**Physics of Turbulence**
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**Lecture - 33**
**Enstrophy Spectrum and Flux**

So now we start. So it is 3D hydro only, but we will look at Enstrophy. Now, it is a useful quantity again not heavily emphasized vorticity is of course heavily emphasized but not as much as it. So, I will tell you few things which are new ok; you will not find anywhere else. So, enstrophy flux is new enstrophy spectrum is in fact trivial.

So, what is enstrophy I defined it before is $\frac{\omega^2}{2}$, $\boldsymbol{\omega}$ is vorticity. So $\omega^2$.

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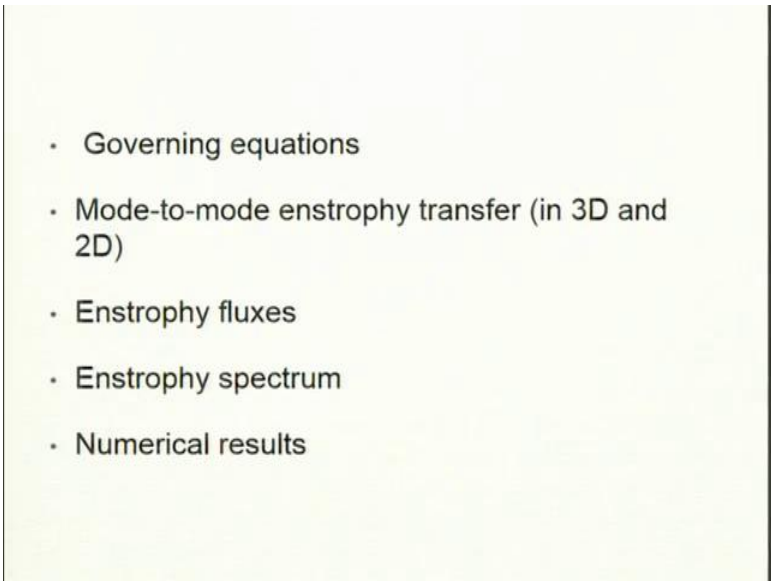


So, we will cover governing equations, mode to mode enstrophy transfer. So, you can generalize what happens for kinetic energy enstrophy transfers; it will fluxes spectrum is trivial which I will tell you and then some numerical results.

So, governing equation I will recap this has been done in earlier lectures.

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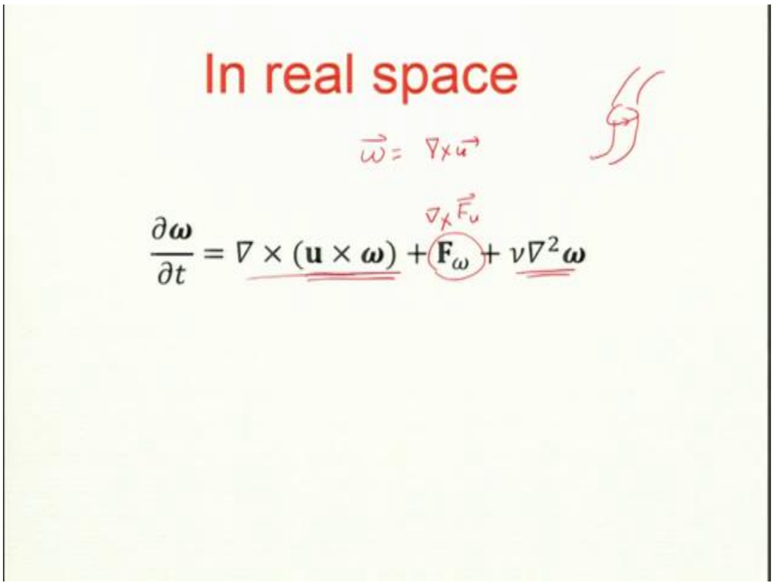


So, real space, vorticity is curl of $\mathbf{u}$ is a vector, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, we are working with 3D. we will come to 2D later, but vorticity in it is like a vortex tube. So, this vortex and vortex tubes.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \mathbf{F}_\omega + \nu \nabla^2 \boldsymbol{\omega}$$

Now, the equation for vorticity is $\nabla \times (\mathbf{u} \times \boldsymbol{\omega})$ ok. So, this is the form and this is coming from $\nabla \times \mathbf{F}_u$ and this a viscous term. So, instead of $\nu \nabla^2 \mathbf{u}$, $\nabla^2 \boldsymbol{\omega}$ ok. So, this has been derived if you look at your notes or look at the notes I gave you; so you find them. So, u dot grad u has become, an pressure has basically disappeared because of curl of pressure is 0, but u dot grad u has written is of this form.

So, in Fourier space; so flux means you work with Fourier space ok, that is best though there is some derivation of flux in real space, but Fourier is most convenient.

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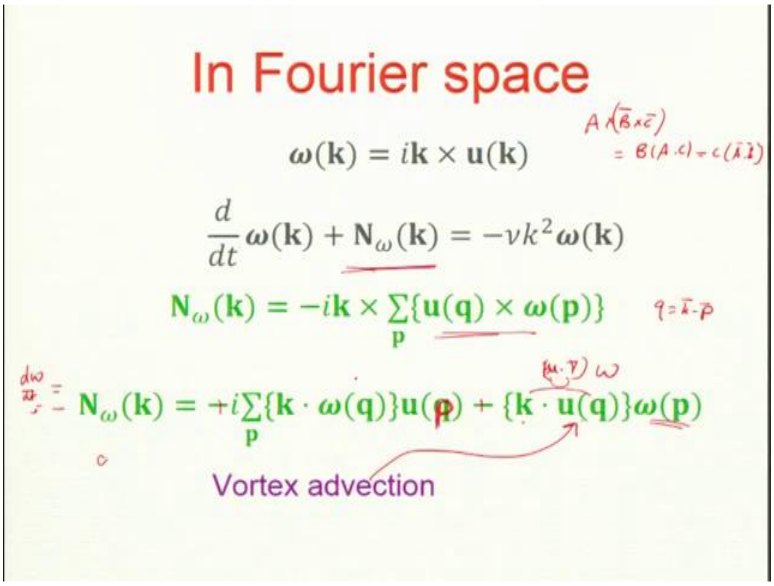


