Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 35
Helical Turbulence

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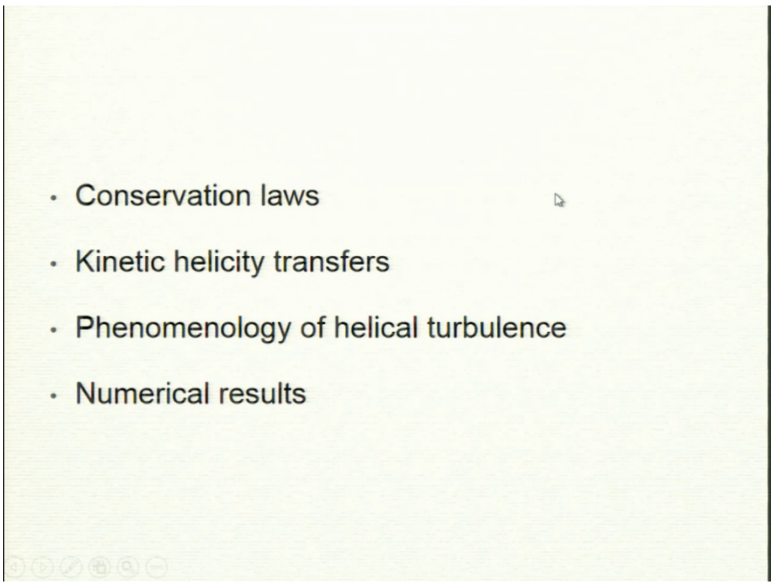
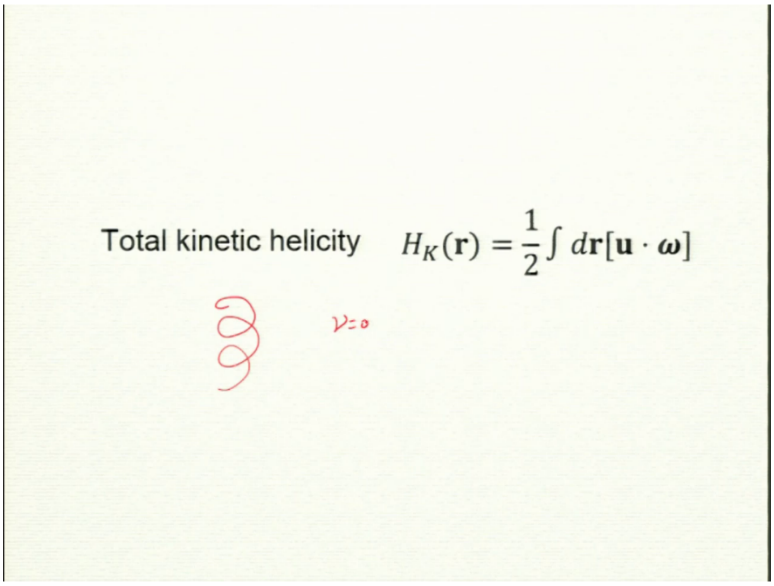
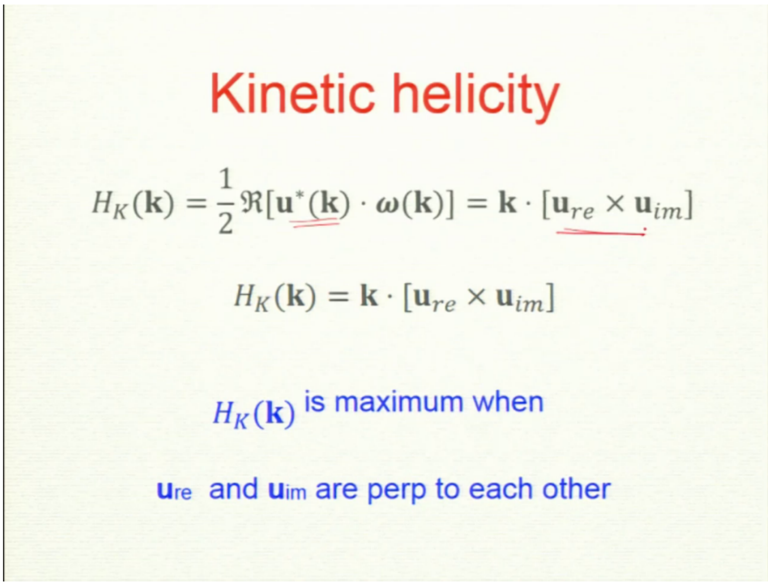
So, now we will discuss about Helical Turbulence fluid only. So, here there will be helicity kinetic helicity. So, flow with kinetic helicity and we will control that. So, let us re brush up our earlier knowledge about kinetic helicity.



So, I will just do these outline conservation laws, kinetic helicity transfer formula, mode to mode transfer then phenomenology of helical turbulence and the numerical results. So, this short discussion conservation laws.

