Physics of Turbulence

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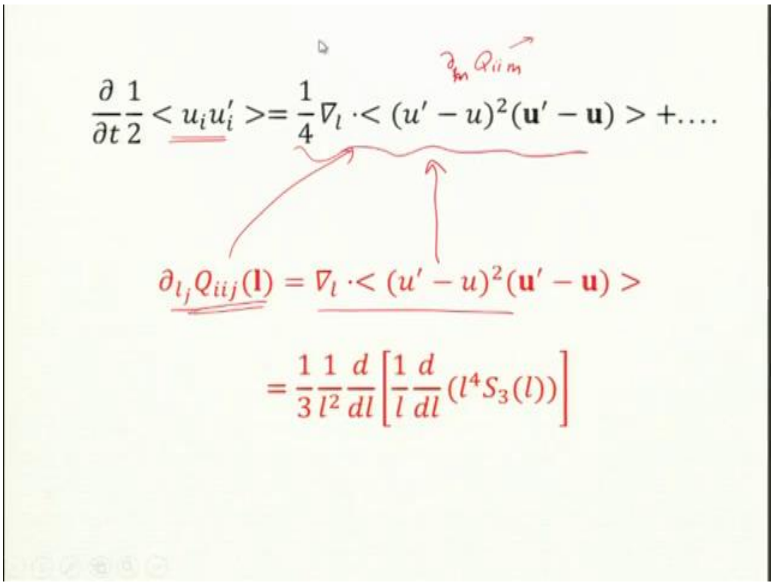
Lecture - 32
Kolmogorov's 4/5 Law
4/5 Law: The Final Step

Ok. So now, we are at the last step. We developed all the tools, second order and third order correlation function, a structure function. So, what I am going to do is the following. I am going to show the following.

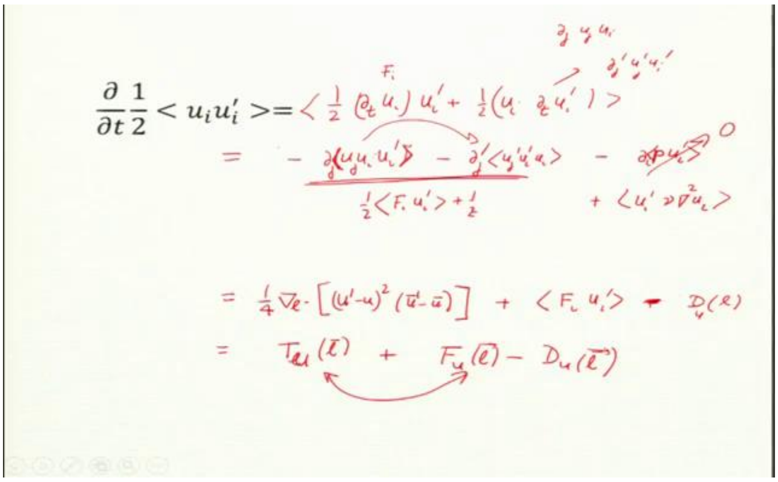


So, these are in layout. So, I am going to, now go through dynamics. So, write down the velocity from the Navier-Strokes equation using that time derivative of the second order correlation function u_i u_i'; earlier you remember that you are doing u_i u_i total energy or total. So, it was not at different point, it was same point but now I am taking two different points.

Now, this can be shown to be this is why we need all this stuff, the right hand side the non-linear term appears that like that u dot u term appear says in fact, structure function. This is $Q_{ij,m}$ ∇_l del m right is which I derived, this was there in my old slide and this is that stuff.

So, this can be shown, now this is connected with a third order structure function and that is the connection ok. So, this one is $Q_{ii,m}$ will which I had done this. This is equal to this one.

Except factor 1/4 but this was one-third this object. So, I will just map it to that and relate; so, is a dissipation rate and we are home ok. Now, let us fill up the gaps.



So, how do I write this time derivative? So, half so, $(\frac{\partial u_i}{\partial t} u_i' + u_i \frac{\partial u_i'}{\partial t})$. Yes, so it is a product rule. Now, what is $\frac{\partial u_i}{\partial t}$? It is $\partial_j(u_j u_i)$ right. Yes, in a ∂_j yeah what?

