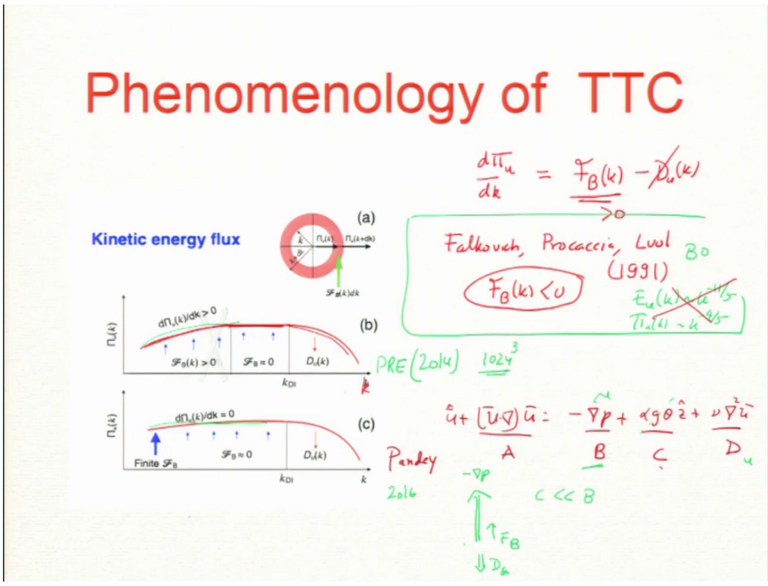

So, what should happen with the flux? So, remember this $\frac{d\Pi_u}{dk}$ in the inertia range by the way for thermal convection we do not apply any external forcing; buoyancy is good enough to drag the flow.

If you just wait for some time it will become steady. So, $\frac{d\Pi_u}{dk} = \mathcal{F}_B$ in the inertia range, I drop the viscous term (Refer Time: 06:08). So, this is positive. So, my flux must increase with k ok. However, Buoyancy is weak to increase the flux noticeably and remain constant throughout the inertia range .

But people did not have this idea. In fact, their names like Falkovich, Procaccia, Lvol. So, the people in 1991; there is two papers where their argued in by using field theory in somewhat complex way, but their calculation has one glitch which was very important glitch that claimed that $\mathcal{F}_B$ is negative. It is following from influence from Bolgiano Obukhov not the physical intuition. So, physical intuition is like everybody will agree that u_z and θ are in the same direction. So, $\mathcal{F}_B$ is positive.

So, it was a mistake and as a result they had argued. In fact, big community everybody is believed that Bolgiano Obukhov theory should also work for thermal convection. So, spectrum must be minus $k^{-11/5}$.

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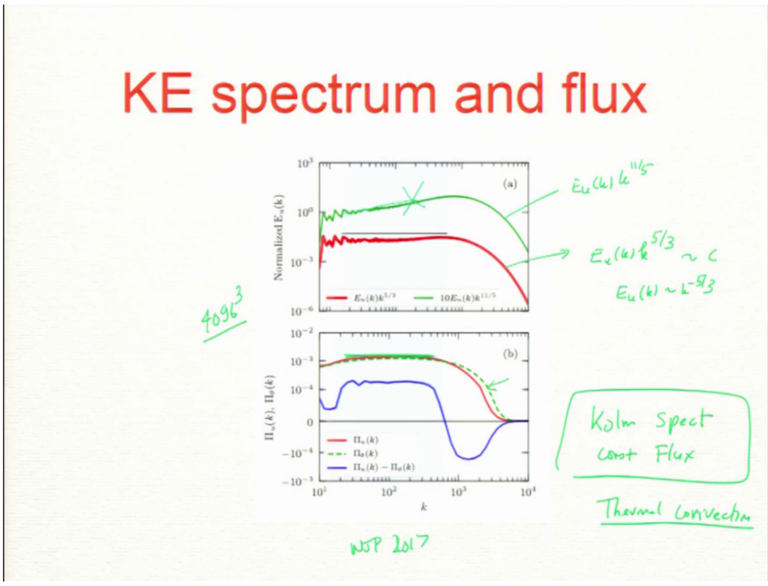

So, this is normalized spectrum $E_u(k)$. So, we you multiply $E_u(k)$, this one is multiplied by $k^{5/3}$ and it is flat. So, $E_u(k)$ vary s $k^{-5/3}$. This one is multiplied by $k^{11/5}$ and the compensated plot is increasing; that means, $E_u(k)$ is not going as$k^{-11/5}$.

So, it is very different than stably stratified flows and it better be because their nature of energy induction into kinetic energy is different.

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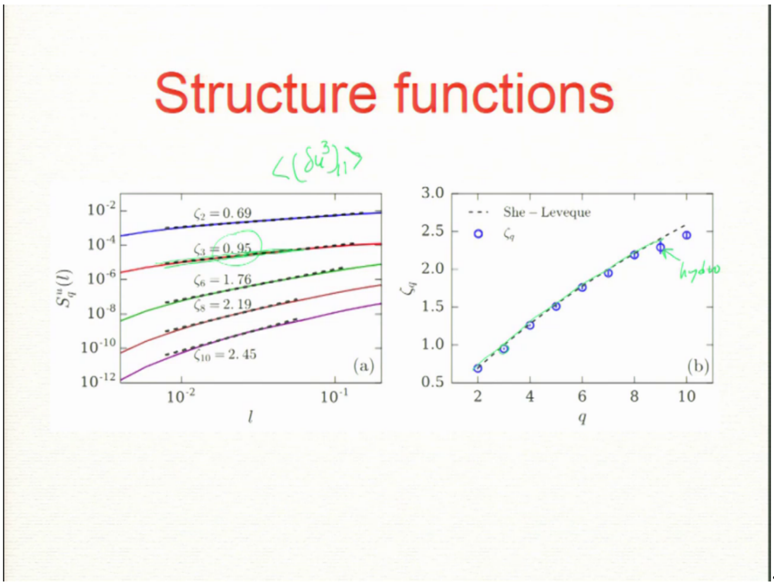

And so, we have done this structure function. So, what do you expect for structure function? So, you look at S_3 .So, S_3 is what is according to Kolmogorov theory.

Student: (Refer Time: 14:16).

So, she Leveque is good for hydro hydrodynamic turbulence it fits very well, but we see RBC is also fitting very well with the She-Leveque model; that means, this structure function is telling us that hydrodynamic turbulence and thermal convection are similar property of turbulence and which is as expected I mean basically in same lines pressure gradient is driving the flow.

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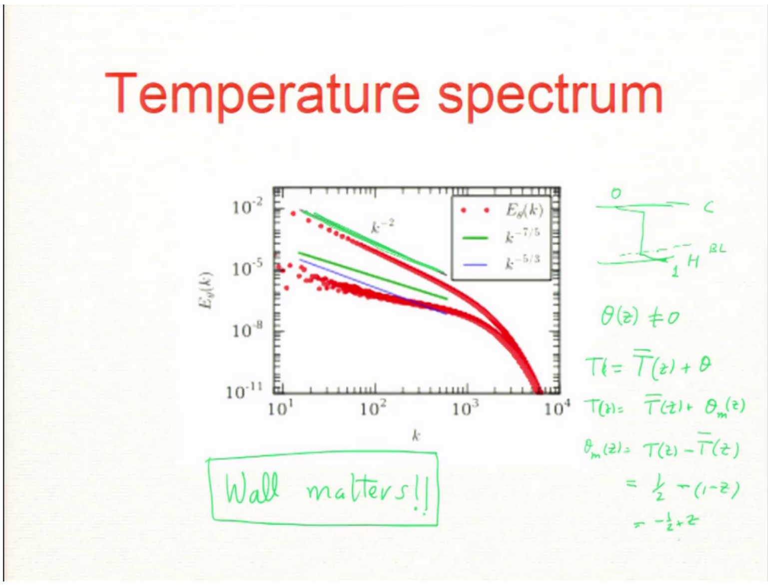

Now, we can also look. So, this is a technical part. So, we will look for temperature spectrum. Now I will not explain this, we get this funny structure these one two lines. Now these two lines bothered me for six months I did not know why these two lines are coming. This coming in the simulation result and you always suspect there is something wrong with the simulation and it turns out this is very physical; this is coming from physical region this is correct. And why it is correct? Because the two walls hot wall and cold wall and the temperature has mean profile.

So, θ if I do the average your planar average. So, theta overall if you do over the whole volume it is 0 if you do the planar average it is not 0 this is not equal to 0 ok. So, in fact, it is easy to see. So, let me just write this. So, $T(x, y, z) = \bar{T}(z) + \theta(x, y, z)$. If I do the planar average means integrate over x y. So, this becomes $T(z) = \bar{T}(z) + \theta_m(z)$

So, $T(z)$ for thermal convection, it looks like this (please see the above slide). So, this is hot plate and cold plate. So, temperature decreases a rapidly, it is constant the middle and then decreases the again ok. So, I need the plate this sharp drop in temperature now this is called boundary layer ok. So, boundary layer temperature drops by half. If the whole drop is one unit; so, half at the bottom and half at the top and in between is constant.

So, what do you expect for $\theta_m(z) = T(z) - \bar{T}(z)$? So, in the bulk, assume this to be one and this is 0 then this $T(z)$ is $1/2$ and $\bar{T}(z) = 1 - z$ So, $\theta = z - 1/2$. If I do the Fourier transform, I am not going to get 0. So, this is the Fourier transform this gives you that top line.

So, this we show it in our paper. So, the mean profile of θ_m is giving me the top line in the remaining modes Fourier modes are giving you the bottom line ok. So, this is what I say that wall matters in turbulent flows you put a wall then you expect something different. So, Kolmogorov theory it is of course, very beautiful, but it does not explain lots of phenomena. So, wall so, my phrase is wall matters for turbulence. You cannot say homogeneous and isotropic flow is the theory and rest all are like minus theory.

So, this is you know we have this grand unification, or we have like universal theory. So, I gave a colloquium which I do not know some of you are not there. So, the one theory is not going to explained everything instead I am strongly disbeliever of that idea and one theory you can explain everything. So, Kolmogorov theory is nice, but is only works for homogeneous isotropic turbulence of course, this gives the idea about how do you use flux, what does flux do, but one idea is to explained everything like what we have in particle physics that does not work ok. Here wall is important ok.

Thank you.

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Lecture - 40
Flow with a Vector

We will deal with Flows with Vectors now. So, velocity is a vector, but it is advecting another vector. So, example being magnetic field, you know magnetic field is advected by velocity field. Of course, magnetic field does something to the velocity field too. So, that is what we are going to solve. So, I will put as a general framework, ok. So, flow with a vector. It could also vector need not be only magnetic field. It could be dipoles, electric dipoles or some vector particle which is travelling. So, one example you say active or like fish population. So, that is like a vector you know going around, ok. So, let us look at the equations.

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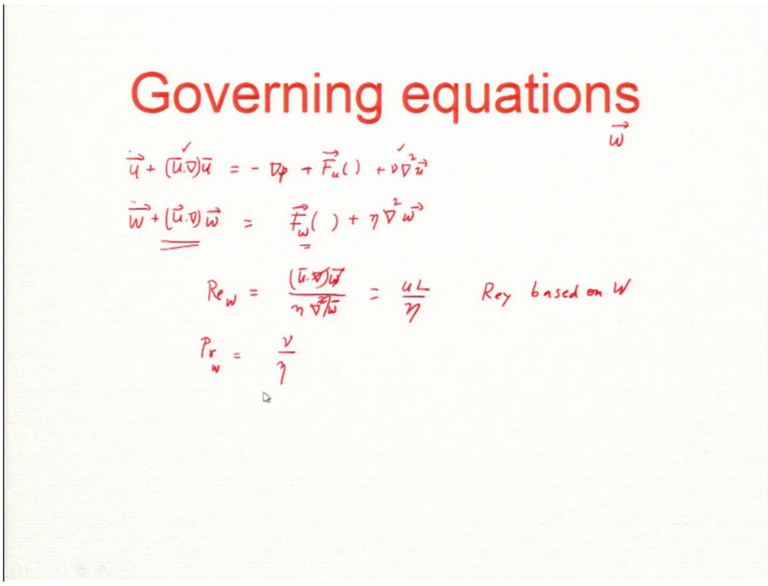

So in fact, is very similar to what you do for scalar, there is a very similar framework. So, let us write down those equations. So, velocity field will be same as before and I am going to put another vector field $\boldsymbol{\omega}$. We write equation for $\boldsymbol{\omega}$. So, $\boldsymbol{\omega}$ is a vector field, ok. So, $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ is the advection of $\boldsymbol{\omega}$. There is no pressure gradient in this, ok, so pressure gradient only acts from the velocity field. So, there could be force on $\boldsymbol{\omega}$. Let us keep a general framework. It could depend on $\mathbf{u}$, no, or it could depend on time it could be general.

Of course, $\boldsymbol{\omega}$ can back react on $\mathbf{u}$ or $\mathbf{u}$ can also influence $\boldsymbol{\omega}$, $\mathbf{u}$ is influencing $\boldsymbol{\omega}$ here, but $\mathbf{u}$ can also influence $\boldsymbol{\omega}$ here, ok. So, we define few quantities Re_w, Reynolds number based on $\boldsymbol{\omega}$, is the Peclet number, so is the ratio of the non-linear term, $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ by $\eta \nabla^2 \boldsymbol{\omega}$, right. So, usual non-linear by diffusion, which is $\frac{UL}{\eta}$.

So, this Reynolds number based on $\boldsymbol{\omega}$, is different then, Reynolds number for velocity field, which is $\frac{UL}{\nu}$. So, you can also define Prandtl number, which is $\frac{\nu}{\eta}$, but this Prandtl number of $\boldsymbol{\omega}$, so, it turns out which will not do in this course but their times when we have both temperature and $\boldsymbol{\omega}$. So, magneto convection, you know which some of your working on. So, then we will have two Prandtl numbers, $\frac{\nu}{\kappa}$ and $\frac{\nu}{\eta}$. So, you need to differentiate that time, also Re. So, there are Peclet number, which is different than Re_ω, ok. Now, so this is clear. So, these are the equation nothing more complicated.

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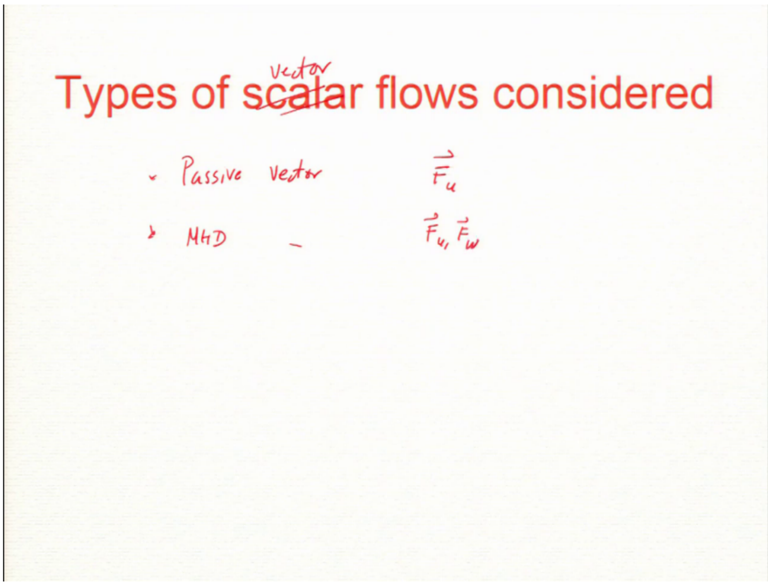

So, what type of you know? I am sorry this is vector flows considered. So, right now for today’s lecture I will consider two types of flows, one is a passive vector. Passive vector is $\mathbf{F}_u$ is independent of $\boldsymbol{\omega}$. So, force is, velocity field is not affected by $\boldsymbol{\omega}$, so it is passive, no. So, $\boldsymbol{\omega}$ is driven by $\mathbf{u}$, but not $\mathbf{u}$ is not driven by $\boldsymbol{\omega}$. So, it is called passive, is same thing is passive scalar, ok.