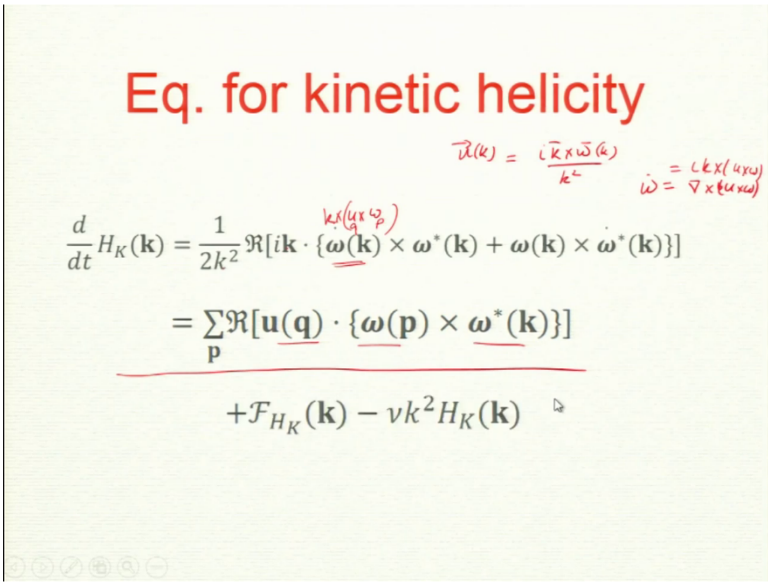


So, let us focus on it. So, total helicity is defined as $(\boldsymbol{u}\cdot\boldsymbol{\omega})$ ok. So, this is I did mention it. So, these are like screw like velocity configuration; so, it is very similar to circularly polarized waves. So, $(\boldsymbol{u}\cdot\boldsymbol{\omega})$ if it is positive then it is positive helicity, negative is negative helicity or it could be 0 as well. And, for ν equal to 0, if you have ν is 0 then total helicity total kinetic helicity is conserved in 3D of course, in 2D is 0 right $\boldsymbol{\omega}$ along z and $\mathbf{u}$ is x y so, it is 0.



So, let us look at a in Fourier space. So, Fourier space these a formula right I mean real part of so, its quadratic becomes dot product of its a model kinetic energy for a given mode. But, one of them must be complex conjugate take the real part of it and I did derive this stuff. So, real cross u measure so, **k** dot real cross imaginary. So, you have to have both real and imaginary and I illustrated by its some examples and, if H_K is maximum when both are perpendicular to each other ok. So, so, they must be perpendicular otherwise you will get $\sin\theta$ component and this was done before.

So, we can look at that time date derivative of H_K, how does H_K change with time; we need that for deriving mode to mode. So, model kinetic helicity changes with time.

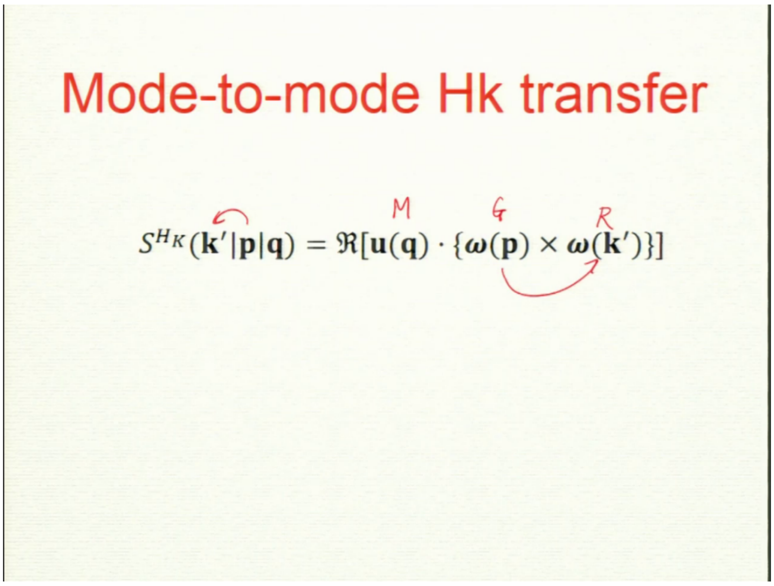


So, I take the time derivative. So, I am turning a viscosity and external force. So, so, well I mean this was done before if you look at your notes. So, $\mathbf{u}(\mathbf{k})$ you write its function of $\boldsymbol{\omega}$. So, if $\frac{i(\boldsymbol{k} \times \boldsymbol{\omega})}{k^2}$, I am not sure about the sign, but you can derive this we did it. So, I substitute this one, this was so, $(\boldsymbol{u} \cdot \boldsymbol{\omega})$ no so, $(\boldsymbol{u} \cdot \boldsymbol{\omega})$. So, you just substitute $\mathbf{u}$ for that and this is a; so, the it is the derivation I have done it before, if you look at the slides ok.

So, these what we is a equation for kinetic helicity. So, we will not write it as a $\boldsymbol{\omega}$ there is a reason for it, I will become very clear. So, giver and receiver are omegas, but actually is very hard to say for kinetic energy we have $\mathbf{u}(\mathbf{k})$ and $\mathbf{u}(\mathbf{p})$, but kinetic helicityk we have $(\boldsymbol{u} \cdot \boldsymbol{\omega})$. So, who is giving and who is taking right because this is $(\boldsymbol{u} \cdot \boldsymbol{\omega})$. So, it is not the same species. So, it so, that is why it is re written as $\boldsymbol{\omega}$ in terms of $\boldsymbol{\omega}$.