So, that is a non-linear term this is of course, pressure term. So, first term is that with a minus sign.

Student: Minus sign.

Minus sign, so one term, so, I am just going to just sketch it ok. So, u_j $\partial_j(u_j u_i)$, but u_i, so, this is operated ∂_j will not act on u_i'. So, I can push it inside right, where this operator it is only on **r** position not on **r** pressure position. Now, I am doing averages, so, this average will come here. So, this is the third order correlation function has come now ok, now that I relate to that order structure function which by the all the derivatives which have done. Now, the second term we just follow the same step. Now, derivative with here we just follow the same steps prime, but this will be on prime variable.

So, this will be ∂_j' because this is the second point u_j' u_i', now u_i will go inside because this thing. So, this is u_j' u_i' u_i' . What about pressure? So, d/dt is the minus ∇P so, minus ∂_jP, now u_i' can be pushed in because derivative acts on P not on P prime variable. So, it is u_j', same thing you derive for the other one. So, they have come in pair with prime switched and then the force as well. So, d/dt is right hand side is force F_i. So, F_i u_i' averaged half and half from there. So, plus half like that and this is the viscous term.

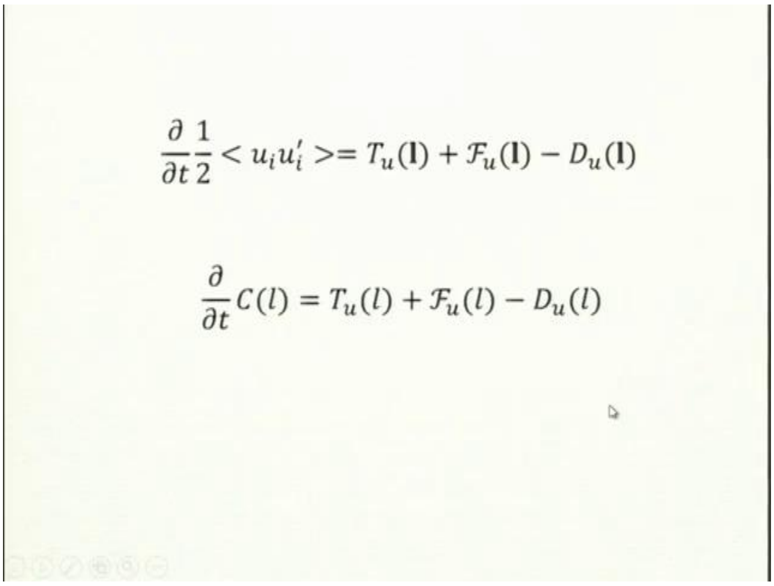
So, this is u_i' ν ∇^2u u_i average ok, is that fine? I mean these how; so, we write that it is a straightforward no problem. What about this one? Pressure with u_j' , this one.

Student: 0.

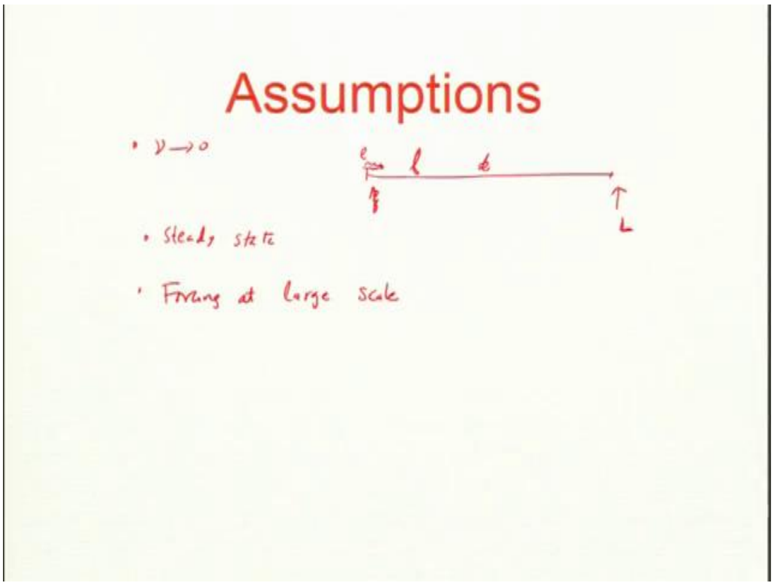
0. So, for given pressure my velocity will fluctuate plus minus plus minus it assigned. So, this goes to 0. So, pressure term does not contribute like our earlier derivation in incompressible flows pressure go with drops out gone. Now, these two by that logic that I can switch these derivatives these two in fact, can be connected together. I will skip this algebra ok. So, I will just write down the final step. So, you can relate this prime, these are exactly remember the third order correlation function and I connected the structure function.