And second, I think some of you will be interested is MHD, magneto hydrodynamics, ok, where $\mathbf{F}_\mathrm{u}$ and $\mathbf{F}_\omega$ are quite complex and which I will try to do later.

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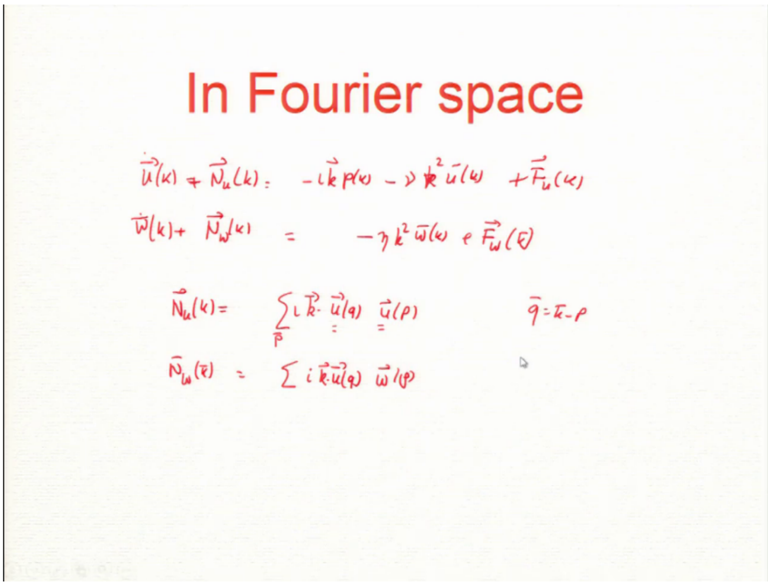

So, your Fourier space how does the equation look like? Please refer to the above slide foe equations in the Fourier space.

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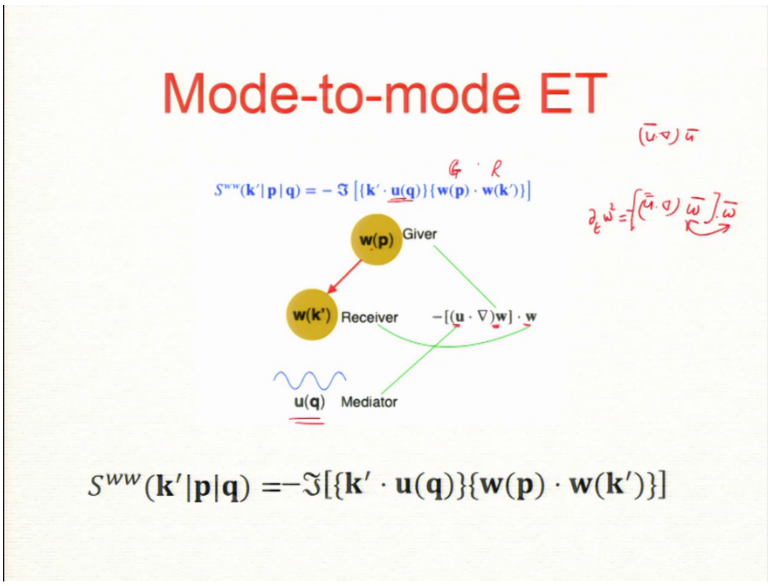

Now, mode to mode transfer. So, for, so there will be quite a few, but $\mathbf{u}$ part is same as before. So, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ will induce energy transfer among u modes, that we done before. So, I will not worry about it, ok, right. I will not discuss here.

But now there will be a transfer from $\boldsymbol{\omega}$ to $\boldsymbol{\omega}$ because there is a term, so $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$.

$\boldsymbol{\omega}(\mathbf{p})$ gives energy to $\boldsymbol{\omega}(\mathbf{k}')$ and $\mathbf{u}(\mathbf{q})$acts as a mediator, ok. So, this is what I have written here. This $\boldsymbol{\omega}$ is the receiver, this $\boldsymbol{\omega}$ is the giver and this $\mathbf{u}$ is a mediator. So, what is the formula?. Please see the above slide for the formula.

Now, this is coming from advection. Now, we will have $\mathbf{F}_u$ and $\mathbf{F}_\omega$that can also lead to energy transfer.

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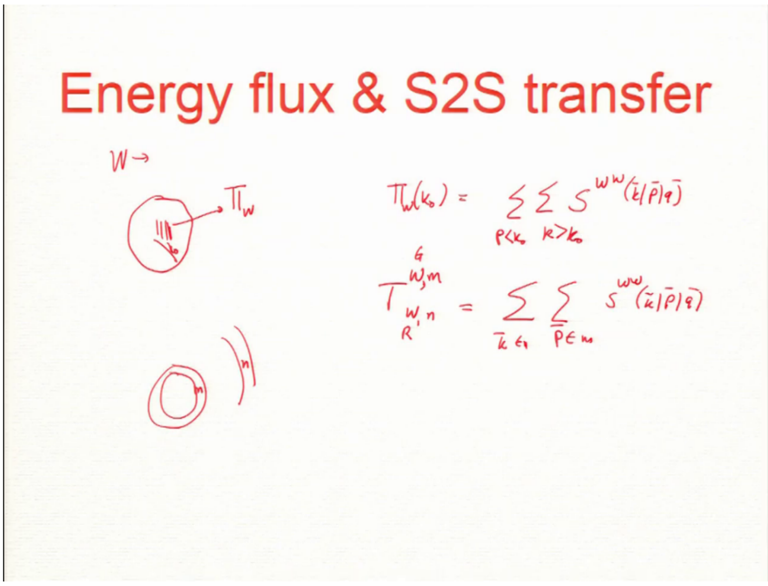

Now, come to the energy flux, Kinetic energy flux same is before, but now we need to worry about is there is a flux for $\boldsymbol{\omega}$ because there is a there is a mode to mode transfer, so there will be flux. In fact, can easily see the flux of $\boldsymbol{\omega}$, Π_ω which is energy going from modes inside to modes outside, please refer to the slide for the formula. from w to w.

We can also define shell to shell transfer. So, T^{ω}_{ω}, so this is my notation. So, giver is above, receiver is below. So, this is a giver, this is a receiver. So, shell, which shell? So, there two shells, no, shell m and shell n. So, shell m is a giver, shell n is a receiver. So, what should be the formula. Please refer to the above slide.

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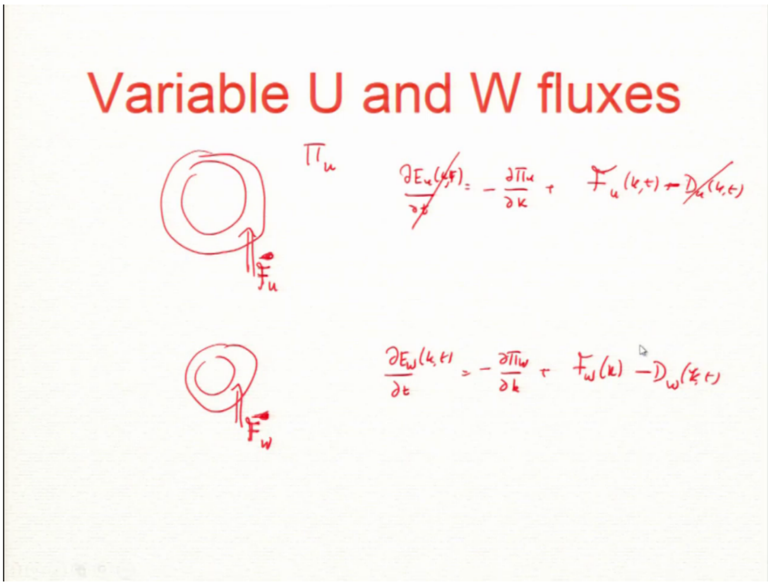

Now, let us work out what happens to the flux. So, what can change flux? We did in the past for $\boldsymbol{\omega}, \boldsymbol{u}$ and for θ as well. So, if we have a shell is this is injection by external force in the inertia range which will come from $\mathbf{F}_u$, then it can change the flux, this for Π_u. similarly, we can do for $\boldsymbol{\omega}$, if there is a $\mathbf{F}_\omega$ then that can change the Π_ω. In the above slide, there are the equations for the variable energy fluxes.

So, in the inertia range this is dropped, this is dropped, and flux can change because of $\mathbf{F}_u$, correct, these we done in the past. So, there is a diffusion term. So, that will diffuse $\boldsymbol{\omega}$, ok. So, we need this for MHD, but for the next slide I will not really, I will make it simple. Now, this is general framework. So, let us think of what happens to passive vector.

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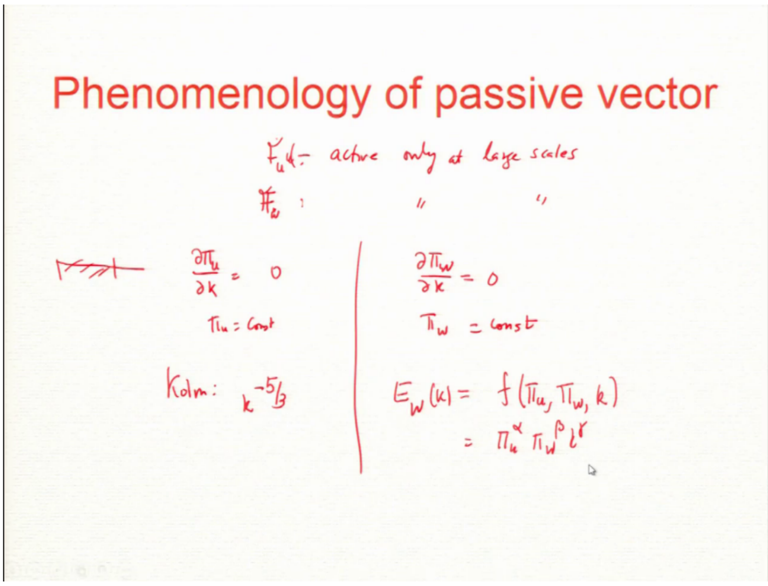

So, what happens to passive vector? $\mathbf{F}_u$ is on is not a function on $\boldsymbol{\omega}$ and let us assume it active only at large scales. Also assume that $\mathbf{F}_\omega$,and the injection by it are active at large scales. So, inertia range there is 0. So, if you look at the equation from the previous slide, what happens to $\frac{d\Pi_u}{dk}$in the inertia range? There is no dissipation, there is no injection, so this is 0. $\frac{d\Pi_\omega}{dk}$, what happens to that? 0 too. So, both Π_u and Π_ω are constants in the inertia range.

So, what is the spectrum for a kinetic energy? Same is Kolmogorov because $\boldsymbol{\omega}$ does not do anything. So, E_u this is same as Kolmogorov. What can you say about $E_\omega(k)$. So, it can depend on function of Π_u, where Π_u is crossing this regime, right, inertial range is seeing Π_u. Inertial range also says Π_ω, it also sees k. So, its function of three quantities.

Please refer to the slide below for the derivation of scaling of $E_\omega(k)$

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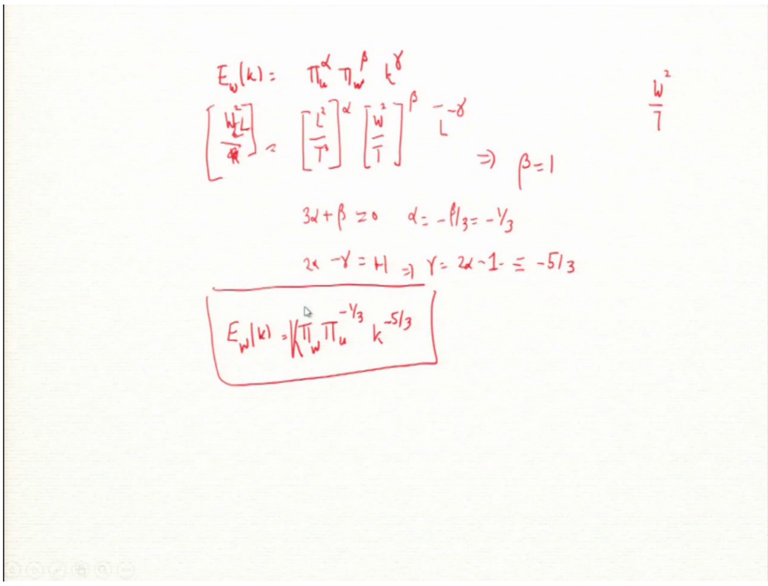

Thank you.

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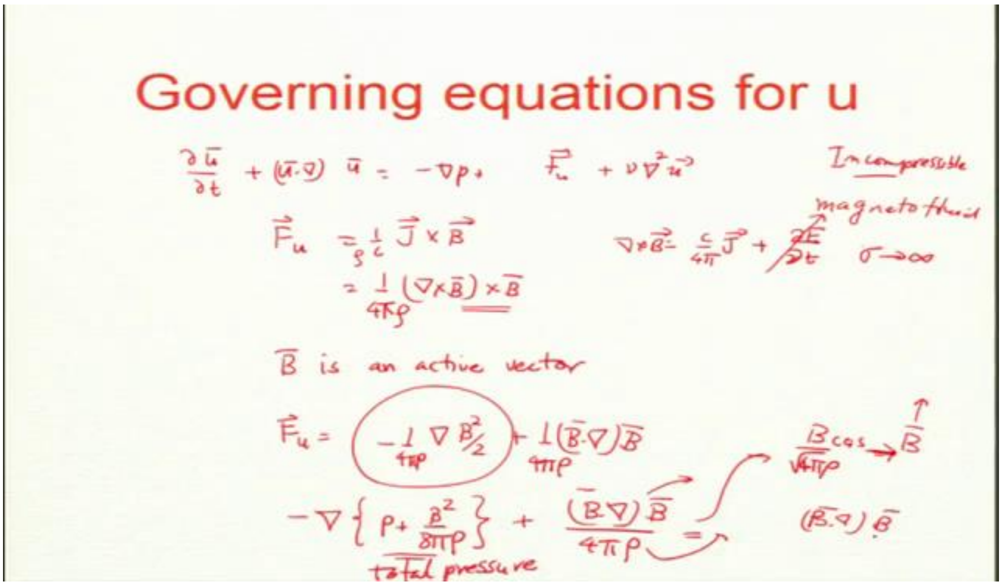
Lecture - 41
MHD Turbulence
Formalism

So I start the new topic MHD Turbulence, but this on the frame work of flow with the vector. Now magnetic field is the like a vector invaded in velocity in the flow; in the flow. So, the two vectors basically $\vec{u}$ and $\vec{w}$, but velocity field I found it is as if it is advecting the magnetic field ok. And, now we will use magnetic field as a $\vec{B}$ variable not $\vec{w}$ there is a standard notation in literature ok.

1. Formalism
2. Energy transfers and fluxes
3. MHD turbulence models
4. Dynamo

So, the topics will be formalism I will set the equation like energy transfer and fluxes. Then I will just describe few turbulence models there are many models, but I will describe what I considered to a considered important one and then dynamo ok. So, we will quickly go through these topics each of it is like one full course, but we will try to do in 1 hour and we will see.



So, formalism so let us first set equation for $\vec{u}$. So, we already have set equation for $\vec{u}$. We need to only worry about $\vec{F}_u$. So, that is there was formalism. In fact, I do like the way it has been written and it is very useful to think in a global way you know. So, let us rewrite that equation for $\vec{u}$, u ∇p plus $\vec{F}_u$ vector u plus ν $\nabla^2\vec{u}$. So, what is $\vec{F}_u$? So, I am assuming constant density here. So, we will assume incompressible which is good approximation for many lessons, not so good for some extra physical applications. But it is quite good for sun or this they work quite well.

Sun, yeah sun is nearly incompressible roughly ok. Quite good I will not say best, but it is quite good. For galaxy also, but I will not discuss it privately we can discuss. So, there are ways to say that incompressible is reassembly ok may be I will make a remark. So, first no I think let us skip this because this will go into various approximations and so on $\vec{F}_u$ just think of liquid metal ok. So, inside that there is a lot of metal and there is a magnetic field. So, and it is like molten metal so it is like water you know. So, it is incompressible. So, that is what you should think in your head. So, what is the force density? So, this is the force density right and all of this is.