Now I write so, what is $\dot{\boldsymbol{\omega}}$? $\dot{\boldsymbol{\omega}} = (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{\omega}$ right, this is what is a equation for omega $(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{\omega}$. So, in Fourier $i(\boldsymbol{k} \times \boldsymbol{u}) \times \boldsymbol{\omega}$ so, substitute it and do some more algebra. So, we get this form, now at this algebra I am avoiding is there in the notes fully. So, $\mathbf{u}$ so, basically so, there is a there is a k here, there is another k here right. So, I substitute here so, this becomes $(\boldsymbol{k} \times \boldsymbol{u}) \times \boldsymbol{\omega}$. So, this $\mathbf{k}$ and this $\mathbf{k}$ will cancel with k^2 and you get this $\mathbf{u}(\mathbf{q})$ and $\boldsymbol{\omega}(\mathbf{p})$ convolution know it becomes convolution.

So**, $\mathbf{u}(\mathbf{q})$ $\boldsymbol{\omega}(\mathbf{p})$ $\boldsymbol{\omega}(\mathbf{k})$**. So, now it is quite easy to guess how to construct the formula for mode to mode. So, it is not $\boldsymbol{k} \cdot \{\boldsymbol{u}(\boldsymbol{q}) \cdot \boldsymbol{u}(\boldsymbol{p})\}$ so, this so, but it turns out I prove it again in the notes that you can. So, you can basically satisfy all those energy or kinetic helicity transfer formulas like a giving to b is same as b giving to a with a negative sign and a b together giving to c is sum of a giving to c and d giving to c. So, these are formulas and it must satisfy this relation for a trail. So, so, there is some subtleties, but I will ignore all that and I can write down formula for mode to mode. So, these with viscosities where this I am ignoring it right now.



So, mode to mode kinetic energy formula is this. So, $\mathbf{p}$ to $\boldsymbol{k}'$, now that is what we want, but its I normally write a way number $\mathbf{p}$ to wave number $\boldsymbol{k}'$ not saying who is giving what because $\mathbf{u}$ and $(\boldsymbol{u} \cdot \boldsymbol{\omega})$. So, $\mathbf{p}$ to $\mathbf{k}$ so, that $\boldsymbol{\omega} \times \boldsymbol{\omega}$. So, this is a giver, this is a receiver. So, omega cross product of giver and receiver and mediator is just $\mathbf{u}(\mathbf{q})$. So, there is a wave number does not appear in this formula and it works well, we computed various quantities and it does the quite good job.

So, this is a formula if you want derivation you can look at the notes, but I do not you want to look at derivation, now this you can use fine. So, this I am giving you as the formula to use. Now, once I have mode to mode I can compute various quantity like flux and shell to shell transfer. So, flux will be useful for constructing phenomenology for helical turbulence.

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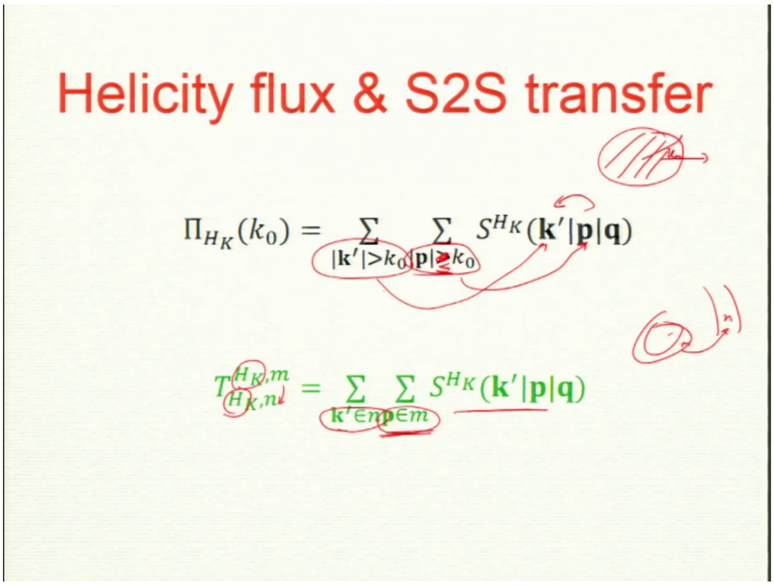

So, let us compute flux. So, what is the flux? We just usual helicity transfer from **p** to **k** prime, but **p** is a giver. So, flux was for its sphere know, all the mode inside this sphere to mode outside this sphere and the radius of the this sphere is $\mathbf{k}_0$. So, all the modes within this sphere are giver this one and all the modes outside are receiver here and I made a mistake. Less than equal to ok. So, all the givers are within this sphere and all the modes outside this sphere shell to shell. So, giver must be in their giver shell, a receiver must be in the receiver shell. So, we have two shells, this shell m is giver and shell **n** is a receiver. So, I want to see how much kinetic helicity goes from shell **m** to shell **n** and we can compute them so, helicity **m** to **n**.

So, a giver belongs to **m** shell, a receiver belongs to **n** shell and you can just substitute the mode to mode formula which I showed you in the last class last slide ok. So, you can compute all this stuff, now these are all coming from equations. Now, I would like to see some properties; what is the spectrum for helicity, what is the energy spectrum and helicity is injected. So, you can have a run or we have flow by helicity 0 and a flow where helicity is non-zero. So, will there be will Kolmogorov theory work for the energy spectrum will it be 5/3 or not.