So, we will work is Fourier space $\boldsymbol{\omega}$ becomes i**k**×**u**(**k**) and so I take the old equation which I showed you in last slide convert into Fourier space. The non-linear term becomes k the u cross omega becomes convolution of the curl. So, it is not a dot product; it is not a cross product. So, **u**(**q**) × **ω(p**) and **q** is equal to **k** minus **p**. So, a sum over all p's and this is a viscous term.

Now, this **A**×(**B**×**C**) can be written in terms of some vector identity. So, this **A**×(**B**×**C**) is a very good thing to remember **A** × (**B** × **C)**; what is that? **B** (**A**.**C**), the first one minus A; A dot 1 and 3 and **C** (**A.B**). So that is what I have done here. So, i**k**· **ω(q)** **u**(**p**) and i**k**.**u**(**q**) **ω**(**p**). This we derived it before, but it just a recap, but you should pay attention; so this is important we need this.

If you look at equation, this is basically d**ω**/dt. So, if I put a minus sign of this it becomes plus and this plus. So, d**ω**/dt – **N**(**ω**) forget the viscous term. So, the first term is **k**.**ω**(**q**). So, that your this p this is the type; so qp qp qp; so I have again made one mistake here, so this is a mistake. This is vortex advection this one. So, is carrying; so **ω** is carried by **u**(**q**). So, this comes as$(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$; this comes from this term $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ right because this will come from grad and this comes from u is that clear to everyone?

A real space there are two terms this $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ and this is the minus sign also this is I made a. So, this is sign error as well know one of them come with a minus sign. So, this comes with a minus sign and this comes with a plus sign in the right hand side ok. Let us just talk about $\mathbf{N}_\omega$. So, I think there is there is error of sign; so which is plus sign?

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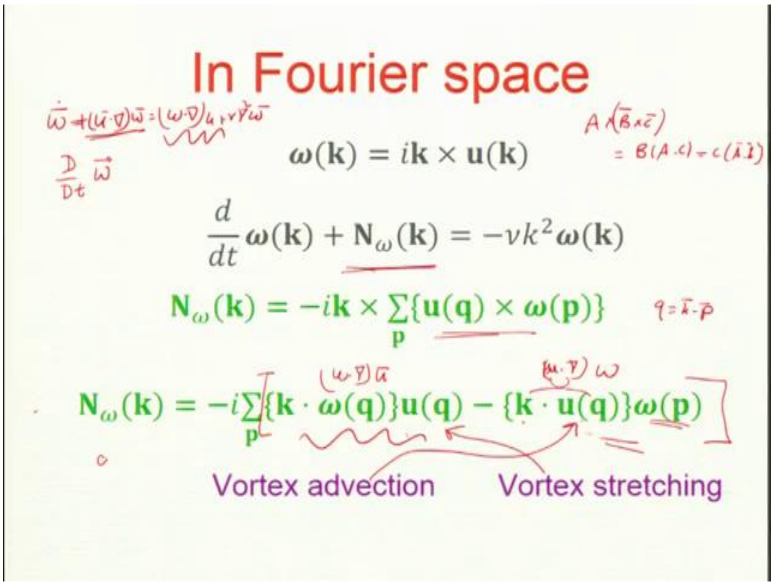


Student: (Refer Time: 05:20).

This plus sign this supposed to be this and there is a minus sign the minus becomes plus ok. So, this term is coming from ( $\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ and this term coming from $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ .In real space, let me just write this real space $\dot{\boldsymbol{\omega}} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$ .This is equation of $\boldsymbol{\omega}$. This term is giving you this non-linear term in Fourier space and this one is giving you this non-linear term; so this term is advection.

What is the advection? So, this is basically total time derivative of omega on left everybody is done this fluid mechanics, So this is material derivative. So, omega if you sit with then when you go along with it. This is pushing this the mean flow is pushing this vorticity is that clear to everyone? So, this term is advecting term and this term is vortex advection and this term is $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ is something Stretching**.**

Student: Stretching.

So, this is vortex stretching. So, I did not look at it, but the other terms are. So, k dot omega u is vortex stretching; ok, Now let me ask a question which one is going to increase omega overall? If I look at omega squared of the whole system you can make a guess.

Student: Sir.

Advection will not increase advection is just porting things; is just taking one same material from one place to other place; it does not give new material. But stretching can increase omega or decrease omega is expected to increase omega. So, vortex stretching is coming from the second term. So, there will be two kinds of energy transfers or enstrophy transfer one is because of advection other one is from stretching.

Now, we can make this quantitative.