ρ equal to it is a force density force per unit volume ok. So, what is force density formula for in a electromagnetic fluid or it is called magneto fluid. You will find this is what fluid which is response to magnetic field water does not respond very well, but salt water will respond weakly. But metals will respond heavily free unit free electrons ok. So, $\vec{F}_u$ is $\vec{J}$

$\times \vec{B}$. Current density it is like it is a Lorentz force ok. Now, if you use S I or if you use CGS then there are different.

Student: Ok.

Constant in front normalization factor; so, I will use SI you know I will use CGS, I will use CGS unlike you had other forces, but I will use CGS ok. So, plasma I think lot of you will use CGS so you will use c 1 by c; c is speed of light. Now under MHD approximation, so this is another Maxwell equation which is d by d t no d$\vec{B}$ by dt is J minus so 4 π by c J minus no $\nabla \times \vec{E}$.

Plus or minus I am I should not.

Student: Sir, $\nabla \times \vec{B}$ I think.

Oh yeah it $\nabla \times \vec{B}$ $\nabla \times \vec{B}$ and these are d$\vec{E}$ by dt right.

Plus.

Student: plus.

Ok, but in MHD approximation we drop this term. So, we assume that electric field is much smaller than $\vec{B}$. Now I will not describe these approximation justification and so on with these what is assumed. So, electric field is much smaller is highly conducting fluid. So, conductivity is tending to infinity and for that this is a good approximation. So, we will focus on energy transfers, fluxes, models. We will not discuss too much into this details ok.

So, if you put that in we will get c c. So, 4 π by c; you know c by I mean too many mistake c by 4 π so c c cancels. So, you get a $\vec{J}$ is $\nabla \times \vec{B}$ curl B by 4 π and then the density is I am assuming one right now, but the density also comes. So in fact, forcefully there is a density should be sitting here first force density, so we divide with ρ ok. So, this formula for $\vec{F}_u$, so force on u is depends on B so; that means, B must be passive scalar or active scalar.

Student: Active.

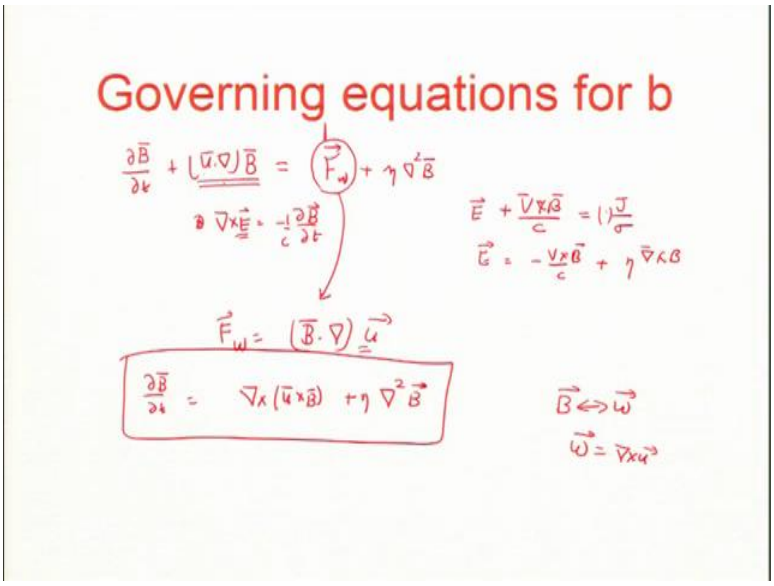
Active scalar, so B affects u. So, B is a active scalar an active scalar you know active vector sorry not active, $\vec{B}$ is an active vector ok. So, our formula which I derived for passive vector will not work ok. So, and moreover the force is quite non-linear know.

Is B B which is a non-linear force unlike buoyancy force which is linear buoyancy was theta you know or density. So, this is non-linear force. So, that makes life reassembly complicated for MHD ok. So, this is what we have to keep in mind that a force is curl of $\vec{B}$ **X** $\vec{B}$. In fact, we can simplify this force let me write it here. So, we can write curl of $\vec{B}$ **X** $\vec{B}$ is minus.

Curl of B^2 by 2 this is 1 by 4 π ρ is outside, plus $(\vec{B} \cdot \nabla)\vec{B}$. So, this is the interesting form. Why it is interesting? Because this part can be observed with pressure right, so we can take this part. So, I say minus grad p plus B^2 by 8 $\pi\rho$ and the other part is $(\vec{B} \cdot \nabla)\vec{B}$ by 4 π ρ interesting know so this is called total pressure.

So, it is a hydro pressure p plus B squared by 4 pi rho which is 8 pi rho which is coming from magnetic pressure. So, it is p is called thermodynamic pressure plus magnetic pressure ok. So, these are total pressure and this is the so I am missing out some vector signs, but these are all vectors. So, this is our force which is non-linear where this is also non-linear know, but I have you remember the pressure does not participate in energy transfers.

So, this term will be silent on energy transfers and we make another change of variable. So, I will make a remark here in CGS you make this variable $\vec{B}$ by $\sqrt{4\pi\rho}$ this as dimension of velocity ok. Now this $\vec{B}_{CGS}$ this is dimension of velocity, so we make a change of variable B CGS to this. So, this has very convenient know now $\vec{B}$ has dimension of velocity. So, we can write this as this part is B dot $(\vec{B} \cdot \nabla)\vec{B}$ and 4 $\pi\rho$ is disappeared ok.



So, now let us go to velocity field magnetic field. So, magnetic field equation you will again borrow the equation for active vector or for the vector. So, now instead of w I am just writing as B. So, d$\vec{B}$ by dt plus $(\vec{u} \cdot \nabla)\vec{B}$. Now the right hand side is F w plus η grad square $\vec{B}$. So, I need to derive this some Maxwell equation. So which Maxwell equation I should use? So, we have one equation is curl of $\vec{E}$ is minus d$\vec{B}$ by dt.

And in CGS E and B have same dimension. So, there must be 1 by c here know ok. Now these are nitpicking on dimensions, but I will not I will basically ignore units I rather I will ignore this constraints, but this there must be c. Now E in I mean that MHD approximation is minus V. So, in the moving frame of the fluid if we go with the fluid the electric field inside is 0 for ideal flow.

Now, electric field inside a conductor is 0. So, $\vec{E}$ plus so by Maxwell not by Maxwell by the relative c transformation electric field inside the fluid is this, under u by c must less than 1 ok. This is a transformation of $\vec{E}$ and $\vec{B}$ under relativity so you will find this is my E prime ok. So, this will assume to be 0, but if the conductivity is large then this is J by σ. Now, if you want to want to get this proper units no I do not derive it here ok.

So, there is a $\vec{J}$ by σ, but there is a some appropriate factor ok. So, I can find E know, $\vec{E}$ is minus $\vec{V} \times \vec{B}$ by c plus $\vec{J}$ by σ which is curl of $\vec{B}$, in fact η comes here made it diffusivity here ok. So, plug this in if we plug this in for electrical field here, you plug this in then I will get a term for F_w. So, this is you define this is how you derive equation for $\vec{B}$. So, I write down the equation now so F_w under my new units, my units were B is in the terms of velocity F_w is B dot.

Student: Ok.

Grad $\vec{u}$ ok, this is just simple right it is nothing complicate. So, you have to use the formula curl of $\vec{A}\mathbf{X}\vec{B}$ or curl of $\vec{u}\ \mathbf{X}\ \vec{B}$ and you will get this. In fact, you can also write in other way $d\vec{B}$ by dt is curl of $\vec{u}\ \mathbf{X}\ \vec{B}$. Now these derivations I am not emphasizing because lack of time, but these are interesting derivations. Now these equation looks very similar to another equation we will just seen in the course.

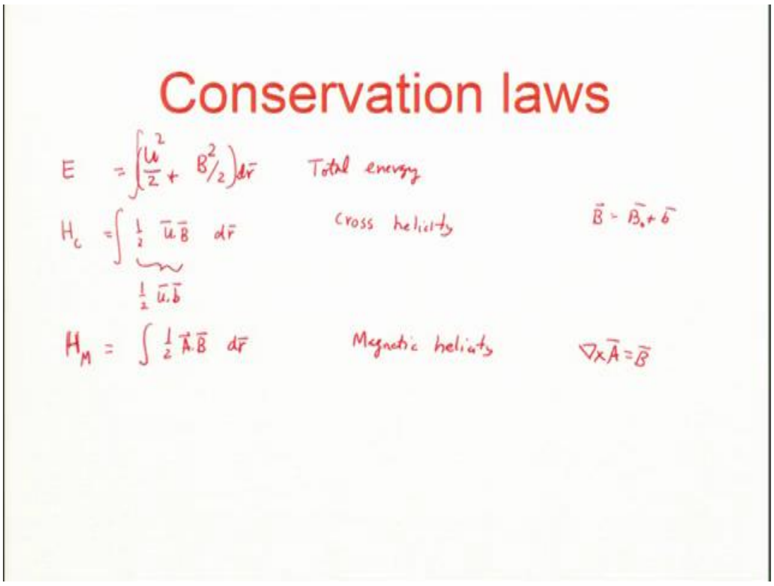
Student: Vorticity.

So, equation was the vortictiy know. So, B and w are similar in fact, they are not identical, but the equation is equation is identical. But they are not same because vorticity is.

Student: Curl.

Curl of $\vec{u}$, but $\vec{B}$ is a independent vector that is the difference, otherwise equation is exactly the same else is some of the dynamics. So, vorticity is stretched by the velocity fluid and this is called stretching of the vorticity you know same thing happens for the magnetic field. Magnetic field is stretched by the velocity field and that some of the magnetic field increases in dynamo. So, magnetic field is generated by this process of stretching of magnetic field ok.

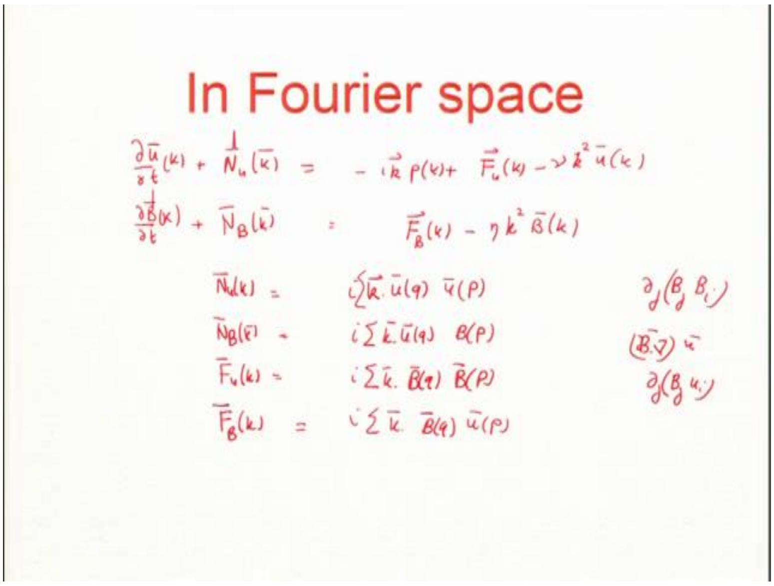
So, may be briefly I will describe it in the end ok, so this is the equation for $\vec{B}$ now. So, this is the advection term and this is u is acting on B of course, so this is like $(\vec{B} \cdot \nabla)\vec{u}$. But it has very similar form is $(u.\nabla B)$ you know except B has come before and u has gone later. So, these are similar non-linear terms.



So what is the conservation laws? Now because F_u and F_b; u squared is not conserved, F_u is not 0 right. I mean if you put in magnetic field I cannot set the force to be 0 and F_b is also not 0. So, they make the conservation law quite complicated.

So, u^2 is not conserved B^2 is also not conserved, but together they are conserved, this is called total energy. So, integral nah integral of this quantity over whole space is conserved. Another quantity called H_c it is called cross helicity which is defined as half $\vec{u} \cdot \vec{B}$. So, $\vec{u} \cdot \vec{B}$ right now I am keeping B as a total field it can have a some mean component. So, B field could be $\vec{B}_0$ plus $\vec{b}$, but when I put a $\vec{B}_0$ u is fluctuating. So, $\vec{B}_0$ will not contribute know such that we will get cancelled $\vec{u} \cdot \vec{B}_0$ will be 0.

So, this is same as half $\vec{u} \cdot \vec{b}$ ok. And there is another consecutive is constrained a which is constant is kinetic helicity, a magnetic helicity. Magnetic helicity which is half $\vec{A} \cdot \vec{B}$ is a vector potential ok. So, curl of A is B. So, proof I will not discuss, but these are three conservative quantities for MHD, u^2 and B^2 are not conserved individually the total is. This is for 3D; 2D has certain modification, but we will skip 2D discussion ok.



Now Fourier space how the equation look like? So, is I mean it is straightforward if you just have to look at your notes and pick up the threads. So, du dt plus Nu, so my notation is the in this book is ok. So, I write like $\vec{N}_u(\vec{k})$ and minus i $\vec{k}$ vector p(k) plus $\vec{F}_u$ of k is force. So, it is function of $\vec{k}$ and minus ν k^2 u(k). Let us something that coming here ok. I am going to write $\vec{N}_u$ and $\vec{F}_u$ soon and for $\vec{B}$ $\vec{N}_B$ of $\vec{k}$ non-linear term coming from u dot grad B and this is a $\vec{F}_B$ of $\vec{k}$ minus η k^2 $\vec{B}$ of $\vec{k}$.

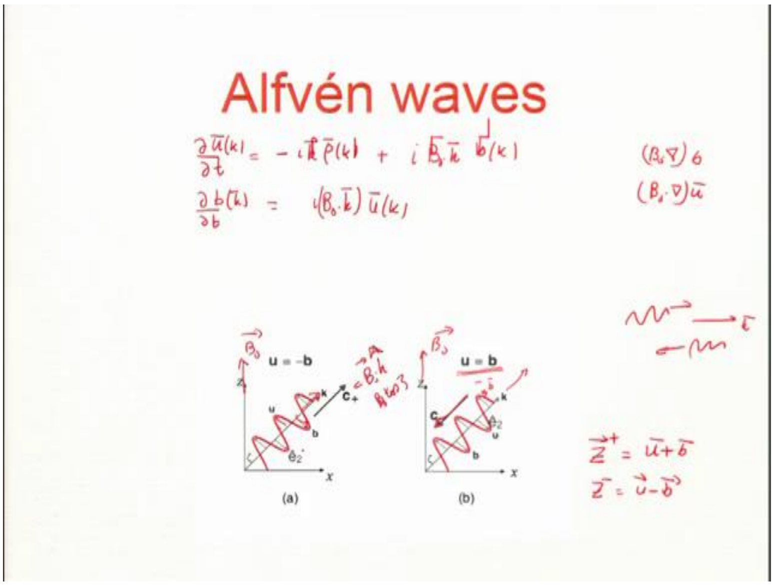
Now, $\vec{N}_u$ of $\vec{k}$ you know already right. So, let us for completeness I write it here $\vec{k}$ dot $\vec{u}$ of $\vec{q}$; $\vec{u}$ of $\vec{p}$, $\vec{q}$ is $\vec{k}$ minus $\vec{p}$ correct. What about $\vec{N}_B$ of $\vec{k}$? All vectors know no minus this is plus i $\vec{k}$ dot $\vec{u}$ (q) $\vec{B}$ of $\vec{p}$; $\vec{F}_u$ it is $(\vec{B} \cdot \nabla)\vec{B}$.

$\vec{B}$ is also divergence free right. So, you can write them as ∂_j B_j B_i.

Yes. So, we know what is ∂_j u_i u_j so it is very similar. So, i $\vec{k}$ dot $\vec{B}$ of $\vec{q}$ $\vec{B}$ of $\vec{p}$ right it is straight forward and F B of E sorry F B of k. So, this comes from $(\vec{B} \cdot \nabla)\vec{u}$ remember I just derived it in the last slide. So, this will be del j Bj u_i. So which is advected?

Student: Bj.