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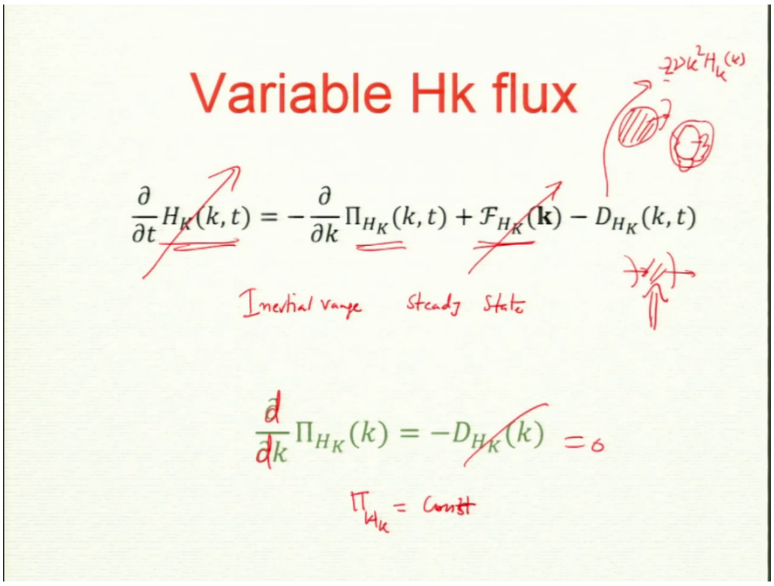
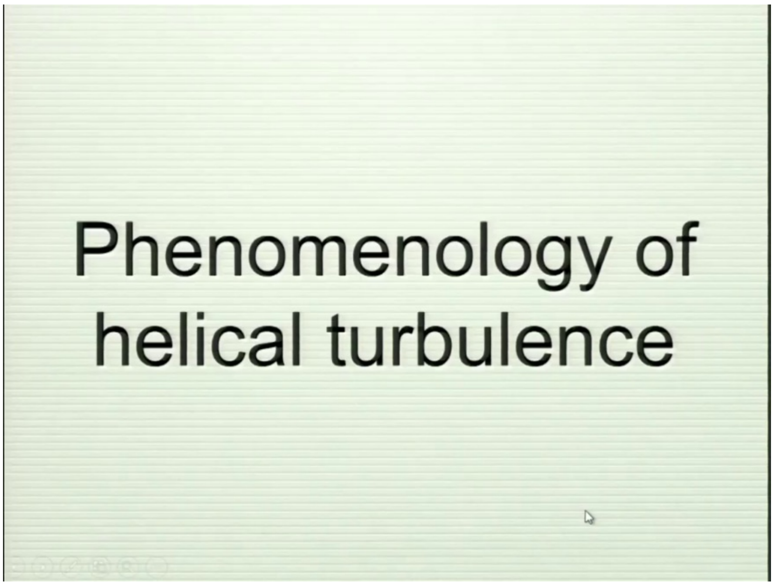


So, it turns out now this so, before we do that; so, I want to still one more formula because I need that one. So, we can also define this variable helicity flux. So, this is exactly like what we did for kinetic energy know. So, we have this sphere and so, energy well actually from this sphere we derive an equation for the shell, this shell of radius k and ok. So, this is helicity in a shell, it can change by changing flux very in fact, identical. So, this argument for energetics are universal, you can apply for all sorts of this energy transfers. So, it is so, this shell can gain energy by come the flux coming in and lose energy lose helicity by flux going out.

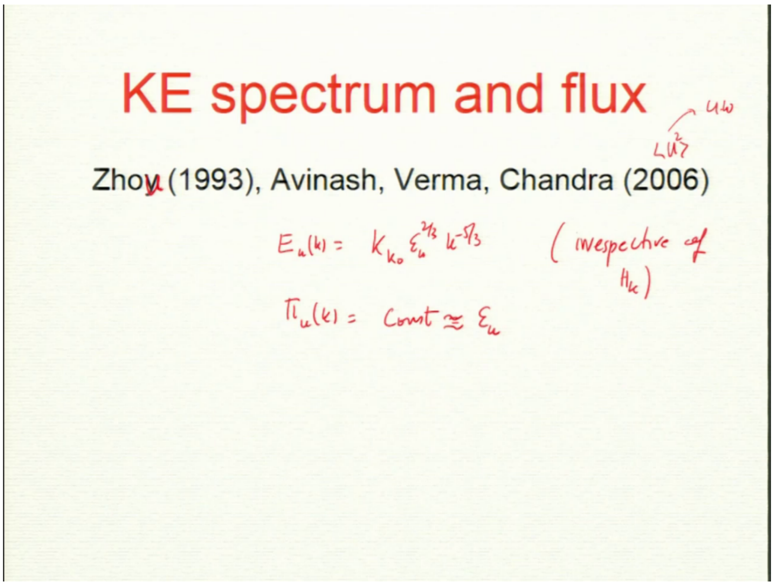
So, total will be in minus out ok. So, this one, these are coming from external source. So, is the external source there could be helicity in the external source ok, rotation is one good source for helicity injection well conviction is ok; so, convection injects helicity. So, there is a external force which can inject helicity and dissipation kill helicity and this one in fact, I should say that $-\nu k^2 H_K$, H_K is helicity for shell k ok. So, this formula for dissipation is very similar to kinetic energy, there is a 2, I believe there must be 2.