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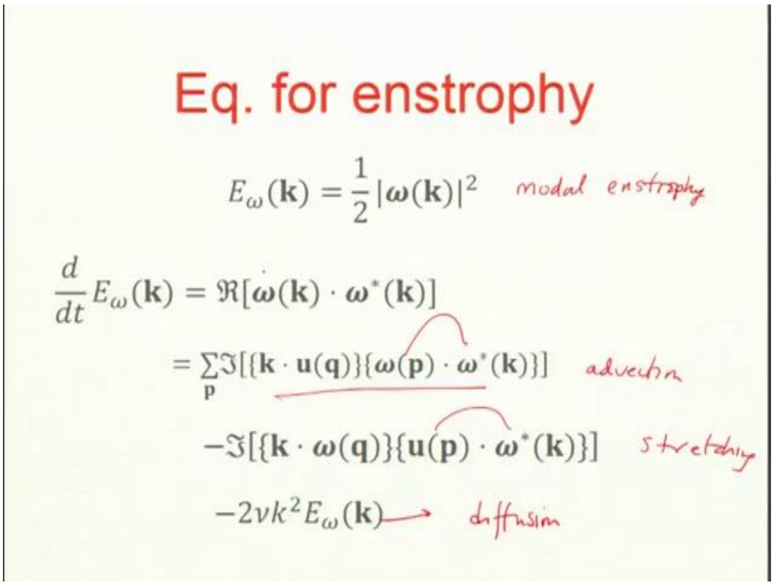


Now enstrophy I define as $|\boldsymbol{\omega}(\mathbf{k})|^2$. $\boldsymbol{\omega}(\mathbf{k})$ is the complex quantity. This is modal enstrophy, modal means for a given mode what is the enstrophy.

I will take the time derivative of this. Since $\boldsymbol{\omega}$ is function of time. The $|\boldsymbol{\omega}(\mathbf{k})|^2$ will also function of time. It will be just straight forward. So, this half will go away and this is very similar to what you do for velocity field. I will get now $\dot{\boldsymbol{\omega}}$ as two terms. So, it is now also will have two terms. So, this coming from advection and second one is stretching.

Now, this stretching and diffusion or dissipation let me call. Now, in advection term $\boldsymbol{\omega}$ is multiplied by $\boldsymbol{\omega}$ and here $\mathbf{u}$ is multiplied by $\boldsymbol{\omega}$ with dot product multiplication. Here is dot product ok.

We can do the same analysis like mode to mode transfer. This is energy now let us do, what happens for a single triad now this is for all triad now. So, let us try to dig further and try to get some picture about how energy is flowing by the way. So, this is this one work

I do not see many papers now, but people look at it, book by (Refer Time: 09:43) and (Refer Time: 09:44) and (Refer Time: 09:45) this is a lot of discussion on mechanism of vortex stretching. So, flow should go like this and it should stretch this is all in real space.

Now, there is interesting discussion when you stretch it frequency will increase just because of mechanics know if you stretch it, then (Refer Time: 10:04) increases. Now, I would want personally to do it in terms of Fourier space. So, understand in terms of transfers. So, I will exactly tell you what we should look for in terms of mode to mode transfer and it can be done.

So, we look for mode to mode enstrophy transfer. So, same formula we for kinetic energy we can generalize this to enstrophy.

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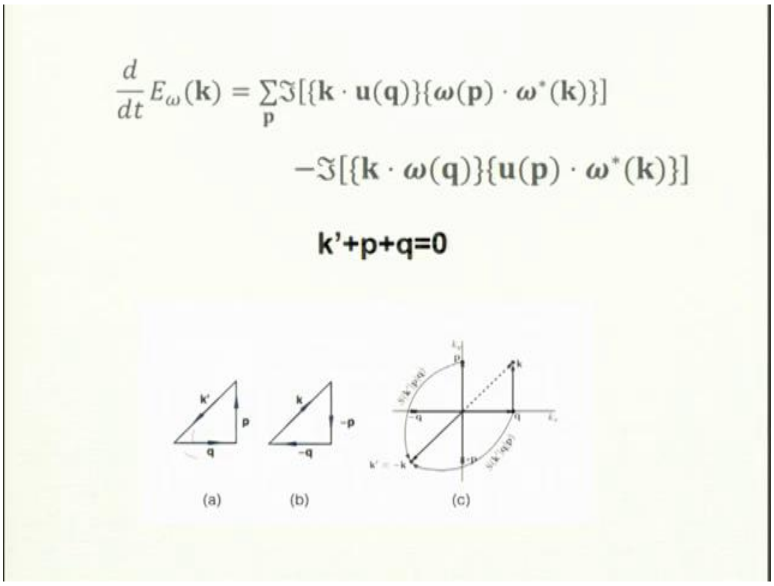


Instead of working for all possible triads in the flow, we focus on a single triad. Single triad and its pair for complex conjugate reality condition requires that also – **k**, **-p**,**-q**. So, we have a **k′ p q** and this is k minus k prime which is k **-p -q**.

So, I have only one triad. So, for a single triad instead of sum I will get few terms you know; the sum will reduces to only few terms.

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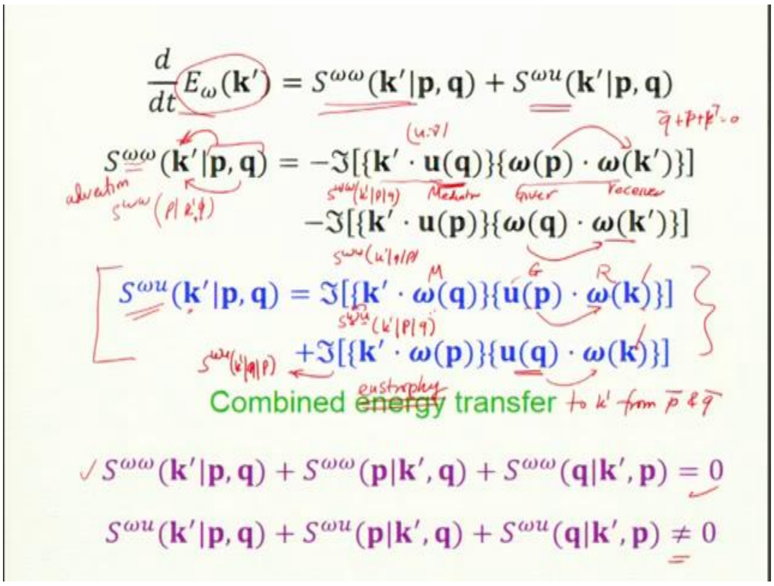


And so this is transport advection and this is stretching ok. So, the sum will involve two terms for stretching and two terms for advection. Straight forward know I mean these are just long formulas, but if you understand physically that is what they mean. So, this is $\omega$ $\omega$ multiplication k dot u. This is coming from $(\mathbf{u}\cdot\nabla)$. So, this is advection term.