Bj is one with cons with thing. So, it is going to i $\vec{k}$ dot $\vec{B}$ $(\vec{q})$ $\vec{u}$ of $\vec{p}$. So, it is straightforward and these are the terms which will exchange energy among modes, they transfer energy from one to other ok. So, you see that very nice way to interpret these transfers and derive flux formulas.



So, I will make one more discussion is Alfven waves. So, these are the waves so in hydrodynamic flows they no really wave. Though some of you have got ODE's which are not like wave solutions, but there are oscillations right. Like we found that three more model has oscillations, but there is no wave really, but MHD has waves.

If you turn off the non-linear term we get wave solution and what are the wave solution. So, let me just make it linear equation. So, B gives you one term know so the pressure part is minus i $\vec{k}$ p of $\vec{k}$. So, this p is total pressure ok, but this is a term comes linear term is $(\vec{B} \cdot \nabla)\vec{B}$, so B is a total B. So, I can write as B naught dot grad B naught dot grad B right; so, I can write this as in Fourier space i B naught dot k b of k.

So, this is the term coming from $\vec{B}_0$ and the equation for B also has the term which is coming from $\vec{B}_0$. In the right hand side remember you have $(\vec{B}_0 \cdot \nabla)\vec{u}$ right. So, these are linear terms because $\vec{B}_0$ is a constant. So, the term which comes from is i B naught dot u of k ok. So, these are two linear equations I turn off viscosity and diffusivity. So, we can solve for them now is there in the notes you solve then you get two solutions.

So, we apply magnetic field along z direction. Then if the wave vector is along any direction wave vector can be along any direction. So, this is a direction k, then you get a solution where u and B are in the opposite direction. Can you see that, u is here and B is there.

Student: Yes sir.

And they will oscillate and also the wave will move wave moves with c plus Alfven's field and the c plus is $\vec{B}_0 \cdot \vec{k}$.

Student: k cap.

So, the Alfven field is B naught dot k cap so thanks to Naren. So, this is because it has dimension of velocity only if it is k k cap. So this angle is ζ, so B naught $B_0 \cos \zeta$ is the magnitude of the Alfven's field which is the different for different zetas. So, that is why it become 0 for ζ equal to 90 ok. This derivation I am skipping lack of time is nice derivation you can do it yourself. But is so c plus moves with B naught k, a solution is u and B are opposite and they are equal. So, there is a beauty of this unit that u and B have same dimension and in fact, they are equal so the equipartition know.

So, if you look at kinetic energy and magnetic energy on the average they are equal. So, this is another solution. If B naught is in that direction same thing, but k is going rightward, but the wave moves in the minus k direction ok. Wave is moving in c minus in that direction with the same magnitude B naught dot k, but it goes in the opposite direction. So, this is the wave which is going in that direction the wave going in that direction for given k and for this u and B are in the same direction.

So, if u is here and B will be same ok, so these are called Alfven waves and these are fluid mode. But these are wave modes in a MHD, and they have very important role to play in the dynamics ok. Now this I am very briefly describing it. Now we also have variable was z^+ and z^-.

Any questions?

Student: Small b dot k naught k.

Small.

Student: Small b dot k it is also 0 or not?

Yeah that is 0 because incompressibility.

Student: Same.

So, small b is perpendicular to the k right. So, fluctuations are perpendicular to k always. There is one subtle point what happens when the wave is along x axis.

Student: k.

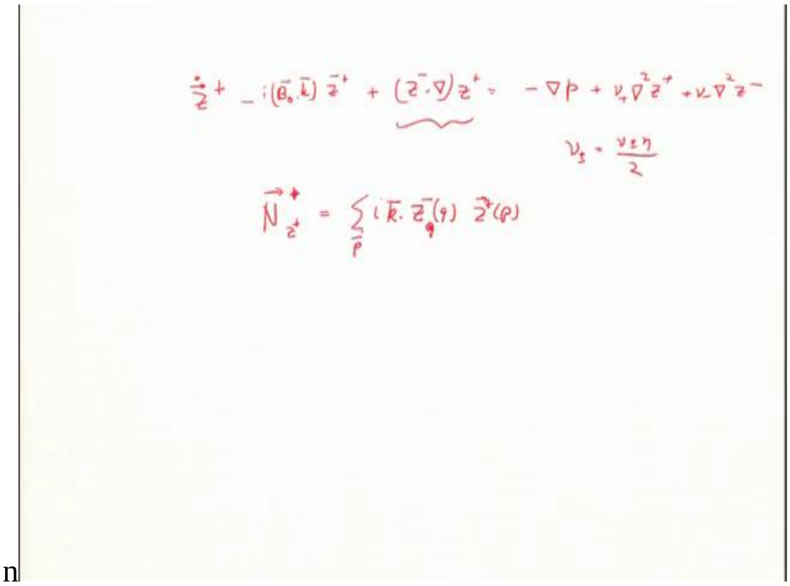
or k is along x axis. What happens to B naught dot k?

Student: 0.

Is 0 know.

Cos 90 is 0; so there is no wave on the plane on the equatorial plane. So, theta equal to 90^o has no Alfven waves ok. And, so they are non-linear term plays a big role on the wave on that on the plane. Now, we have two variables z^+ and z^- we define it for convenience u plus B and u minus B ok. So, given B you can write always z^+ z^-

Sum in difference and the equation is a much nicer in z^+ and z^-. Ok so let me write z equation in fact, this is just for, in fact we do lot of work with that.



n

So, let me write the equation for z^+ z^+ dot vector. So, this is a mean magnetic field ok.

It is a minus i B naught dot k z^+ plus z^- dot grad z^+ is minus grad pressure plus viscous terms ok, but I am not. So, let me write this I will not really discuss this terms $\nu_\pm$ r ν plus minus eta by 2. Now, equations are nice know because now we do not have 4 non-linear terms. We have one z^- dot grad z^+ this is for z^+.

What happens to z^-? We just flip plus to minus is symmetric and plus minus and except this sign becomes plus, plus wave moves in opposite direction ok. So, these are convenient non-linear terms which I will discuss in the next set of discussion. So, what will be the non-linear term by the way N in Fourier space N^+?

In fact, I write this is the N z^+ to differentiate between helical plus and helical minus, but too many plusses if you write for everything. So, the helical plus and helical minus know. So, that so I take N z^+ which is going to be i $\vec{k}$ dot z^- of q and z^+ of p ok. So, this is my equation sum over p ok. So, I think this is the last line.

Thank you.

Physics of Turbulence
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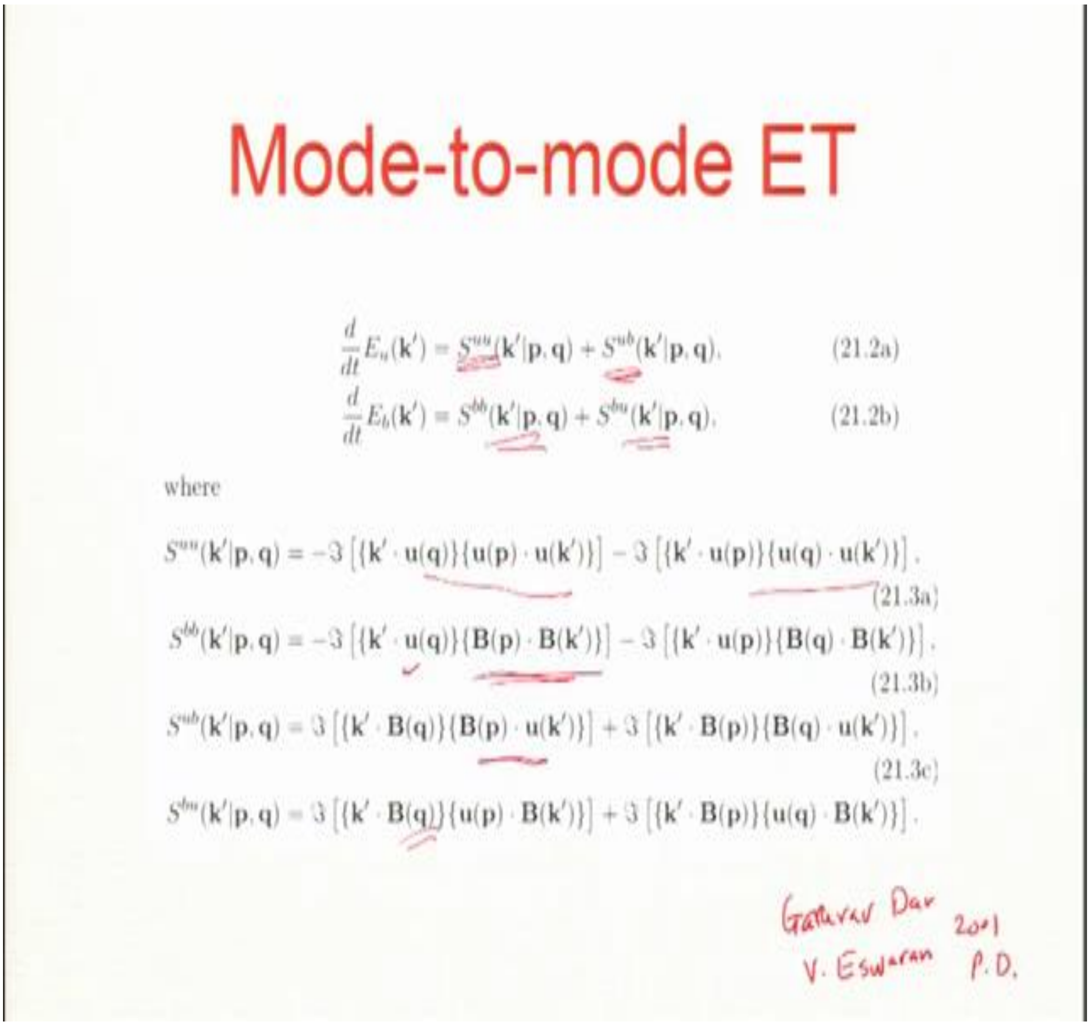
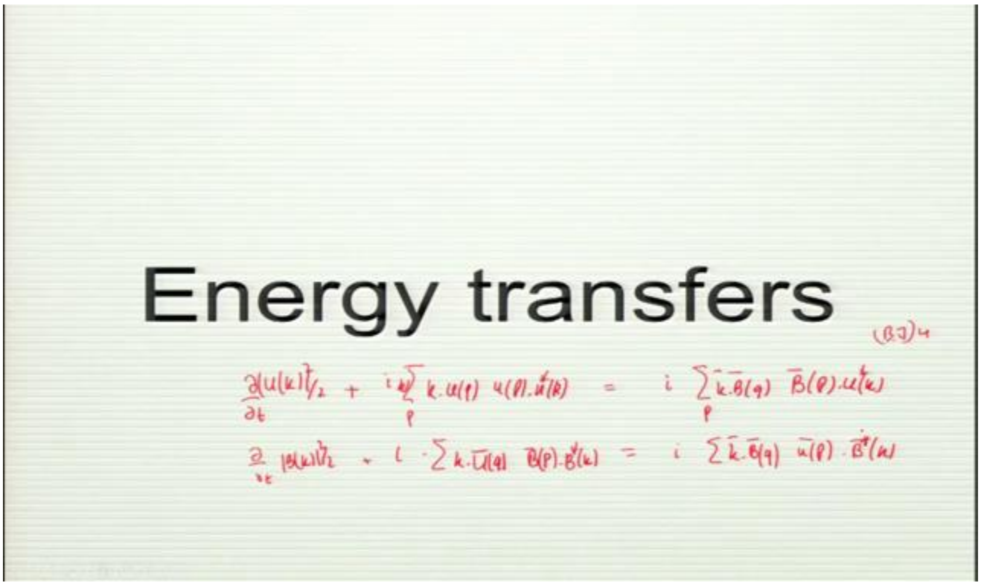
Lecture – 42
MHD Turbulence
Energy Transfers

So, we will do energy transfers in MHD now ok. So, similar well, generalization of hydro where is quite complicated, but if you are got a good hang of hydro, then you can easily do all of it.



So, the equation for energy; now, maybe I should write down equation of energy. So, we wrote the equation for the non-linear term. So, I think let me write down and go we go back. You know, I do not have a slide.



So, let us right down here, equation for the energy.

So, what is equation for kinetic energy? So, it is $|u(k)|^2$ by 2. Modal energy you know this one. So, the first non-linear term we will give you u dot grad u. So, that will give you k dot u(q) u(p) dot. This in fact, have you done it $u^*(k)$ right. Sum over ps right, Is that correct? I mean this will come.

So, non-linear dot u star the next term is right hand side is minus ∇p. So, ∇p gives you 0. Right? Because, pressure is in the k direction and I am taking dot product u(k) is 0. So, you get only the term for B dot grad B, which is, this is i sitting here ok; i k dot B(q), B(p) now dot with what?

Student: u.

Dot $u^*(k)$ sum over p's, q is k minus p. So, this is I drop the viscous term and this equation for energy. Now, we write down equation for magnetic field. $B(k)^2$ by 2 and you get i k dot, k you know sorry k is inside, in the above k dot u of q.

B of.

Student: p.

p dot.

Student: B.

B star of k, and the right hand side is i k dot.

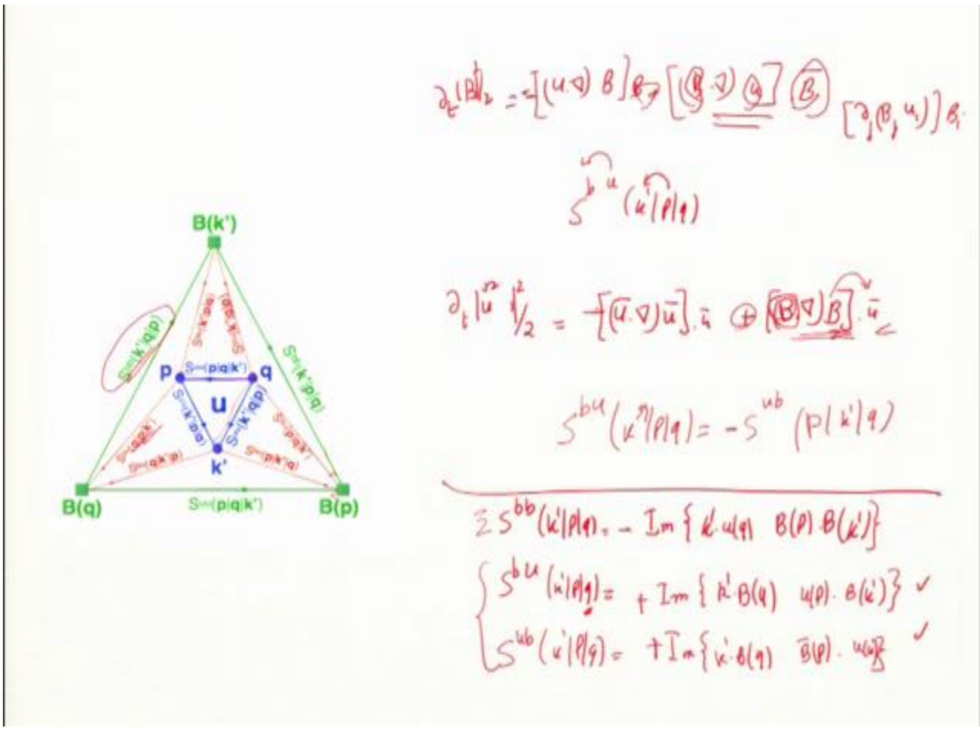
No, it is remember is B dot grad u.

Student: B.

So, k dot B, B of q, u of p. So, the difference here is who is the mediator? u of p dot B star of k, just straight forward ok; so, this is what I wrote in the next slide. So, you have two terms, now I just rearrange them. You take left hand side, right hand side. There is a u u transfer, which is coming because of u u terms. This is like hydro. There is no difference between hydro and formula and this formula. There are u B term which is this term and at this term u B term. So, this is coming off $(\vec{B} \cdot \nabla)\vec{B}$. Now you got, B B term which is here. Here u(q) advector. Remember if it is in the left, $(\vec{u} \cdot \nabla)\vec{B}$. Now that last one is.