So, shall we go back let me just look at here dH_K factor 2, so, there is a factor 2 here ok. So, like $2\nu k^2 \mathbf{u}(\mathbf{k})$ ok, now what if the steady state; for steady state this one goes away, you assume inertia range, inertia range means this goes away. So, initial range steady state for these two cases, I get basically two terms with only function of k_o function of time. So, I get actually this is not partial this ok, that is function of; so, flux will change because of dissipation ok.

Now, if dissipation is week in the inertia range then this becomes 0 and then $\prod_{H_K}$ will be constant; similar to what happens for kinetic energy, kinetic energy flux is constant ok. So, this is what we get. Now we are ready with this for our phenomenology.



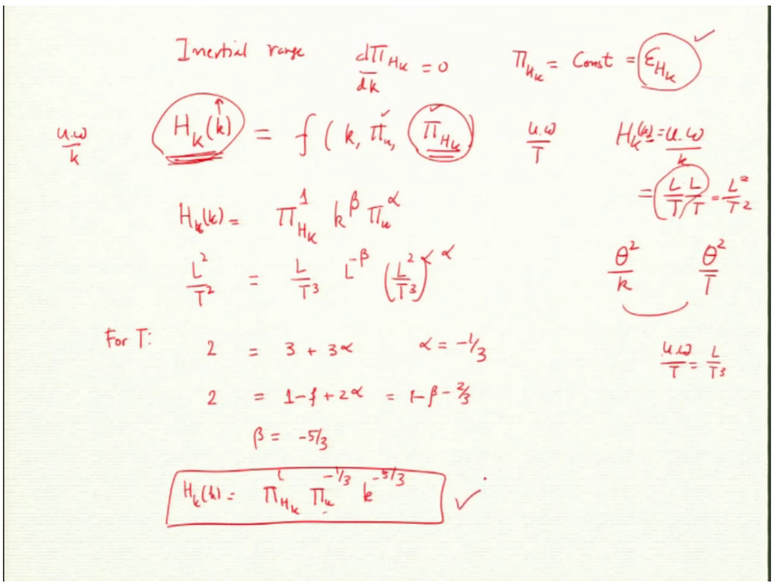
So, phenomenology turbulence can helical turbulence. So, please remember I am injecting some helicity. So, how to inject helicity? So, there some kind of this helical velocity field like a circular polarized, one direction more than other. So, that could be positive circular polarized and negative circular polarized ok.



It turns out for kinetic energy there is a no I am sorry I missed **u**. So, there is a page set of papers which we also did one work in long back. So, it turns out it is shown in this papers that kinetic helicity does not change the spectrum. It does not alter the non-linear term significantly, there actually it is not the non-linear term; it is basically the diffusion does not change. So, there is something called do normalize viscosity because, of turbulence it gets changed, but the change is same for helical as well as non-helical.

So, the example which I gave in the earlier class I believe right I mean; so, the diffusion of dust particles is by turbulence or heat diffusion is by turbulence it is not by turbulence diffusion. So, this turbulence will diffuse faster we will help diffuse faster, but if I put the same energy in the flow, but make it helical with the same energy. So, u^2 is the same total, but u^2 with could have $(\boldsymbol{u} \cdot \boldsymbol{\omega})$ or may not a $(\boldsymbol{u} \cdot \boldsymbol{\omega})$. So, u^2 is the same for each mode then this diffusion does not change ok; now this is a theory which I cannot tell right now, but this is a theory you go.

So, it does not really change the spectrum. So, this was what was proven and so, a spectrum is same as Kolmogorov. So, $E_u(k)$ irrespective of helicity ok, k Kolmogorov $\boldsymbol{\varepsilon}$ it will be turned k^{-5}. So, irrespective of H_K is a clear the statement is clear, I have not proven it now this statement and the flux will be same as what it was without helicity. So, energy only decide the flux, helicity does not for kinetic energy flux ok, use same constraint approximately. Now, what about kinetic helicity flux and its spectrum which Kolmogorov theory did not say anything about it? So, let us try to derive helicity flux ok; now I will apply same kind of logic which I did for kinetic energy spectrum and flux.



So, we are focusing on in the inertia range its steady state. So, inertia range means there is no forcing and there is no dissipation. So, what about the flux? So, I showed you that $\frac{d\prod_{H_K}}{dk} = 0$. So, $\prod_{H_K} = const = \varepsilon_{H_K}$ so, matching. So, whatever injected at large scale is dissipated small scale, but it is crossing this intermediate range, same thing cascades is that a clear. So, it is very similar to what I inject energy at large scale it cascades, it is not generated in the intermediate range and it dissipated small scale fine.

So now, let us talk about spectrum now of helicity. So, we denoted by $H_K(k)$. So, this **k** is a shell spectrum so, energy in a shell k. So, dimensional analysis what all could it depend on? So, it is function of k as well as $\boldsymbol{\varepsilon}_u$ like before kinetic energy flux or kinetic energy dissipation rate and wave number. But it could also depend on this now or if you like in I think notes $\prod_u$ and $\prod_{H_K}$ both are constants this constant this constant.

So, I have three quantities which it could depend on. What is the dimension of H_K by the way? $H_K(k)$ so, H_K has dimension of $(\boldsymbol{u} \cdot \boldsymbol{\omega})$, if I put this total helicity if I put k as n I divide this by k yes or no. So, anything spectrum we divide it by k you can do it blindly, because $H_K = \int H_k(k)\, dk$.