So, $\{\mathbf{k}'\cdot\mathbf{u}(\mathrm{q})\}\{\boldsymbol{\omega}(\mathbf{p})\cdot\boldsymbol{\omega}(\mathbf{k}')\}$ and $\{\mathbf{k}'\cdot\mathbf{u}(\mathrm{p})\}\{\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(\mathbf{k}')\}$ and $\mathbf{q} = \mathbf{k}\text{-}\mathbf{p}$ So, $\mathbf{q}+\mathbf{p}+\mathbf{k}' = 0$. Let us keep this notation vectors. Now what about $\omega u$? It just follows from the sum. So, sum as the u is replaced $\omega$ and product is u with $\omega$.

So, $\mathbf{u}(\mathbf{p})$,$\boldsymbol{\omega}(\mathbf{k}')$ I missed prime here $\mathbf{k}'$ $\mathbf{u}(\mathbf{q})$,$\boldsymbol{\omega}(\mathbf{k}')$ this is for a single triad. But kinetic energy has only one term the first term and these are all us everything was u. Now with vorticity of course, there will be more complication. So, now we have vorticity to vorticity and velocity to vorticity. So, these are the terms and this should be telling you is a stretching term. So, we should analyze this for stretching when you stretch it how does the vorticity change. So, mode to mode and from the DNS; we should pick the data you know daily see whether it is stretching.

Let me just do few more steps and close my derivation; this is enstrophy increase rate of change of enstrophy at a wave number $\mathbf{k}'$ by two terms so and two wave numbers. So, there is increase in $\mathbf{k}'$ from both $\mathbf{p}$ and $\mathbf{q}$ so, it should be combined enstrophy transfer. So, this is a combined oh actually sorry this should be combined enstrophy transfer ok.

So, combined enstrophy transfer to $\mathbf{k}'$ from $\mathbf{p}$ and $\mathbf{q}$ and now this satisfies some properties. Now as I said you can do it the first one you have to write for $S^{\omega\omega}$ $(\mathbf{p}|\ \mathbf{k}',\mathbf{q})$ and same thing for q by q; sum them up incompressible condition has to be used and if you use that then you get this is 0.

But intuitively it is obvious because this is not stretching is just advection. So, you just porting from one place to other place; you just sweep your floor, but you just transferring the dust from one place to other place you differently seen the dust unless you take it out. So, this is just advection and does not change the enstrophy the first term.

But this one can change. In fact, you can just do the sum you will; you cannot show that it is 0. So, if I have again three terms from $\mathbf{p}$ and $\mathbf{q}$ just add them up this is not equal to this. So, sum is not equal to 0 ok. So, that is the result for combined enstrophy transfer, but now can we think of mode to mode transfers?

Now, same question this is this two terms together give the rate for $\mathbf{p}$ to $\mathbf{k}'$ and $\mathbf{q}$ to $\mathbf{k}'$ is total, but can I say individually how much goes from $\mathbf{p}$ to $\mathbf{k}'$ and how much goes from $\mathbf{q}$ to $\mathbf{k}'$. It turns out this is the proof which I will not prove it right now is that in the book. This term is very similar to what we do kinetic this is $\mathbf{p}$ to $\mathbf{k}'$ whatever is product of ω to $\omega$ is a giver and receiver. So, this is a giver and this is a receiver.

So, $\mathbf{k}'$ is a receiver right So, I identify whatever is dot product $\boldsymbol{\omega}(\mathbf{k}')$ with a giver for this term this is a receiver, this is a giver is that clear. And here what is this $\mathbf{u}(\mathbf{q})$? Is the mediator because it is multiplying with $\mathbf{k}'$ which is $\mathbf{u} \cdot \nabla$ this term k with u. So, this is a mediator it is a advector something advecting; is only driving the thing this is its taking things one place to other place, it does not participate in the transfer.

So, this is $\omega$ to $\omega$ transfer right; so giver is $\omega$ receiver is $\omega$ . So, $\omega$ to $\omega$ transfer. So, this is basically $\omega\omega$ $\mathbf{k}'$ $\mathbf{p}$ $\mathbf{q}$ and this one is $\omega\omega$ $\mathbf{k}'$ $\mathbf{q}$ $\mathbf{p}$ because k $\mathbf{q}$ is giving and $\mathbf{k}'$ is receiving.

Now, same idea here $\mathbf{u}(\mathbf{p})$ is giving and $\boldsymbol{\omega}(\mathbf{k}')$ is receiving. So, these are giver and these are receiver these are mediator. So, this is $S^{\omega u}$ u to $\omega$ right. So, the notation the second argument here in the super superscript, u is a giver $\omega$ is a receiver, $\mathbf{k}'$ is a receiver, $\mathbf{p}$ is a giver and $\mathbf{q}$ is a mediator.