Student: B dot.

$(\vec{B} \cdot \nabla)\vec{u}$, now which is here. Now, B I keep as total B. But, today's lecture I will not really emphasize, but it is very convenient to keep total B to see the effect of B naught, ok. But we will not delay with your on it, ok. Now from this, you can easily see that how to get to mode to mode energy transfer, right? That is what we did long time back in 2001 with my student Gaurav Dar. Sorry Gaurav Dar. His full name is Gaurav Dar and Eshwaran ok, 2001 Physica D, ok.



So, let us see the how the modes look like. So, let us focus on a triad, right. We focus on a triad only mode to mode is in a triad. So, we have three magnetic fields. Modes magnetic modes; B(k′), B(p), B(q), and three velocity modes u(p), u(q), and u(k′). Blue is velocity and green is magnetic term. Now, these are exchange of velocity among themselves, right. u to u transfer. Now, this is the B to B transfer, which is coming from which term?

The u dot grad B term; there has to be in B^2, mod B square by 2. So, this term will be the term responsible for B to B. It does not talk to you for energy transfer, its talking to u for mediation. So, here u is the mediator. So, for these transfers u is the mediator, but there is no exchange from u to B. Now, this red lines are energy going from u to B.

So, energy magnetic energy can grow or can decrease by this red lines, and that is coming from which term.

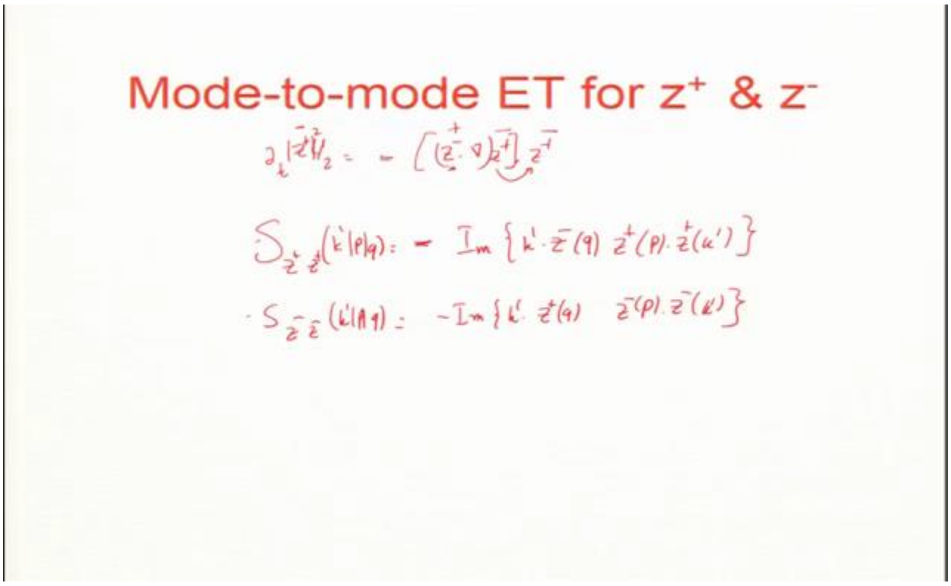
The other term B dot grad u dot with B, these are responsible for this. So, we write this S_{bu}; so, our notation is right to left. The giver is in the right, receiver in the left, ok. That is a notation we need to keep. And we write k′ p q. So, p is a giver k′ is a receiver. And we have red lines which are u to B. Is the energy transfer from B to u?

Student: B to u.

Yes, because if we write equation for u squared u(k) mod square by 2. Then we have minus u dot grad u dot this with u plus B dot grad B dot this with u. Well, I should not really write k, this is this one. This is in real space. So, this is the energy transfer from B to u. B this is a giver and u is a receiver. It is from green to blue. In fact, that should be equal and opposite know? So, you can see that, you can easily see from here if for mode to mode by property, k′, p, q. Here, u is giving to B. You can write this as u b.

Student: p.

p is the receiver, k′ q with a minus sign straightforward know. So, this is better with, this is like energy is here like a quantity, it is like probably money or any this is a property of transaction, any transaction; transaction by scalar quantity. That is what it will happen, you should follow this laws. So, you can follow similar derivation. So, what I have in the previous slide is combine energy transfer. Previous slide was, u(k) was getting from both p and q.



Now, we can derive in the same lines which I will not derive it here, but we can I will just give with the formula, no. So, I will just write down the formulas, ok. So, what are those formulas in fact, you can easily write them yourself.

So, let us write down the formulas. S_{uu} and S_{bb}, I do not need to write. Correct? Because you done it in the past. S_{uu} you have done before. S_{bb} will be like what?

Like passive, the vector which I did because that is advection, coming from u dot grad omega w. So, let us write for completeness i write, S(k′|p|q) is minus imaginary, k prime. So, is between B and B; So, B(p) dot B(k′), and u is a mid-vector or is a mediator so, u q; so, this as B(p), fine. Now this, we can write down how much goes this guy S_{ub}, S_{bu}. This is the S_{bu} term, this one. So, who is a mediator?

Student: This B.

This B is a mediator, this B right. So, to the left of grad is a mediator ok. That is what i, so.thing term coming to the left of grad. So, this as if you write this ∂_j B_j u_i dot with B_i. Correct? So, who is the mediator? Bj.

Student: Bj

Bj is the mediator. So, you will write k′; B is a mediator so, B of q. Mediator we write as q. That is our standard notation. q is the mediator here, fine and we get.

Student: B.

Who is the receiver?

Student: B(k).

B(k); B(k′). So, B(k′) and who is the giver?

Student: u(p).

u(p). So, u(p) is the giver; u(p). So, this is intuitive advector is to the left of grad and giver is to the right of this is the giver and this is the receiver, ok it is straightforward. What about S_{ub}? k′ p q minus imaginary. So, this coming from here right, S_{ub}, u is getting from B. So, who is the mediator?

Student: B.

This B is the mediator. So, k′ dot.

Student: B of.

B of q, and who is the receiver?

Student: u.

u of k′. This is the receiver. The u(k′). And who is the giver?

Student: B(p).

B(p), This is B is the giver, ok. So, these are the formulas which we can write down.

From this, we can get fluxes which I will describe in the soon, ok. So, this is the terms here and here they are responsible for energy going from one to other. Now, so you can see that there are u to u transfer, B to B transfer, u to B transfer and B to u transfer. But, if we look at z^+ z^-, the equations are simpler.

Now, without proof I am also telling you that if I look at I sum up over a triad of S_{uu}, this triads, they sum up to 0. Now, I am not proving it. These sum of, these S_{bb}, if I sum over all the modes in a triad they are 0 and if you sum these two together, then they are 0, ok. It has important consequence, but I will not repeat. I will not discuss, sorry, not repeat, I will not discuss, ok.

Let us look at energy transfer for z^+ z^-. So, here for z^+ mod square by 2. What is the form? It is z^- dot grad z^+ z^+. Correct? So, who is going to whom? z^+ is giving to z^+. This is giving to this. And who is the mediator?

Student: z^-.

z^-. So, it is straightforward. We can write down energy going from S_{z+z^-} plus k′ p q is here. There is no minus sign, this is the plus sign. Where is the plus sign?

Student: Already minus sign.

This is the minus coming from, this minus or this minus here, but these are plus signs know?

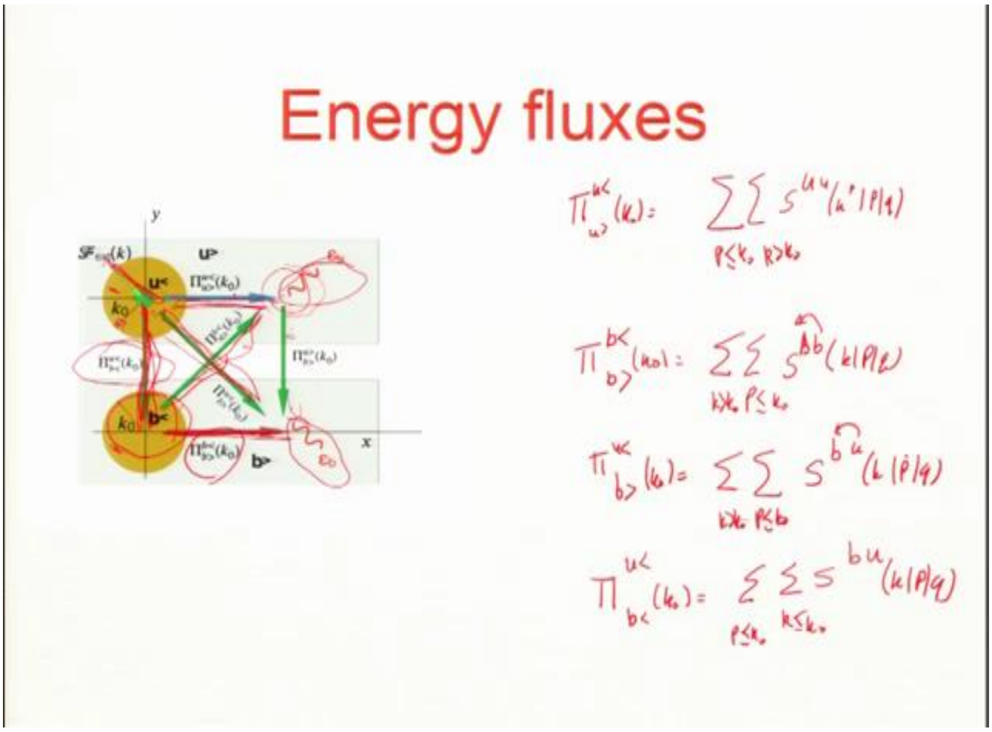
Student: Ok.

These are plus signs. So, the sign difference between S_b and Su and S_{bb} and S_{uu}, ok. Here this is a minus sign here. So, this is a minus sign here as well, the proof is bit involved, but, you can see the signs we have to be careful. I am doing mode to mode know, so minus imaginary, k′. Who is the giver? z^+ is the giver; z^+ is the mediator, z^- is the mediator.

Student: z^+ is the giver.

z^+ is the giver, p and z^+ k prime is the receiver. What happens with the minus sign, z^- z^-? Exactly same with minus flip to z reverse. So, let me just write this equation. So, these becomes minus, these becomes plus, these becomes minus, these becomes minus. This is total symmetry, and whether the B naught term does not do any non-linear transfer. These are Alfven term know, B naught dot k. That term has no non-linear, because there is only is a linear term. So, there is no non-linear transfer coming from that. This is interesting linear transfer, but we will not discuss today.

So, you can write down equation for z^- as well, k′ p q is minus imaginary, k′ z^+ of q, $z^-(p)$ $z^-(k')$, ok. So, there is energy transfer from z^+ to z^+ and z^- to z^-, there is no transferred from plus to minus. So, this is convenient. So, in fact, these are better variables for turbulence modeling, if you are looking at energy transfers and flux then ub. Though u b are measurable, that is what you measure, magnetic energy kinetic energy, but they are very useful variables.



Now, what about the flux? So, these are the fluxes of ub, ok. So, we have two spheres. You can see there is u less sphere this one and b less sphere. So, these are, so basically there is only one Fourier space, but I am writing, I am giving you on b sphere and u sphere or the same radius ok, same radius. The both of them are radius k naught. Clear?

Now, so we can define many many 6 fluxes. So, what are those fluxes? So, modes inside this sphere for u to modes u modes outside this sphere. That is Kolmogorov. So, $\prod_u$ less to u greater. $\prod_u$ less to u greater of k naught that will be S_{uu}, right. That is going from u to u, correct, Suu k prime. Let us not use k prime k p q. Where is p and where is q? Where is k? p is less than k naught.

Student: k is greater than.

And k is greater than k naught. So, this is usual Kolmogorov. This is velocity stuff and the formula is exactly same as what we did for hydro. But, now we have 6 arrows. You can see they are red and green, the 6 arrows. So, this is b to b transfer as well, this is b to b transfer. So, what is b to b transfer? So, we write Π b less to b greater, this one, k naught. What is the definition? So, b less to b greater which is S_{bb}; so, b to b know. So, giver and giver, this is giver and this is a receiver.

So, k p q; giver is within this sphere, receiver is outside this sphere, fine. You can define from, let us say this one, this guy u less to b greater. So, what is that? ∏u less to b greater. So, now, what should I use now please tell me?

Student: u.

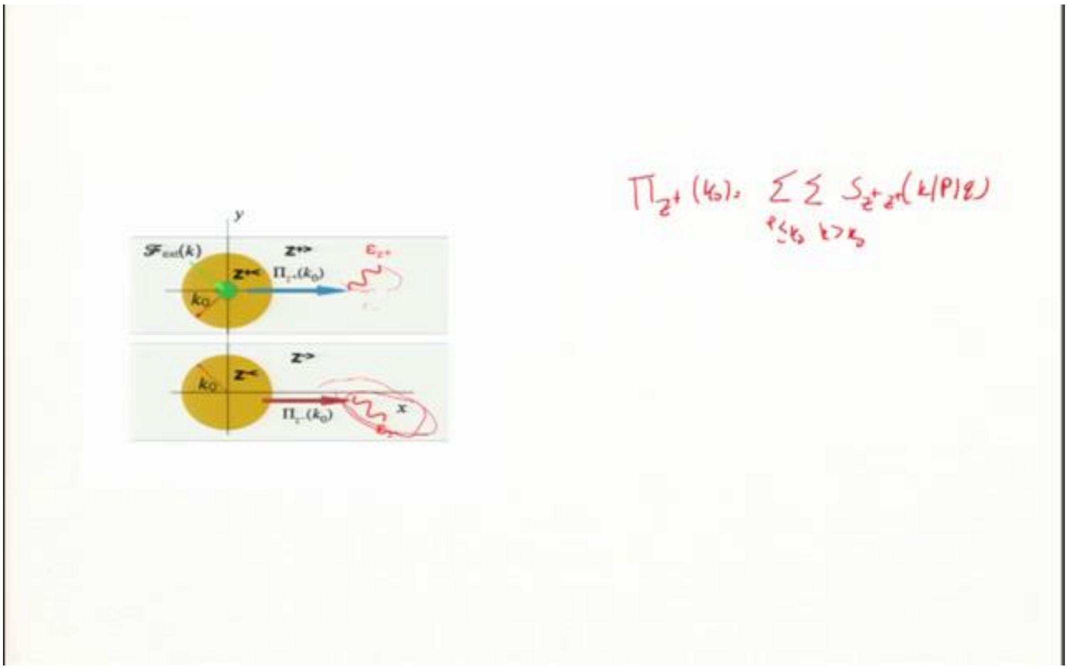
Yeah S_{bu}, u is the giver, so b receiver k p q. p less than k naught, k greater than k naught. Similarly you can define these guys, b less to u greater so, b is the giver u is the receiver. Now all these seems I mean why are you doing it, but I will tell you why we are doing it in 10 minutes, but this is interesting transfer this one, this one.

So, who is the giver and who is the receiver? ∏b less to b, no sorry, ∏u less to b less. Remember this one, you can see this in the screen is this small, this one, pi u less to b less. So, what should I write here? Is u is giver; b is the receiver, k p q. So, k is a less than k naught, because receiver is also in this sphere, but it is a different field. This is a b field in the same house. We have English, let us go we have some x people and y people and they are exchanging ok.

So, they are two different so, k less than k naught and p less than k naught, less than equal to I put, ok. So, just to and now you also define energy going from larger to larger. u greater to b greater, than the most probably outside this sphere. So, you just change the sign to greater, ok. And they are useful for dynamo ok, which we will discuss in 10 minutes, fine. Now, so these are six fluxes actually for u b variable, the 6 fluxes. What about z^+ z^-? How many fluxes do you expect?

Student: 2.

2, great, very good.