So, you follow the step and this is $\frac{1}{4}\nabla_l[(u'-u)^2(\boldsymbol{u}'-\boldsymbol{u})]$. This is this non-linear term ok, this is just very few steps which I have in my notes and the second term is the half and half, a force in that equal F_i u_i' average. So, there are two forces coming they are equal and the viscous term. Let us assume viscosity going to 0. The viscous term is small in the regime of my interest, but right now, let us keep this viscous term; I will call it dissipation D(l).

I put a minus sign here following the notation. Now, this one is called T_u(l) is a non-linear transfer term and this is coming as force and this is a D_u(l). So, these how we connected the second order correlation, time derivative with the non-linear transfer term. This is that s if we call S^{uu} (kpq) and this is what is exactly similar to what we did for Fourier transforms. Now, we are almost there; so, we will make some assumptions and connect these two ok. So, I make one more step. So, these are in fact, function of vector l, since it is isotropic it becomes function of l only magnitude l. So, that is my next slide.

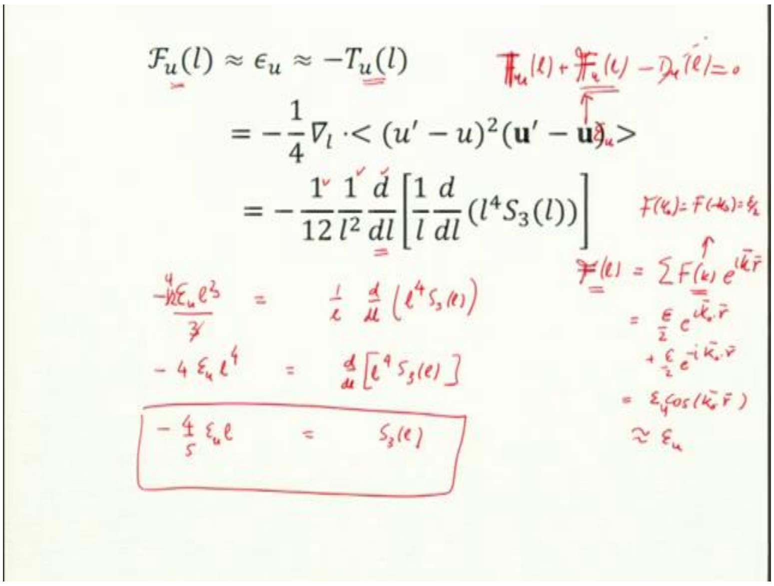


So, this is magnitude l. So, instead of writing vector l I just write l.



Now, the next step let us make assumptions. So, same assumption like before we make viscosity tending to 0 ok. So, we are making the flow very turbulent, but we have very strong a very large inertia range. So, I have forcing somewhere here and I have l here that forcing is large scale no; so, we are looking into it. So, if l is increasing I have l here and viscosity is very small l. These I forced my I have to flip it, because the fourier space I forced at small scale but I am forcing it very large scale.

Steady state what happened to a steady state; d/dt is gone. So, set d/dt to 0 and assume the forcing at large scale forcing at large scale, the force set very large scale ok. So, these are assumptions which we made already before I make the same assumptions, the same theory what is in real space ok.



If we make that assumption then d/dt term is gone. So, if we remember so, d /dt is gone. So, T_u (l) plus F_u(l) math cal actually because, this is force times velocity minus D_u(l). I am interested in the inertia range which is bigger than whose length scale is bigger than the dissipation range. So, viscosity is very small-scale dissipation range. So, go to inertia range because intermediate scale. So, this term is negligible, dissipation is effective only it small scales. So, this term is gone. So, we get you relate these two will with the minus sign.