And what about this one is $S^{\omega u}(\mathbf{k}'|\mathbf{q}|\mathbf{p})$ because $\mathbf{q}$ is a giver; $\mathbf{u}(\mathbf{q})$ is giver and $\boldsymbol{\omega}(\mathbf{q})$ is the receiver. Here this is like giver is somebody else is not $\omega$ is getting from u field is stretching. So, it is not among themselves but it is some other party. So, this is what it is ok.

So, the proof I will not do it here is quite long, but it is very similar to what you do for kinetic energy. So, complete as I read in the book, but is just intuitively this; this is what it is.

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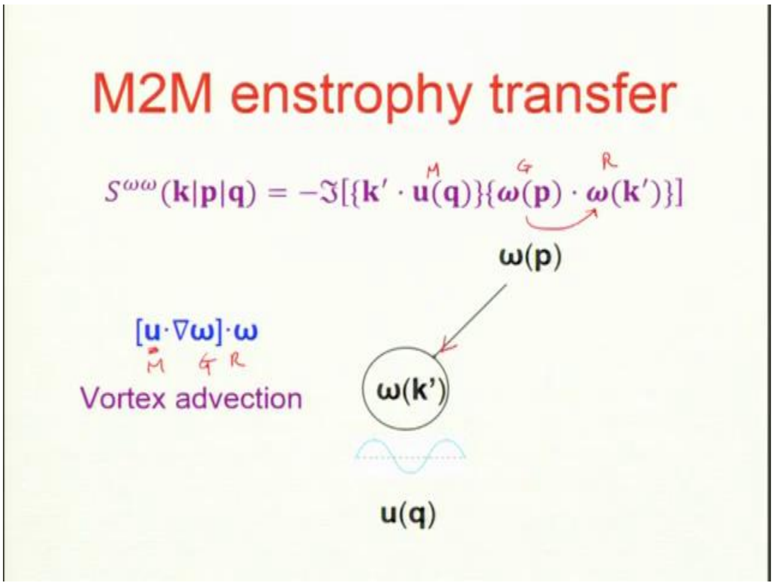


Let us see this result. So, without proof I am just stating that this is $\boldsymbol{\omega}(\mathbf{p})$ to $\boldsymbol{\omega}(\mathbf{k}')$ with $\mathbf{u}(\mathbf{q})$ is a mediator, giver, receiver, mediator and the picture is it comes from u is advecting mediator; this is this is giver and this receiver and the picture which is; so it is coming from here $\boldsymbol{\omega}(\mathbf{p})$ to $\boldsymbol{\omega}(\mathbf{k}')$ with $\mathbf{u}(\mathbf{q})$ is a mediator. So, this is mediating wave is mediating.

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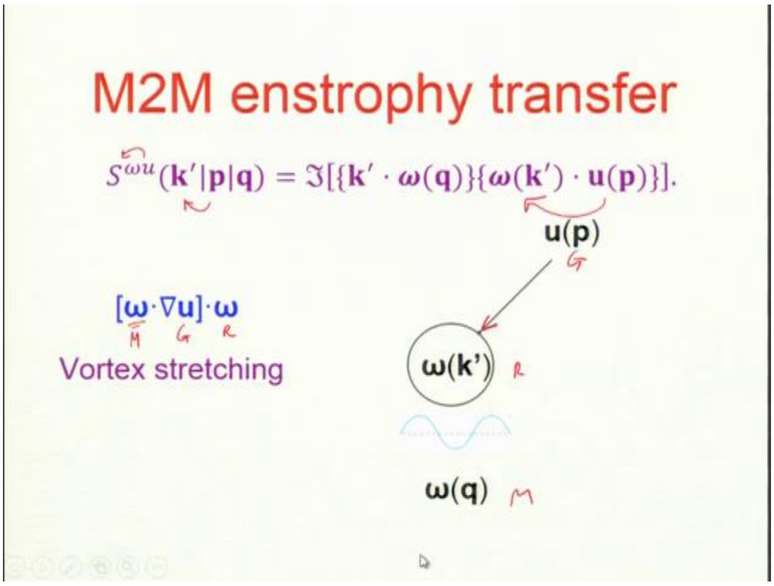


Now therefore, other one $\omega$u; here **u**(**p**) is giver and **ω**(**k**) is receiver. So, this is here and its comes from $\omega$ these are mediator giver, receiver and this is a vortex stretching and up is the arrow is not very up is a giver this is a receiver and this is a mediator. So, this is mode to mode u to $\omega$; We just discussed mode to mode transfer. So, this vorticity to vorticity and this is a velocity to vorticity.

Now, we can define flux with that you know. So, non-linear term that was in a triad, but you have many triads. So, that will define energy going from one wave numbers sphere wave number sphere inside to outside. So, it is just like Kolmogorov flux very similar.