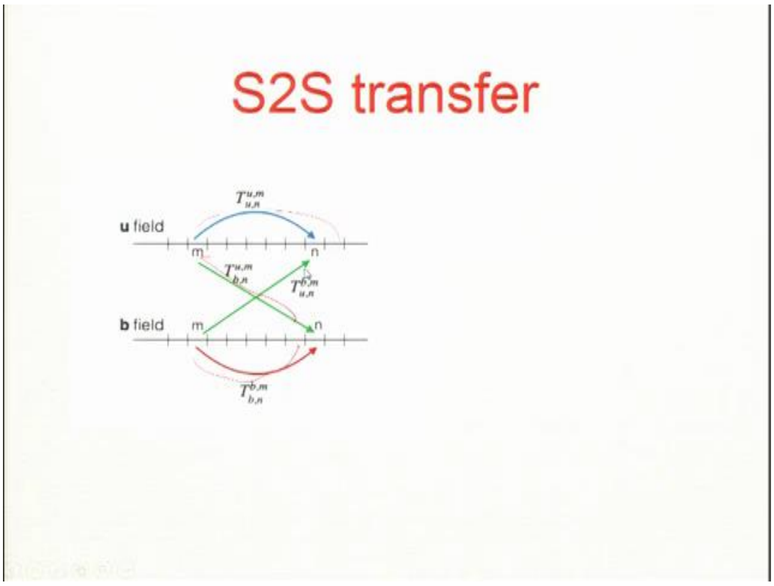
So, for z^+ z^-, there is only 2 fluxes. Now, this is z^+ channel and this is z^- channel. There is nothing across, so there is across ones are not there. So, we have $\prod z^+(k_0)$ which is Sz^+z^+ (k|p|q), k less than k naught. And now sorry p less than k naught you know, p is a giver and k greater than k naught. So, this is for z^+ and z^- will be just change z^+ with z^-. This is convenient this is only two variables.

Now, by the way these variables get dissipated. So, the dissipation here in fact, I am quite incorrect. Yeah, so there is a it is ok. So, these are dissipation here of z^+ and this is dissipation here z^-, ok. So, in the earlier slide, there is a dissipation of magnetic energy and kinetic energy.

So, let me just make a remark, because actually we need this. Here you find and all the magnetic energy here is dissipated at large wave number, right. So, that is why they are effective dissipation. A kinetic energy is dissipated here. So, how much energy is dissipated? Kinetic energy, is a just ∏ u less to u greater? This one or more?

Is more, because energy coming in is under steady state. You force at large scale here and here. The energy is going from this one, but then also this is another source of energy. B is giving to u. So, under steady state all the energy coming at large wave numbers must be dissipated. So, energy coming in here will be this plus this minus this, but the according to this notation is going away. So, whatever coming here net must be zero under steady state. So, there are lots of formulas we can derive on the properties of this fluxes, ok. For example, under steady state whatever energy coming in here, you know b is getting some energy; b sphere under steady state, this sphere should not get any net energy.

So, means whatever comes in is going into this and this channel. Understand know? So, if I have my income does not change, then whatever I get must be given away or dissipated. So, right now it is not getting dissipated because this small k and just distributed. So, you can define these laws for the fluxes.

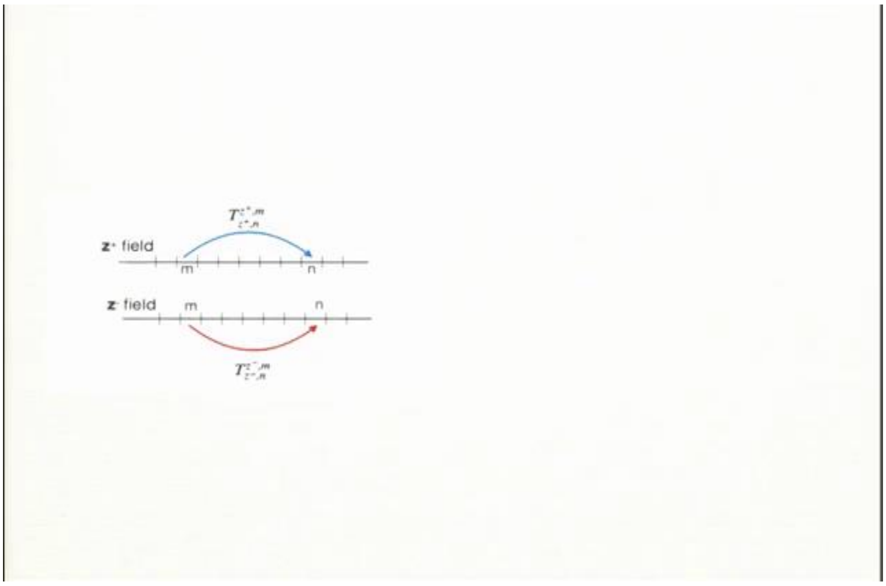


Ok, now we can also define mode to mode transfer, sorry shell to shell transfer. So, this is S2S shell to shell transfer. So, there is u to u transfer from shell m to shell n. There is a b to b transfer shell m to n shell, also we are transfer from u to b and b to u.

So, the 4 shell tran; shell to shell transfer into the ub variables, ok. We can define the, we already defined it, but the four of them, you should keep in mind. And for z^+ z^-,

Student: two.

Only two of them so, there is nothing cross.



And z^+ z^- is only two of them, ok. So, this is a nice way to compute these quantities, ok. So, we stop.

Thank you.

Physics of Turbulence
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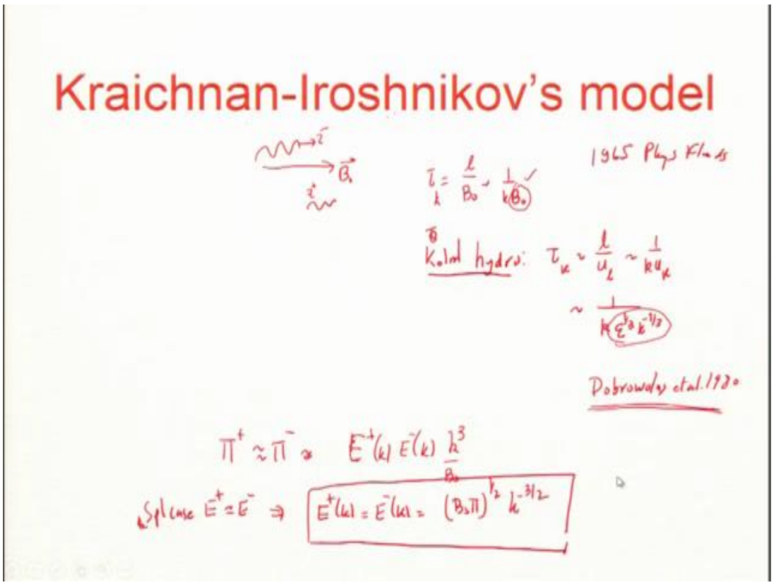
Lecture – 43
MHD Turbulence
Turbulence Models

So, in the last slides we discussed energy transfers and define fluxes. So, these are all coming from non-linear terms ok. Now, we can use these ideas and also earlier ideas done by various researchers; to talk about energy turbulence models for MHD. A Kolmogorov model is great for hydro; it is very good as long as we force at large scales. Of course, we need modification if the energy is injected at different scales. And, there was generalized for buoyancy driven flows.

So, we see an MHD what kind of generalized models exist. But at the outset I say that there are still unsolved problems, there is no convergence. Though I am a believer of some theory which I am strong believer but there are there is no convergence, among everybody does not say well this is a correct model ok. There is no one correct model because there are large number of variables and so on. But there are still lot of miss I mean divergences.

So, one thing I can say it at the outset that this not a passive vector. So, five third is not expected really for this fb and fu which are active know. So, they can change the fluxes. So, we cannot use five third theory ideally. So, we let argue from different angles and we have to do that. So, in this slides at present some leading models, which are 50 year old model, some are 30 year old models. So, I will just present them very scaly sketchily.



So, first model is my Kraichnan -Iroshnikov's model. Kraichnan paper is one-page paper in 1965; Physics of Fluids it is just one-page paper that is why it is, I think it has been misinterpreted and is hugely sighted but it is incorrect ok. So, what does Kraichnan say? So, according to Kraichnan these are mean magnetic field and that leads to Alfven waves right. So, we discussed Alfven waves. So, that is why Alfven wave is a key for modeling imaginary turbulence.

So, remember so, let us assume that $\vec{B}_0$ is in this direction and Alfven waves travel. And, the two does Alfven waves z^+ and z^- and one of them travel in fact, they travel in opposite direction. So, one with u B in opposite direction travel in along B_o and one with ub parallel travel along u minus. So, z^+ should travel backward in minus B_o or minus $B_o \cos\theta$ it may not be in the same direction ok.

So, these are is one-page paper. So, this huge amount of hand waving you know the slight of approximations and, but the ideas are an interesting. So, there are two waves or two packets going in opposite direction. So, they will interact briefly, according to Kraichnan. And what is the time scale for interaction? So, length scale is l will come from wave number and the time the velocity is v_o, $v_o \cos\theta$. But there is one model which tries to model cos theta as well; but let us assume that we ignore theta effects. So, l by B_o will give the time scale right, from divisional arguments a time scale for a wave number k will be 1 by k B_o a B_o is a constant ok.

Remember for Kolmogorov what is the time scale Kolmogorov theory? Tau k is Kolmogorov hydro tau this is please remember this ha; So l by u(k) u(l). So, 1 by k u(k). And what is u(k) in Kolmogorov theory? Epsilon one third k minus one third ok, this comes from some initial arguments from five third we can easily derive this which I have done in the class. So, this will give you k2.

So, this is k dependent and in Kolmogorov in Kraichnan theory is k independent; because of Alfven effect. So, B_o cannot be B_o effects are real what about u_o can somebody may say well. If I apply mean velocity field for hydrodynamic turbulence, then I should get similar feature right. Why is in the time scale 1 by k u_o for hydro ok?

So, it turns out velocity field can be illuminated by Galilean transformation know, we can go to a frame where u_o is 0; but we cannot go to a frame where B_o is 0 B_o cannot be gotten rid of ok. So, that cannot be gotten rid of by Galilean transformation. So, these effects exist and they are real. Now we can do divisional analysis which I will not do it here. This is a nice paper by Dobrowolny et al in 1980, 1980 or I think.

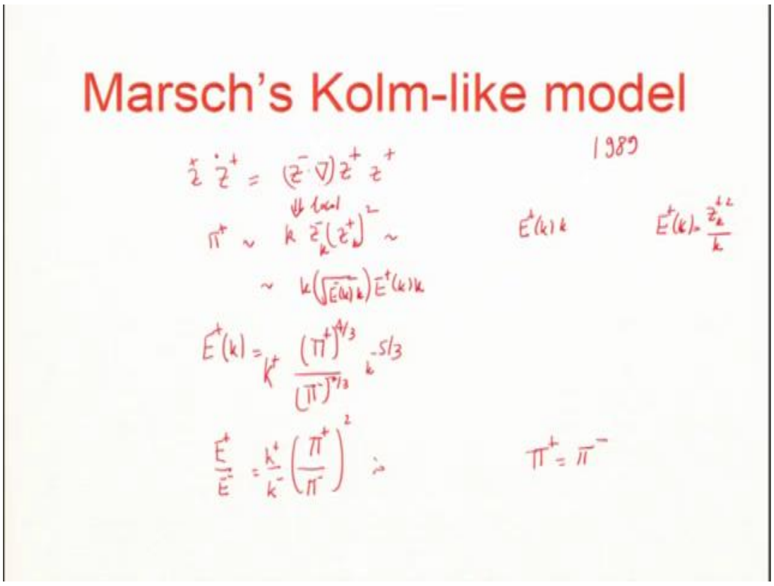
So, around that and there it is derived in more systematic manner a Kraichnan idea because it is one-page paper. So, in this theory, it has been argued so, there is z^+ and z^- are basic variables. So, I am basically studying Dobrowolny's result; by Iroshnikov's also very short paper a similar idea. So, that is why it Kraichnan Iroshnikov's Russian and American 1965.

So, in this theory pi plus a flux of z^+ approximately equal to pi minus and that goes as there approximately equal ok. And so, we can write well I mean there is a formula for in terms of $E^+(k)$ $E^-(k)$. So, it's a product of E^+ and E^- this generalization by Dobrowolny; Kraichnan assumed that if it is E^+ equal to E^-. But it need not be equal there they are unequal these are B_o below in k^3 ok. So, these proved by Dobrowolny.

Now, if you assume E^+. So, one thing you can see that if E^+ and E^- are not equal it could be any number it is a product but the fluxes are equal right. So, fluxes pi plus and pi minus are always equal irrespective of ratio of E^+ and E^-. These are prediction of these phenomenology ok; the proof I am not giving you I mean these are bit detailed proofs.

Now, let us take a special case, when E^+ equal to E^- then what happens $E^+(k)$ both are equal. So, I can write down equal to E^- (k). So, B_o will go there, B_o pi square root right because there is a square k minus 3 half these are theory of Kraichnan and iroshnikovs.

So, it is not five third is minus 3 half ok. So, there is a difference in exponent, but the numbers are quite close 1.5 and 1.67 now let us go to. So, this is old theory and people believed it very strongly.



So, there is a paper by Marsch in 1989 and it says the following says, well, let us ignore the B_o effect ok. So, you can assume the B_o is 0 or some of it is just trial error and we just let us ignore B_o effect. If you ignore the B_o effect, then we have z^+ dot is z^- dot grad z^+ right these are some. Now, for scaling arguments you do not keep all the terms you keep only this. So, if I look at the fluxes what happens and multiply by z^+ both sides.

So, this left-hand side is like flux, you can also argue from what the flux would derived but this is easier than by dynamics. So, this $\prod^+$ under steady state you can put a force in the right side. $\prod^+$ and this one is grad will give a k z^- give $z^-(k)$ $z^+(k)$ squared. Now what is $z^+(k)$ squared? $E^+(k)$ times k is that correct? This is a definition of, what is E^+ (k)? Is z plus square divided by k dk is k right I mean that is what we said is power law physics so, d k is k.

Student: sir there is a star and

So, is a real space. So, there is no star.

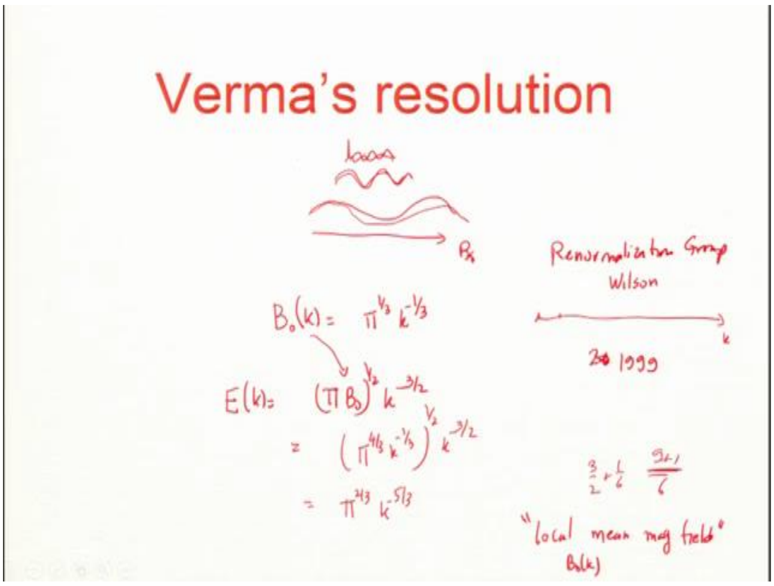
Student: Ok.

First, we so, unless you have to put the star in fact, you have to, but I am doing an estimating the scaling argument. So, you are doing a real space so, where the Kolmogorov theory can be derived exactly in the same manner.

So, from here this from here I am going to real space a Fourier space. So, here I assume local approximation local interactions. So, modes of similar sizes are interacting. So, first equation is in real space second equation is a Fourier space. So, let us write this is k $E^+(k)$ k and the other one will be square root of $E^-(k)$ times k. So, the second term is z^- square root. So, there is a asymmetry in E^+ and E^-.

So, if I so, this equation for $\prod^+$ $\prod^-$ will be similar equation with plus minus interchange. So, if you workout for E^+ and E^- inverted you get the following. $\prod^+$ four third $\prod^-$ two third k^- five third; there is a constant k^+. So, you get five third law, but the two fluxes $\prod^+$ and $\prod^-$ this is for $\prod^+$ E^+ and you can get equation for E^- as well right. So, what will happen to E^-? Just replace plus and minus.