So, dk has dimension so, this has basically come on come down here is that clear. So, I will erase. So, u = L / T, k =1/L, ω = 1/T so, $\frac{L^2}{T^2}$ so, right side now. So, there so, there are two mass is absent here. So, L and T we are two fundamental variables, now three quantities. So, I cannot determine all three right, we can only two equations where matching dimension as L and dimension of T. So, what do I do? I will make some arguments.

So, these not well this is only argument which works, there is argument from field theory as well, but we will not get into that. So, I said this one $\prod_{H_K}$= H_K/k right. What about $\prod_{H_K}$ the flux, is total H_K/t. So, you make I you make an answer that H_K $H_K \propto \prod_{H_K}$. So, this and this point these are proportional because so, by the effort for things like temperature spectrum which would be very useful, temperature is a new quantity, it has additional dimension.

So, temperature flux and temperature spectrum must be proportion right because temperature spectrum is $\frac{\theta^2}{k}$, θ is temperature ok. I mean many of you done this before and

T is a temperature flux will be $\frac{\theta^2}{T}$. So, these should be equal in fact, I want motivation from this argument, you will find that this argument will also go to a scalar like temperature spectrum. So, we will make this one thing is that I will make $\prod_{H_K}$. So, we will do it bit later. So, this is $H_K(k) \propto \prod_{H_K}$, we just make it like that yeah.

I mean this is coming from one phenomenological argument phenol, this is not rigorous proof, but it ricks looks reasonable. And, now we are two unknowns $k^{\beta}\prod_{u}{}^{\alpha}$ and which we can determine yes.

So, this was an assumption that this exponent is 1. So, the left-hand side is I said H_K has dimensional $\frac{L^2}{T^2}$, this one has dimension of H_K/T. So, this is what is $\frac{(\boldsymbol{u}\cdot\boldsymbol{\omega})}{T}$. How much is that? L/T^3 right $\frac{L}{T^3}$.

So, 1/T here 1/T here and L/T^3 L/T^3, this is $L^{-\beta}$ and $\prod_u = \left(\frac{L^2}{T^3}\right)^{\alpha}$. So, what do I get? So, let us see let us do the T first, T has one quantity less. So, T^2 is 2 equals to for T for T $2 = 3 + 3\alpha$. So, what is α? $-\frac{1}{3}$, now what would β? So, $2 = 1 - \beta + 2\alpha$ So, 1- β = -2/3. So, β equal to?

Student: 5/3.

5/3. So, 2 goes $-1 - \frac{2}{3} = -\frac{5}{3}$. So, $H_K(k)$ its $\prod_{H_K} k^{-5/3}$. So, let us put $k^{-5/3}$ to the right $\prod_u{}^{-1/3}$ $H_K(k)$ $k^{-5/3}$. So, these are my spectrum for helicity. So, it is similar to Kolmogorov, but this was flux to the power 2/3, but now over all flux has the dimension of 2/3. This is plus 1 and this is -1/3, but they are not divided.

So, the two fluxes helicity flux and kinetic energy flux, helicity flux has the exponent 1 and kinetic helicity a kinetic energy has flux -1/3 ok. So, these what we can derive one-dimensional analysis, you can also do from field theory, but we will we will we will not do it here ok. So, I derived all this quantity kinetic energy spectrum, kinetic energy flux which is same as Kolmogorov no change. But kinetic energy flux is sorry kinetic helicity flux is constant and kinetic helicity spectrum is 5/3 with dependence on both helicity flux and helicity a kinetic energy flux.

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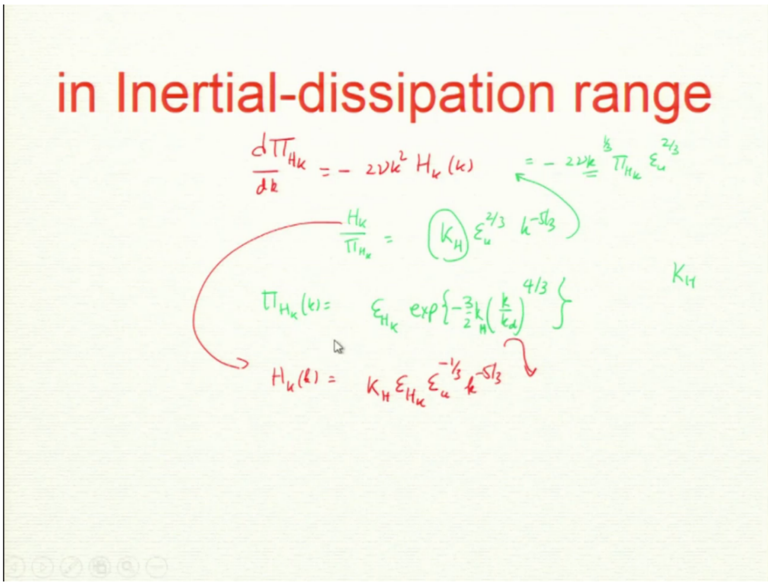