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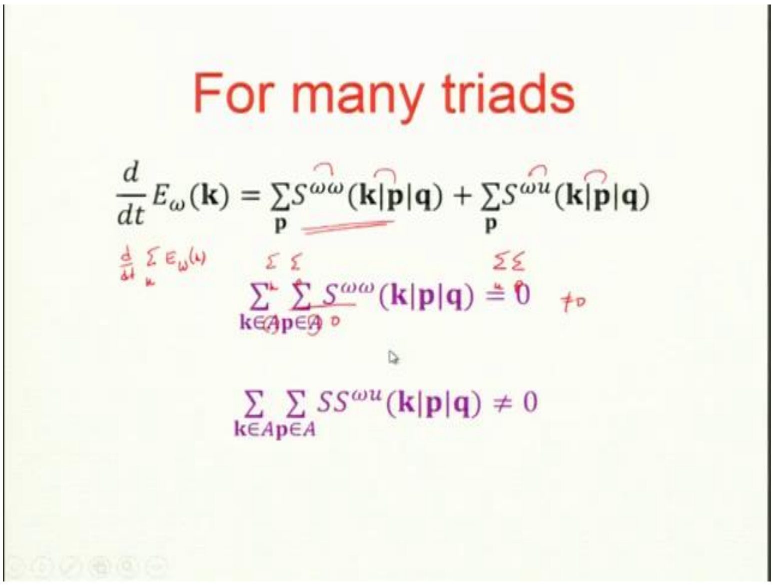

So, this was $\omega$ to $\omega$ advection and this is u to $\omega$. Now, I do it for many triads; so I derive it for single triad mode to mode, but just do it for many triads. So, this will give you. So, $S^{\omega\omega}$ is basically sum over all **p**'s. This is coming to **k** from **p**.

Now as I said if I sum over all k's I sum over all k's d by d t of this. So, this is sum over k sum over p sum over k sum over p this is 0 sum over p is 0 because advection is basically conserving enstrophy. And this is this sum is not equal to 0 this is enstrophy coming from velocity field by stretching.

So in fact, this is only for all, but is true for any region, but giver and receiver in the same region you know you sum over. So, $S^{\omega\omega}$ is 0 this you can show for like from the conservation law which I told in the last set of slide and this is not equal to 0. Now from this; so again you can do more formal thing which I did in the verticity, but velocity field, but I will skip all that you can define flux.

And definition is once you have the transfer from one region to other one mode to other mode then you can define flux that is a formula is any no need to motivate further.

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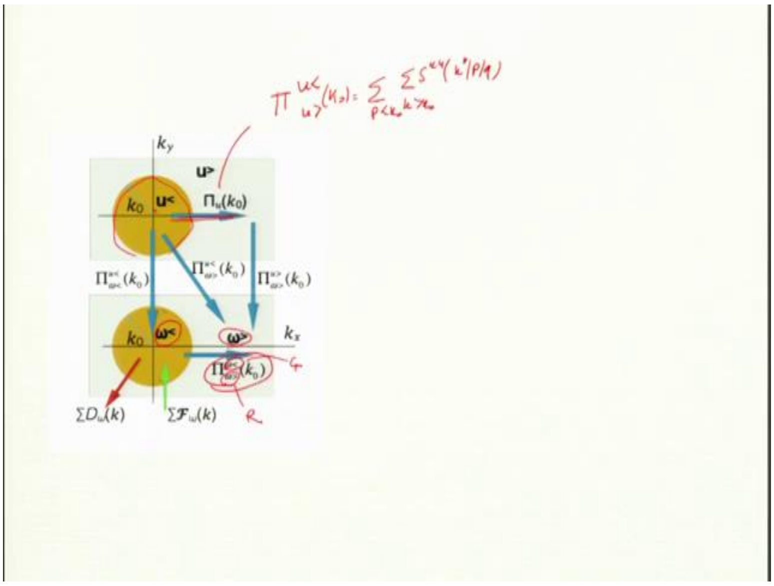


So, there is the picture looks very complex, but if you look at the picture slightly more carefully then you will find the idea. So, these are velocity sphere the brown is velocity sphere the mode inside is u< mode outside is u> and we also define vorticity sphere. So,

$\omega<$ is the modes within this sphere $\omega>$ is the modes outside this sphere, is that clear? Though $\omega$ and u are related, but we define two different spheres.

Now, u sphere u< sphere can give now this is Kolmogorov flux right modes inside giving more energy to modes outside and this was written as $\Pi^{u<}_{u>}$. In fact, it can be written as like this we did in the last lecture radius of this sphere is $k_o$. It is sum over p which is giver. So, it is less than $k_o$ and the receiver is outside. So, k greater than $k_o$, $S^{uu}(\mathbf{k'}|\mathbf{p}|\mathbf{q})$ where, $\mathbf{k'}$ is -**k**. So, this we define now we can also define flux for enstrophy. So, let us look at the simple one first this one. So, what is this? $\omega^<$ and $\omega^>$, who is receiver and who is the giver?

Student: (Refer Time: 23:23).

who is the giver? $\omega^>$ is the receiver and $\omega^<$ is the giver. So, giver modes must be within this sphere and receiver must be outside this sphere.

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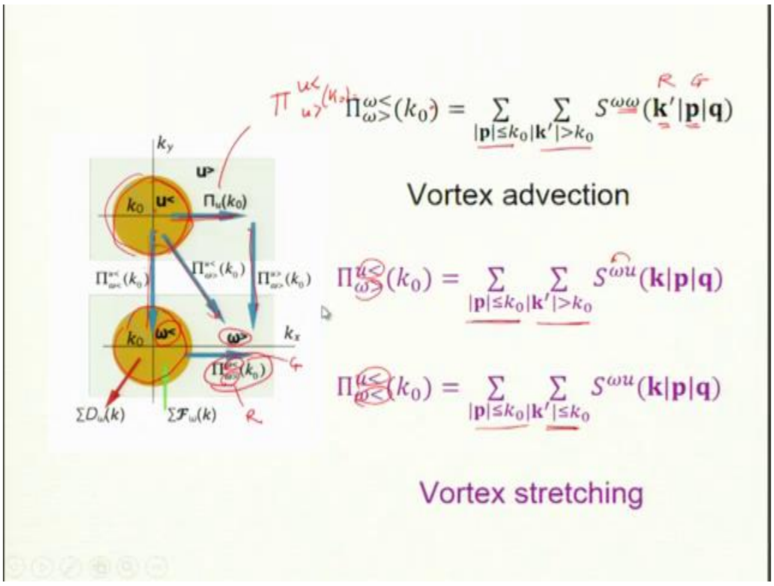


So, I just define it here; so let us erase this one. So, $\omega^<$ to $\omega^>$ is defined as receivers. So, $\omega$ to $\omega$ transfer which are defined mode to mode giver is **p**, receiver is $\mathbf{k'}$ so this receiver giver. Giver must be within this sphere and receiver must be outside this sphere, you just sum it up. So, this is $\omega$ to $\omega$. In fact, this is within $\omega$ there is like Kolmogorov type very similar, but $\omega$ to $\omega$ and this is advection vortex advection.