Now, you can also derive E^+ by E^- is $\prod$ so, five third will cancel. So, $\prod^+$ by $\prod^-$ square k^+ by k^-. So, here if E^+ you know and E- are not equal, then $\prod^+$ and $\prod^-$ are not also not equal if they are connected by square. So, this relation differs from Kraichnan and Iroshnikov's pi plus and pi minus are equal ok. So, this is a second phenomenology given by Marsch in 1989. Now, looks kind of odd know that we have five third and three half which of them is correct



So, this is 5 by 3 and this is otherwise three half which is correct. So, that is where I come. So, I said let us try to understand this bothered me for it still bothers me. So, there is an intuitive idea is a mean magnetic field B_o and there Alfven waves are various scales. So, it is turbulence so; that means, there will be Alfven waves have many many scales right.

So, the B_o there is a one big Alfven wave a smaller Alfven wave smaller Alfven wave like this they travel in all directions. Now, for this Alfven wave does this c only this B_o or does it see collecting effect of all waves it should be collective effect right I mean. So, if you are trekking in the Himalayas or in a mountain you are not looking at the mean slope of the or the mountain local ups and downs.

So, same way the wave should be seen local mean magnetic fields. So, this is are you convinced with this. So, at least intuitively it seems possible that though there is a mean magnetic field effect which you will probably effect of the phase being changing like Bo effect. But, the non-linear interaction the local mean local fields must also affect the Alfven waves.

So, for these Alfven waves what should matter is undulation here and for these Alfven wave what should matter is b coming from here and so on. So, this procedure is called is addressed by a theory called renormalization group ok. So, this is a scale by scale theory. In fact, if you look at quantum will low dynamics. So, if you have seen this you might have heard this there are beard charge of electron is infinity heard of this term or no?

So, electron has infinite charge and infinite mass, but what we see is not the infinite charge and infinite mass why do not we see this? Whether there is no single electron, there is electron will create a electric field; electric field will create positive electron pair because of Heisenberg. So, electric field will if you quantize it you will get various virtual particles we call virtual particle know.

So, electron is covered by these lots of particles which are being created. So, what we see the single electron or what we think is a single electron is collection of centered electron plus lots of virtual particle. So, like this horse riding in a very dusty environment. So, lot of dust collected around the horse so, that is the analogy some people make. So, there are lot of virtual particles around it. So, if we go close to the electron, you are not going to the center of the electron, but you are seeing collective effect shielding of the infinite charge by this lot of virtual pairs am I making myself clear or not.

So, what we seeing are if collective effect. So, if a charge will increase, if you go closer and closer in you understand. So, log so, what we are seeing is a charge at some scale. So, similarly here my magnetic field effects will change when I go to different and different scales. So, the B_o which is postulate by Kraichnan is not a single constant B_o is function of k. So, when I go to inertia so, if I make this wave number scale k; then I have small wave number here if you see the mean magnetic field large scale now there is mean magnetic field.

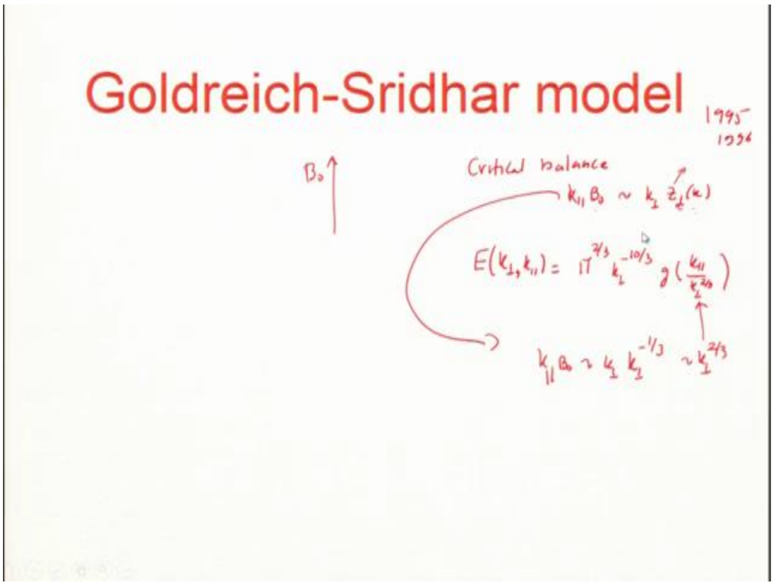
If you keep going down then you see effects of other fields like by going closer to closer electron the effects are changing of the shielding ok. So, this is solving the problem scale by scale and we follow this procedure by Wilson. And, you can show whether this is a big calculation it published in 2000 1999.

So, B_o is k dependent and it is ∏ one third k^- one third of this kind of nice know it is same as E(k) of Kolmogorov theory. Kolmogorov theory is ∏ one third k- one third I wrote that. So, B_o what is affecting the wave waves nonlinearly is this. So, now if I use this formula for Kraichnan so, what is Kraichnan formula a Kraichnan and Iroshnikov's formula k- three half. So, let us put this ∏ half ∏ one third.

So, B_o let us put the B_o here, but I is replace B_o of Kraichnan by $B_o(k)$ yes. So, they will give you so, inside let us put inside. So, this ∏ and ∏ one third will give you ∏ four third right; ∏ four third k^- one third here to the square root k^- three half. So, this will give ∏ two third k^- one six and one six and three half is three half plus 1 6, 6, 9 plus 1. 10 by 6 is.

5 by 3 ok. So, we recover Kolmogorov theory, by just replacing B_o by k dependent B_o, but here assume z^+ and z^- say equal. Giving you run equal and calculation is very complicated which I try to do it there is some work, but is more detailed I am not confident of that calculation. Now this calculation is reasonably correct and clear. So, this is how we make a consistent theory of Kraichnan theory and five third theory.

And, show that mean magnetic field is not b_o, but it is a local mean magnetic field. And, this is a word which is used quite often now in literature local mean magnetic field which is $B_o(k)$ ok. So, this I would like to verify this numerically ok, this has not been done in a computer simulation. So, there are ways to measure B_o and we need to do it ok. So, we move on so, three models done. So, let us go to fourth one Goldreich Sridhar model, which is highly sited paper, but which is weaker than my B_o effect paper.



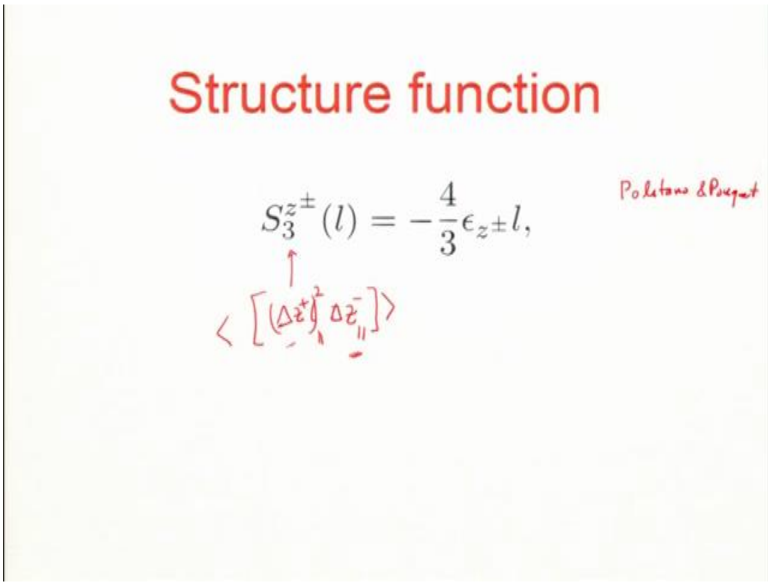
But what this Goldreich says is, if this B_o then it brings a non-isotropy. Its true I agree B_o will give, but if you make it too strong B_o, then there are some more complications. And a very strong Bo will make it 2 dimensional. So, all these theory will just disappear. Now, each one bring in 2 D inverse cascade all that thing comes. So, B_o we assume model one and then isotropy is reasonably good approximation I do not know.

So, this is where I think there are lot of divergence. So, B_o effect and this is a highly influential paper accepted by lot of people. So, according to them this is for critical balance. So, I am just stating them ok; so I am not taking any side except my side of my $B_o(k)$ ok. So, critical balance is k parallel B_o is k per z per k. So, these are relation of the Alfven wave with B_o is also correcting k perp and k parallel.

Now, this is a postulate some people argue against it of course, many people believe it as well and this also should be tested. If you put that then you get different spectrum for k parallel and k parallel k per. So, you get basically formula for E(k) perp E(k) parallel. So, this is a formula with anisotropy right, which is a good formula, but I mean this some people claim that it has been verified. But I mean there are still issues specially with very strong bo.

And, now if I just average over k parallel is integral over k parallel, then you get again minus five third. The formula is bit complicated if you like I can write that formula, but it's combining both I do not know I mean you can look at this there is a formula with k parallel k perp and is there in my in my notes. So, it is here ∏ two third k perp minus 10 by 3 g which is a non this g is a function this is coming from here if you put k perp z perp is k, but minus one third you will get this right ok.

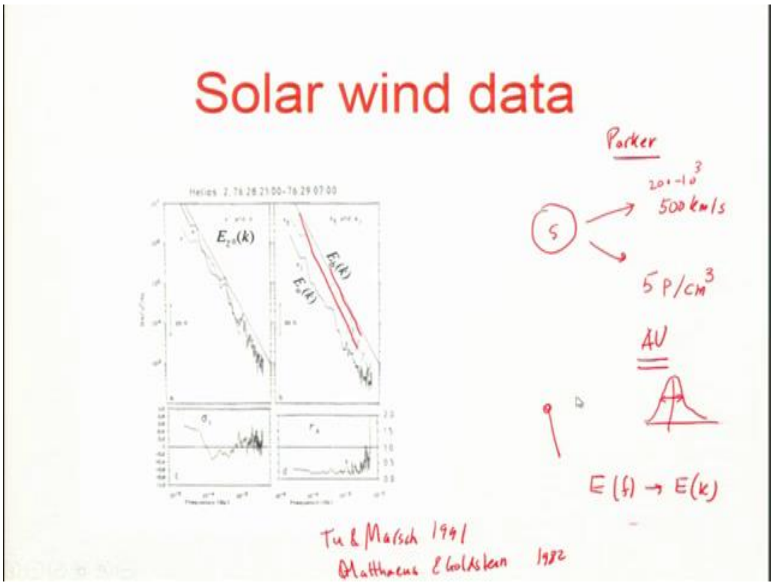
So, because is five third z perp. So, z perp will give you from here k parallel Bo(k) perp k perp minus one third know five third will give this. So, this gives you k perp two third. So, this is this is exactly come from here ok. So, this is a formula from this you can derive 5 by 3 and various term. And this is paper in 1995, 1996 ok. So, this is fourth model a quite a few more I will skip the paper by Gaultier there is a paper by Boldyrev, but I will skip there is a endless story there is lot of models



Unless look at structure function. Now this is kind of nice if you ignore the Bo effect and you get five third. So, this means what is Kolmogorov theory? Is minus 4 by 5 epsilon l; now here we get this S_3 it is a interesting combination of z^+ and z^-. So, it is z^+ square delta z^- take the component parallel components. So, this is average this is S 3. So, is a combination of z^+ and z^-.

So, z^- is a mediator and z^+ and z^+ are receiver and giver. So, mediator is coming with power 1 and giver and receiver are coming together with power 2 and this is exactly same as passive scalar formula which I discussed. So, the derivation is identical for passive scalar for this derivation and this is by Politano Politano and Pouqat this is their paper. So, this is a bit later in 2000 you find them in the in the book ok.

So, this is 2000 something or 99 late 1990. So, this is telling us that structure function of course, with Bo effect is not included in this, gives you a five third. If I go to spectrum we expect five third I mean not quite derive it, but five third.



Now, So, how to prove this? It turns out doing lab experiments, though some of it has been done, but very few getting turbulent MHD is very difficult; the reason being a that u as. So, eta is large there is a diffusivity. So, you need very large l. So, like solar wind so, this is one good setup which lot of people I mean including I will also I mean I did little work on this. So, Supratik does lot of work and I think you may be doing some work. And suns atmosphere is not static so, there is a sun.

So, this was proved by parker, nobody believed him for many years many years means several decades I think they did not believe. So, he says earth atmosphere is not stationary like earth at sorry suns at atmosphere like earth has a study site atmosphere, but nothing is blowing out of the earth. But according to parker suns is emitting electrons and protons and some helium from which is atmosphere and there is a wind blowing and this is called solar wind.

And its mean speed is around 500 kilometer per second huge speed know 500 kilometer per second. Of course, it ranges from 2000 to 1000 there is lot of variations, but there is huge speed. And the density of this plasma is how much is, very rarified is 5 particles, 5 protons per cubic centimeter is very very verified ok. I mean if you take one once when you cube out the gas here, there are I mean I think ten power twenty atoms molecules.

But still so, one question is can you apply MHD model. So, MHD model requires that. So, remember I made one remark that you need continuum approximation. Then what is continuum approximation that the mean free path length is much smaller than the system size then only we can ignore these collisions. Now for the solar wind the mean free path length is, I mean I mean the interaction is there is something called one AU Astronomical Unit.

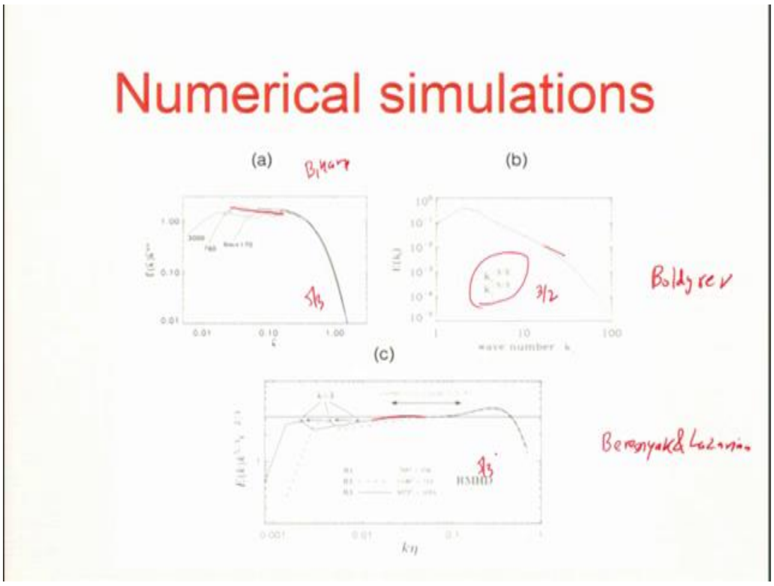
So, distance from the sun to earth is called one AU. So, one collision is verified when it is on the average from sun to earth it hardly collides. But still if you look at the probability distribution function is not too far away from actual length. If you look at so, this way to check whether it is thermalized; so, what is Maxwellian distribution we have this Gaussian distribution the Vrms know.

So, how does it become Maxwellian, if it is not colliding. So, reason it is becomes Maxwellian I think I agree with this, but mostly argued by researchers, there is ambient magnetic field. So, the ambient magnetic field acts like a scatter is a random magnetic field. So, that is scattering there you do not require a magnetic field to scattering. Just imagine that there is electron which is going in a magnetic field and for turbulent magnetic field you do not need another electron to scattering. So, magnetic field will scatter and if it is turbulent then it can make it Maxwellian stop ok.

So, this is what we will make it Maxwellian. So, we see this solar wind this is paper by T u and Marsch, I think in 1991 this is a paper by Goldstein and Matthews and Goldstein 1982. So, all of them show five third to good approximation. So, these lines are five third lines spectrum ha. So, this line I am did not draw it properly. So, these are so, that is well I mean. So, spacecrafts have been sent. So, there are spacecrafts like voyager pioneer sent way back in 60s 70s 70s 1970s and they collect data.