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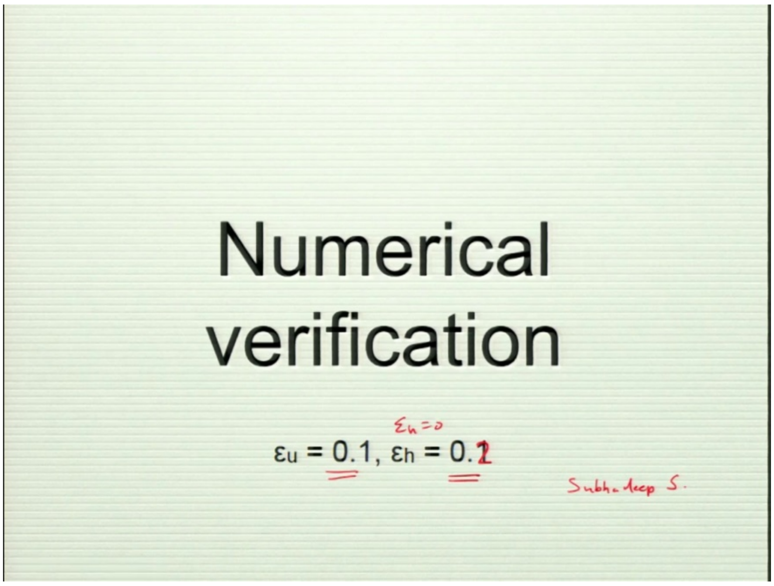
So, now let us try to generalize it to inertial and dissipation range. So, what will I do? So, I will just give you the idea and it is in fact, it is very nice we can derive he formula and it fits with simulation data very well. So, we again use this power formula ok, I mean pushing his power formula very much off late $\frac{d\Pi_{H_K}}{dk} = -2\nu k^2 H_K$ ok. So, what is the what will power say here? Power will say that $\frac{H_K}{\Pi_{H_K}}$ is functionally of ε_{H_K} H_K and k.

So, I know only H_K already so, H_K -1/3. So, $\prod_{H_K}$ was 1 know power 1. So, if I divide then that goes away say $\boldsymbol{\varepsilon}^{2/3}$, I will just get this ok. So, so this is what I will get substitute it on there. So, if I just substitute here what will I so, $-2\nu k^2$. So, this is going to be $\prod_{H_K} {\varepsilon_u}^{2/3}$. So, 2 this 2 and -4/3 will give you 1/3 one is 5/3. So, $2 - \frac{5}{3} = 1/3$ and sorry and the $\boldsymbol{\varepsilon}^{2/3}$ ok.

So, it is the one first order or ordinary diffuser equation right easily solvable. So, this $k^{1/3}$ integrate this we will get - $k^{4/3}$ you know. So, I likely write the answer $\prod_{H_K}(k)$ is exponential you can easily make a guess $\frac{3}{2}\left(\frac{k}{k_d}\right)^{4/3}$. I forgot to say that the constant in front for helicity it is called k_h, is not for Kolmogorov is k_h new constant k_h. I will not go back to old slide. So, $K_H k^{-5/3}$ so, these are K_H here ok. So, these constants are basically from simulations, we compute them from simulations and also from some theory ok.

So, there is some theory which and there is a ε_{H_K}. So, these are exponential drop; what about H_K? I can use this formula. So, there is just straight forward $K_H \varepsilon_{H_K} \varepsilon_u{}^{-1/3}$. So, I am replacing the fluxes by $\boldsymbol{\varepsilon}_u$ and ε_{H_K} $k^{-5/3}$ and this exponential part. So, we got now formula which works for which should work well, which I expected to work in both inertia range and the dissipation range, I just invoking Pao's model.



So, we are ready to test. So, this was simulated by primarily by Subhadeep Sadhukhan in the lab. So, let me put his name Subhadeep Sadhukhan. So, so, kinetic energy injection rate is 0.1 and now this was 0.2 I think so, this 0.2. So, there is one run without helicity. So, there is one $\boldsymbol{\varepsilon}_h$ this injection rate. So, how much I am injecting? Ok. So, either point so, this kinetic energy is 0.1 and we do two runs; one no kinetic helicity injection and where injection is 0.2, I think it is 0.2.

So, I will know in the next slide. So, we make these runs we wait for steady state. So, this is a force runs, we inject all the time. So, you get a steady state and the steady state we can compute spectrum flux and various.

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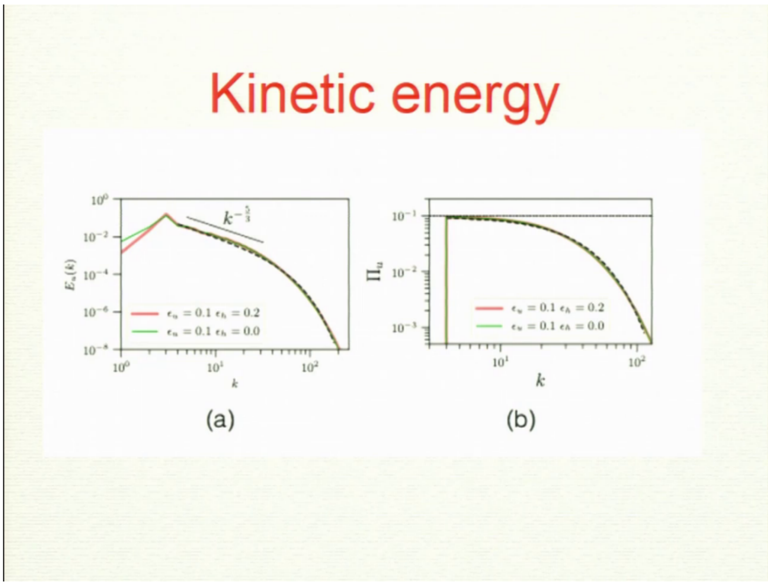

So, these are spectrum of kinetic energy this one and this is a flux of kinetic energy you see. So, the one run with 0 injection of helicity epsilon has 0 and one with 0.2. So, red is with 0.2 and green with no injection. So, green is no helicity and red is with helicity. So, there is some change in the in the left, what is 5/3 is does not changed and these by the way this is part which is coming from power and this is the same it does not, it does not change ok. So, a speculum is kinetic energy spectrum is the same.