Now, $\omega$ get from u by how many channels? You got one two three channels. This is important point u cannot give to u itself within this sphere, but now $\omega$ within can get from u within. So, this blue, this line is u< to $\omega^<$ , it can be nonzero and so we have u to $\omega$, but both receiver and giver are in this sphere, but giver is in this sphere and receiver is in this sphere oh; sorry this other one let me focus on this one first this one both are within. So, this is u< to $\omega^<$.

Now, we have u< to $\omega^>$. So, here to here this is one; so this p is giver less than $k_o$, but receiver is more. So, this vortex stretching plus there is one more which I did not write it here this one; giver and receiver both are outside. Now, this what needs to be computed in simulations and this work is incomplete. In fact, nobody has done this computation; as far as I know I mean no. In fact, mode to mode $\omega$ transfer nobody has given the formalism. So, we are the people who construct all this formalism.

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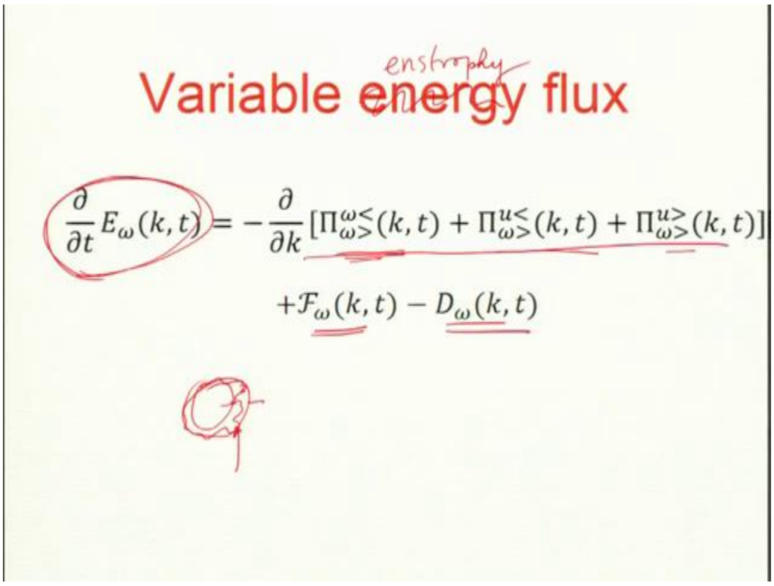


So, we can also define enstrophy flux. So, This is very similar for variable energy flux. But if I look at enstrophy of a shell, it can change by fluxes, but they are not free fluxes. There is one flux within itself, we had what kinetic energy flux here minus flux there, but now these are the flux coming from outside. So, this derivation I am not doing it here, but I derive in my notes.

So, variable enstrophy flux it involves three fluxes. So, there this one is coming from external force and this is dissipation and this one. So, if this is 0 under steady state then we should get and even initial range; then this must be constant. But I am not sure whether this is study this also should be studied whether energy enstrophy keeps increasing or it saturates I would like to see in simulation. Now, this is I am just flashing, it is a something to test in simulations.

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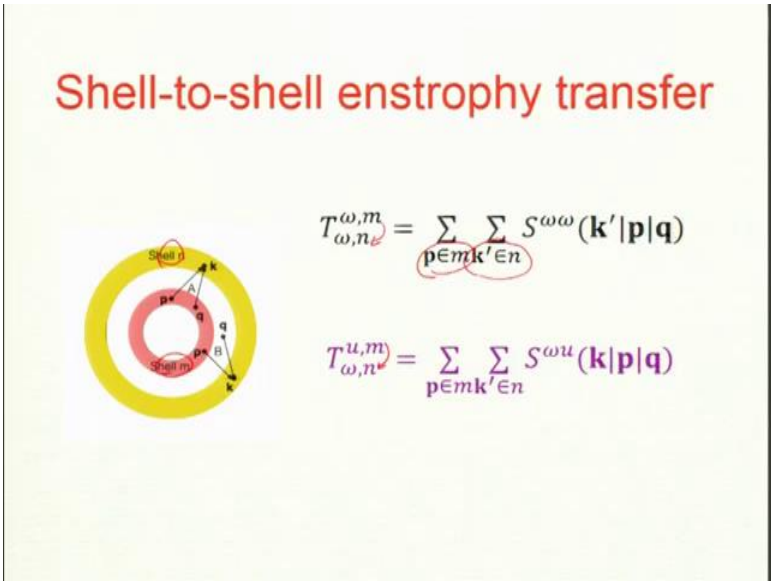


We can also define shell to shell transfers. Now the two of them $\omega$ $\omega$ and u to $\omega$, Remember the shell this is the giver shell and receiver shell. So, **p** belongs to m giver shell and $\mathbf{k}'$ belongs to receiver shell. This is shell to shell; so is m to n $\omega$ to $\omega$; we also define u to $\omega$ from shell m to shell n. So, there two channels of transfer; $\omega$ to $\omega$ and u to $\omega$ and they give you shell to shell enstrophy transfer.