So, what does they do? They put a probe so; they are just going in the free space outer space. And they are probed which measures magnetic field speed temperature and so on. And I but the speed of the probe is much smaller compared to 500 kilometer per second. So, you can use Taylor approximation Taylor hypothesis according to which if I am stationary here, the wind is blowing then you can treat as if the wind is stationary and I am scanning along the line. So, you need to convert the frequency spectrum to wave number spectrum I made this remark before know.

So, you can convert E_f which is what is measured by the space craft's E_f; you can make it to convert to $E_f(k)$. And, these are Eff not $E_f(k)$, but using this Taylor hypothesis we make this conversion. And, they show reasonably reasonable convergence with five third though 5 by 3 and 3 f are very close one point the difference is 1.6 and then enough error bars. So, people can get I mean if a believer then you say well I do not believe you. So, we can plus there error bars are when the difference is too small ok; so, but this has been reported many years.



So in fact, Kraichnan was believed, but you see there are solar wind was saying that it is not numerical simulations. People have measured this especially after 2000 spectrum, but this may be not very clearly visible, but this is minus three half theory by Boldyrev; this is by the Biskamp he believes its five third five third line. But these lines are not very this is by Beresnyak and Lazarian and these are not very convincing either way so, these three half five third, five third. And now so, what we did was we said by let us look at indirect predictions.

Indirect prediction was look at this $\prod^+$ and $\prod^-$ this we wrote in 1996 see this $\prod^+$ and $\prod^-$ and E^+ and E^- are unequal; this is the way to test know if E^+ and E^-. In fact, their issue was roughly 10 and look at $\prod^+$ and $\prod^-$. If they are equal then Kraichnan is right.

If unequal then Kolmogorov theory should work. And we found that Kolmogorov works better ok. So, Kolmogorov works better this is 1996. So, these are indirect proofs which are more convincing than looking trying to fit a line with this five third or three half. Now I think more convincing will be this B_o effect whether we can show that this Alfven wave scattering is by B_o. And there are ways to do it, but nobody has done it so far ok.

Thank you.

Physics of Turbulence
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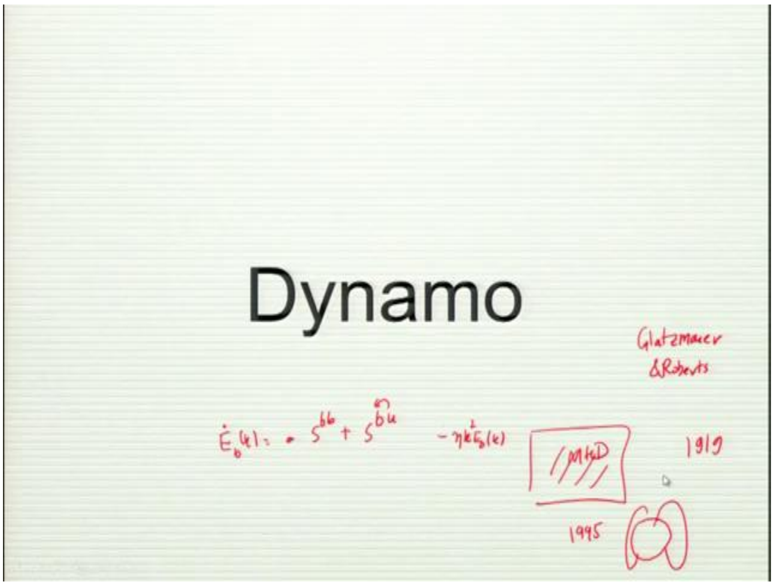
Lecture – 44
MHD Turbulence
Dynamo

So, this is the last part for MHD Turbulence is Dynamo ok.



So, what is dynamo? So, it is not like cycle dynamo. So, cycle dynamo you have seen know so in fact, I opened it up. So, you put a so we in generator electricity, but there is a magnet already. So, is a magnet and there is armature moves in the magnet and generates current. But you should generate current without any external magnetic field.

So, you only have a plasma or MHD plasma and do some stirring or do something on it. But do not put any external magnetic field or no bar magnet and like the cycle dynamo and now generate magnetic field. So, is possible so it was for proposed in 1990 for the sun.

Suns magnetic field has been a puzzle or even earth magnetic field. So, earth has no bar magnet inside anyway it is too hot. So, any bar magnet will melt inside the sun, or the earth. So, that it has been a big puzzle for long, but most we believe in this MHD self induced. So, it has induced by on it is own. And how it is induced by own? Can you I mean you can easily so, let us let us figure out how it is induced on it is own.

So, if you look at equation for E_b, E_b dot $E_b(k)$ let us let us put wave number. So, there is one term which is S^{bb} and there is a much. So, which is b to b it turns out as I said remark S^{bb} does not get any energy from u, it is among themselves. So, two sets of people and once that are be just exchange things among themselves. So, that is not going to increase b and always remember that there is a diffusion term. So, if you do not get anything from other source.

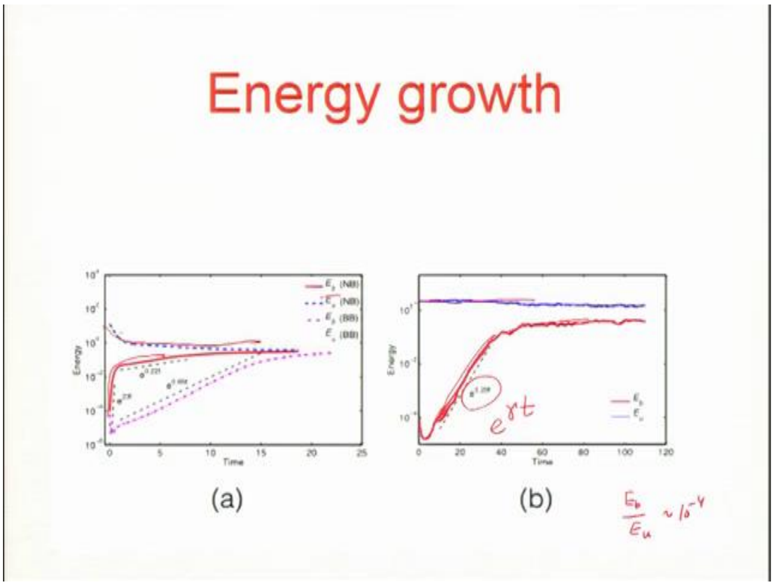
Student: E_b.

E_b will die, but there is other source which is S_{bu}. So, u gives to b and this is the source for magnetic energy which is the mode to mode transfer which I derived. So, from the mode to mode transfer there is a flux and this is what leads to dynamo which you understand at least on numerical simulations, lot of analytical work and this can produce magnetic field.

And so, there lots of evidences that this is the theory and I think one of the let me just make a remark that one of the leading which tilted the believers or well which tilted. I mean so if so well fine MHD will work these are idealized equations. But there may be other things in inside the earth, but in 1995 there is a paper. There is a simulation done after the earth magnetic field.

Somewhat realistic the numbers are not same as earth parameters. Now earth has nu, eta density and so on. But not quite same realistic parameters, but as best as possible and they found this is by Glatzmaier and Roberts; Glatzmaier and Roberts in 1995. He has found that the magnetic field of the simulation in a spherical simulation it is switched, is flipped.

Now, unfortunately I do not have a slide here, but is a very nice simulation where the magnetic field of the earth in the simulation flipped similar to what happens in the observations. So, then people said well I mean this is really the model to try and after that there is huge amount of work on planetary dynamo solar dynamo from simulations and these kind of stuff. So, it is a huge field with lot of activity and I will only show some well I am only going to show some applications based on energy transfers ok.

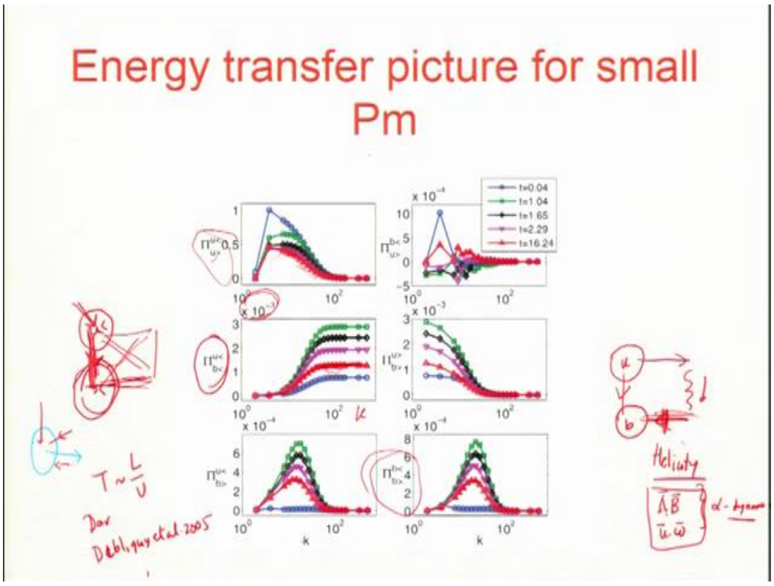


So, these are the simulations done in our lab. So, in these simulations we keep lot of kinetic energy u(k) and some seed magnetic field very small magnetic field. So, E_b could be much less than E_u, you can a spectral simulation. So, we can scatter it in different range of waver numbers E_b by E_u is may be 10^{-4}.

But we need a seed field we can show that if E_b 0 it will remain 0 throughout. You need some small seed field and that is also one of the puzzles in the universe. What cause the seed field and for me well any small fluctuation is enough. So, there are so many charge particles there is small fluctuation we will create that seed field ok.

And anyway there are debates on that as well. So, you can see that magnetic in this these are this b, his red line is a magnetic energy E_b. Now there are various plots you do not need to worry about all of them. So, E_b is increasing with time see in fact, there is a similar plot. So, it grows exponentially e to the power γ t. So, γ is called growth rate and then it saturates.

A kinetic energy look this is very tiny E_b is 10^{-3} E power minus 4. So, E_u does not change very much u is approximately it can decrease by factor 2. So, here it is decreasing somewhat u, but u is approximately same. E_b only grows from nearly 0 to finite value. Some simulation show it because equipartition, some simulations find that they are not equipartition ok. Now question is what makes it grow? Now this has been I mean you want final models what is making in grow it you need is growing. These are box simulation you also do in spheres you find that simulation.



Now, the earlier models now I think I need to say this for completeness. So, this is a u field at a large scales. Now remember is five-third theory you believe that at least without magnetic field it is five-third large scale forcing five-third. Now one set of people said well energy will cascade like what Kolmogorov says. So, it will go to low wave numbers and so low small scales high wave numbers then it will cascade to b field b.

Let us small scales from here it will transfer then there is inverse cascade to b. And these require some helicity which we did not discuss in my MHD discussion as well. So, helicity is A dot B magnetic helicity, and kinetic helicity which is not conserved for MHD. But that also plays a important role u dot omega and there is a papers called alpha dynamo ok. So, these are big thing in solar dynamo, alpha dynamo these relates the fluctuation with the mean field.

So, these use to be the models this is highly popular model called alpha dynamo model, you get my point you need this inverse cascade. So, you needless magnetic energy large scales how can you get energy in large scales? So, well there is cascade of kinetic energy is small scale this thing then it grows backward. Now in our lab you have done lots of simulation without helicity ok, and you find the magnetic energy always not only in our lab. I mean the everybody if you turn off this helicity still get a magnetic field to grow and saturate.

Now, question is what makes it grow if the helicity is so, important, where the I must also say that there are people who say well this is not required. But this helps, but the question is if there is no helicity what makes it grow. Now it turns out that we can easily answer that question by energy transfers. In fact, this is what we discovered first.

So, you remember this flux formulas u less than b less. So, this is a simulation done by Gaurav in the same paper which we wrote in 2001, this is a huge transfer. So, kinetic energy was forced here so which way and will it transfer. So, there are 6 of them know here, here, here, here, here, here. So, there is one guy which was very very good very powerful and that was this one.

So, u small to u less to b less which is large scale transfer. So, if nobody had talked about. So, this is a transfer which is large scale u to large scale b. So, we do not need these kind of transfers we just need these transfer right. I mean and you using this we can make models of galactic dynamo and estimate the time scale.

So, time scale from these models we can say is l by u; l is a system size, and u is a RMS and you get reasonable numbers, I mean I am not really deep into galactic dynamo. What you can get numbers which is one it written over time which is a 10^8 years or so, and so.

So we can make interesting predictions by this flux arguments, now we computed this flux in for I mean we have done this flux many many times. So, there is one paper by there there is one paper by Debliquy, this is our Belgian collaboration Debliquy et al 2005 ok. Now, these are simulation for dynamo first with seed field.

Now what I would like you to now there is another parameter called small Prandtl number, magnetic Prandtl number. Now for the earth and the sun may be Prandtl number is small and this is for really small is 10^{-7} or so and that cannot be simulated. So, best simulation we could get is 0.2 and people have done some tricks they got to 10^{-2}; 2 but I think that using lot of modeling, but you can seal this u less to b less this one.

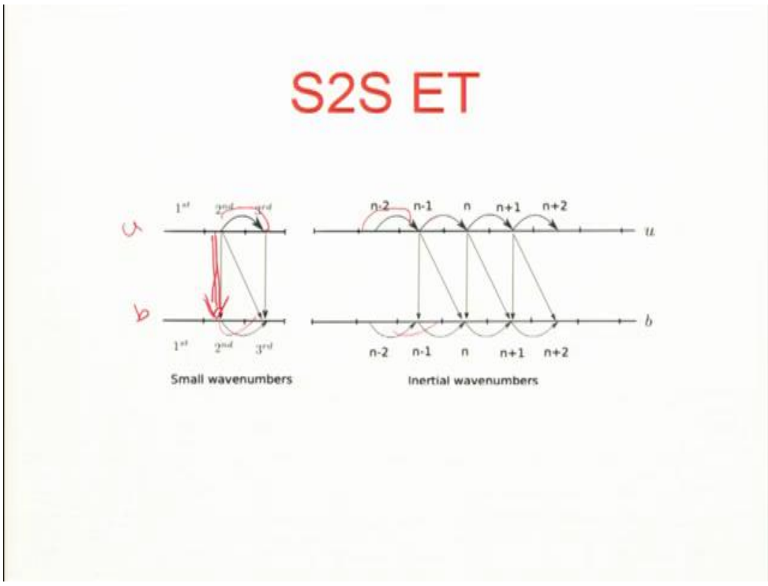
Now, this is wave number now x axis is wave number k k. Now this is at final time u less to b less is one of the most significant transfers among b. Whether a u less to u greater this kolmogorov flux which is quite significant, because kinetic energy is still dominant right. Kinetic energy as I said is giving tiny bit you know this guy rich guy is just giving some small things to magnetic field.

So, u less to u greater is stilled a large, but u less to b less is the most significant transfer for small Prandtl number dynamos. And this is that final time which is quite large ok. Now these are details which you can look at paper, but I mean we are believers that this is a significant thing to look at for dynamo. And of course, we should look at other things like these and these and it turns out one more point that b less to b greater this one. So, b less to b greater is which one this one is positive.

Student: Right.

Right all of them they have come with positive sign. So, when this those are the inverse cascade of b. If b large is losing all the time then there must be somebody to give that otherwise it cannot be sustained cannot be grown. So, this is the critical part which is well there is one option is so you see there is not much choice. So, this has to grow I just want to be emphasize my point if this has to grow and if this is positive. Then how can it grow impossible know.

So, one way is to something comes from the top and something comes like that ok. So, these are the things which come and that, not that is not inverse case. So, this is this sign is definitely incorrect these are helicity. With helicity this sign may come with negative there is evidences are there is a contribution from helicity which makes it this plus helicity effect that gives negative ok. So, these are observations one can make from fluxes.



You can also look at shell to shell transfers. Now I am talking about the small Prandtl number now which is good for modeling sun and earth. So, these are the picture of shell to shell so u less so there is u to b. So, u less to b less is like this and u to u is forward, like forward is going down in Fourier space a magnetic is also forward. But this is what is sustained in the magnetic field and now these are critical input from fluxes. So, the flux formula are very useful for these computation.

Thank you.