So, by the I derived last one is for helicity spectrum, kinetic energy spectrum is unchanged; in the dissipation range as well is the same formula we did in hydro, Kolmogorov theory and then I derived. What about kinetic energy flux? Is constant in the small region, but then it decreases, but it is unchanged again for both the runs. Red is helical and green is non-helical is unchanged. So, this is a very clear demonstration of the theory.

You said that helicity does not change the kinetic energy spectrum and kinetic energy flux in the inertial range at least. There is some change here, but in the inertial range is unchanged ok. I hope you are convinced, it is a good simulation. There are not many simulations done the some but not very many. What about helicity? So in fact, 5/3 with some corrections in that in the dissipation range so, helicity is here.

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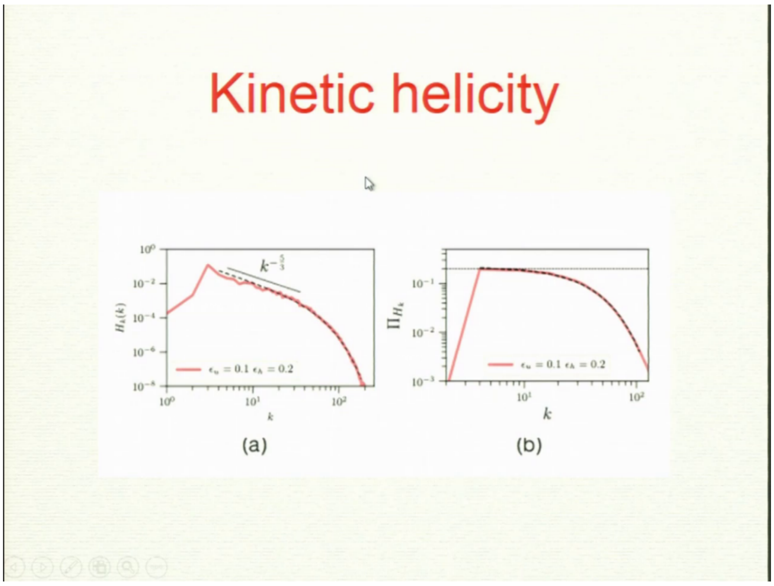
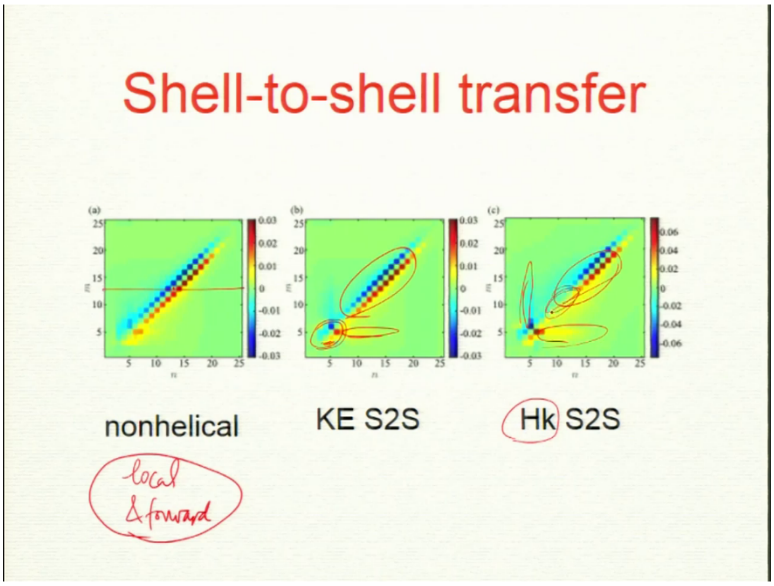


So, so, these for same run we compute helicity spectrum. So, this is five-third, but then there is a dissipation part exponential part 4/3. So, it is fitting very well right I mean this is really nice. So, pause from model you just magically work and the helicity flux is also very nicely fitting. We can will also look at shell to shell so, we look at shell to shell.



So, non helical which we did before its forward right I mean for any shell diagonal is 0, off diagonal next neighbour is positive and lower left neighbour is negative. So, that is called local and forward this property of 3D. So, these flour kinetic energy, the kinetic energy non-helical we computed, now kinetic energy with helical is in this region it is essentially same; except it probably you cannot see, but I can see from here maybe you can see. There is something happening here can you see the red, yellow thing. So, there is some non-local thing, but there is a forced range, the forcing is somewhere here.

So, that is not the inertial range, but in the forcing region there is some helicity effect and a here two kinetic helicity is local and forward in the inertial range. But here there are some non-local, non-local means long distance, local is you talk to your neighbour you know what no otherwise I talked to long distance neighbours ok. So, this kinetic helicity has bit a non-local component from the forcing band. And, this is typically is interesting as concept which we need to investigate in more detail; for forcing range with some non-local interactions ok.

Ok. So, I think this is end of kinetic helicity.

Thank you.