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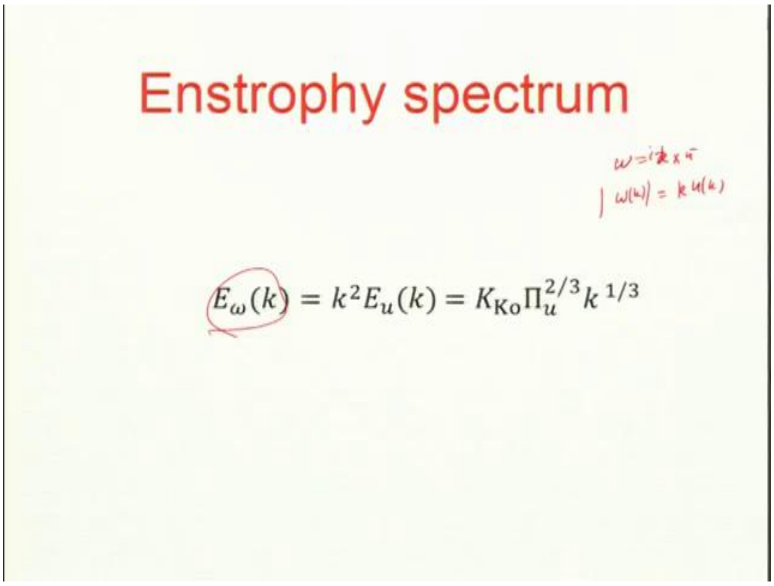


What is enstrophy spectrum? that is by dimension not dimension because $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. So,$i\mathbf{k}\times\mathbf{u}$; $\boldsymbol{\omega}(\mathbf{k})$ is; mod of this is ku(k). So, spectrum of $E_\omega$= $k^2 E_u$.So, its spectrum is straight forward just Kolmogorov theory. By the way velocity field follows Kolmogorov theory; this is a funny situation my velocity field is Kolmogorov, but enstrophy is giving somewhat new well I mean it has not been studied that is why I mean enstrophy has not been studied so carefully in Fourier space. So, spectrum is this is hundred percent sure because it just follows of Kolmogorov theory, but the fluxes are what we need to compute.

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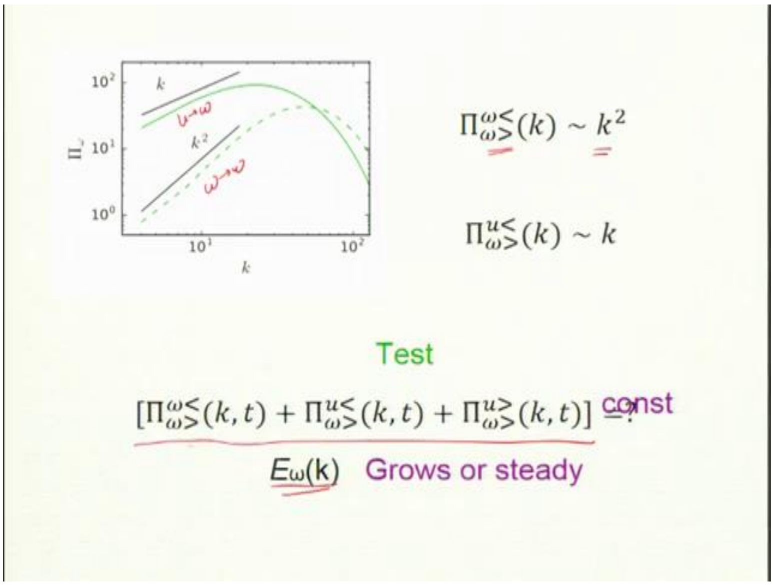

So, spectrum is fine. So, in numerical simulation; this was done by Subhadeep who is in our lab student. So, he computed this fluxes has been there in Tarang, so somebody has to run it. These are fluxes from $\omega <$ to $\omega >$ this that $\omega$ channel is $k^2$ and $u<$ to $\omega$ channel only k, one channel has been studied and this is $\omega$ to $\omega$ and this is u to $\omega$ and they are not constant.

Now, my question is, it does what about the time derivative of this is it when this $\omega$ is growing or become steady in the under steady state I believe should become steady. But one should check and what happens to this in the initial range and also as a picture of stretching in Fourier space. How does it look? So, which mode we can make a triad, but we can construct triads you know examples which I have constructed in the class.

You try to do it not well many of them do not give stretching in fact, they give you decrease in $\omega$. So, this is important point let me make this point clear if I construct any arbitrary example for sin cos or cos sin which you have been playing around compute; the stretching or just see whether $\omega$ will increase or not mode to mode transfers it turns out $\omega$ may decrease in fact, I find $\omega$ decreases.

It tells us one important point, it is a organized structure of the steady state that will give you stretching any arbitrary field composition will not give you stretching. So, stretching increase of enstrophy which is say that stretches is for flows which are already reach steady state or is evolved states not all any combination of velocity you does not give stretching; is it clear to all of you?

We need to look at the simulation data construct triads and then look at stretching. In fact, we just we need to analyze the data of the developed flow. So, the fluxes are so if I take any arbitrary flow field a compute Kolmogorov flux what will I get? Not from the simulation, just generate any velocity field.

You will get if basically get 0 flux random, but is evolve field this is it its phases are in such a way that it is organised itself that gives you constant flux. So, organization comes by evolution this is equation is leading towards that organised flow. So, this stretching is what we should look at from the developed data. So, these are some things one needs to work out. So, I think we stop.

Thank you.