Now, what about this F_u(l)? What is it equal to? This is force injection net any position, if you look at it. These equal to energy supply rate is same at all positions approximately. Now how do I see this? I have argument but we can write this as I think I have to skip the argument. You just look at the proof. I think I do not want to make you more tired. So, this is equal to is constant everywhere approximately and equal to energy supply rate. So, this is a constant which is a ε_u (l), because the energy supply rate equal to dissipation rate so that means, minus T_u must be ε_u this equal to 0.

So, this must be equal to it right. So, this is how we connect dissipation rate with the energy term non-linear transfer rate. Now, you can see the proof is almost there. Now, T_u(l) is connected with third order structure function. Now, we can we can just finish off the proof. So, this is T_u (l) is minus one quarter of this one and this was I proved this one;

one I taken the already the divergence in the spherical polar coordinate. Now, how do I do it? They just straight forward, so, multiply with take that $(12\ l)^2$ to the left. So, let me do the last bit of algebra now minus ε_u $(12\ l)^2$ equal d/dl. Shall I do the integral again?

Student: Absolute.

Integrate the two d b/dl. So, so let us do the first integral. So, this will be l^3 / 3. So, I have done this one, this one and this one; this is the operations $(12\ l)^2$ went to the left and the d by dl integral is (1/l)(d/dl) l^4 S_3(l). So, what is this? This 4 right; now the one l is sitting here, again take it to the let because minus 4 ε_u l^4 is (d /dl) l^4 S_3 (l). Integrate this 4 by 5 right.

So, 4 by 5 ε_u l^5 is l^4 S_3(l) and now divide by l^4. So, this goes and ok. So, you can clap. So, these how we get S_3 (l), S minus Fourier 5 ε_u (l). So, I have say few more words, but if you want to ask some questions you can do it now.

Student: Personally at large scale but in absence.

So, let us see. So, so I can I can I can do it now. So, large scale forcing means you are forcing everywhere. So, what does it mean? Forcing everywhere; so, if I take this bucket and stir this stuff is forcing everywhere.

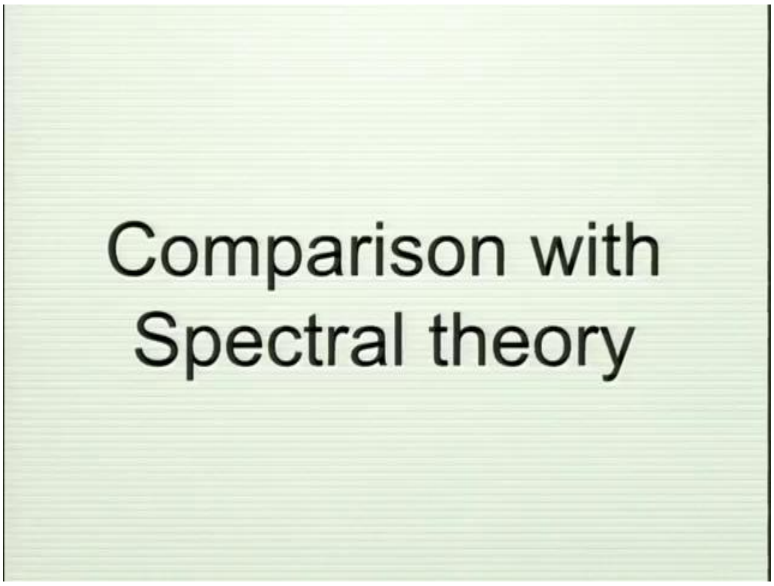
Student: In real space.

In real space, the Fourier space it is banned is small wave number, but is forced everywhere. So, if I multiply the velocity everywhere. So, large scale velocity versus large scale force it is affects everything, is like in the central government know; something happens it takes up is affects everything, in real space. A Fourier space is only the low band. Now, if you want to prove then it is also easy. So, let us let us do quickly F (l) math l is F (k) Fourier. So, there is a theorem that a real space correlation is Fourier transform of the spectrum, now this you have to believe me. So, this is the theorem you can relate. So, this is the correlation a two point.

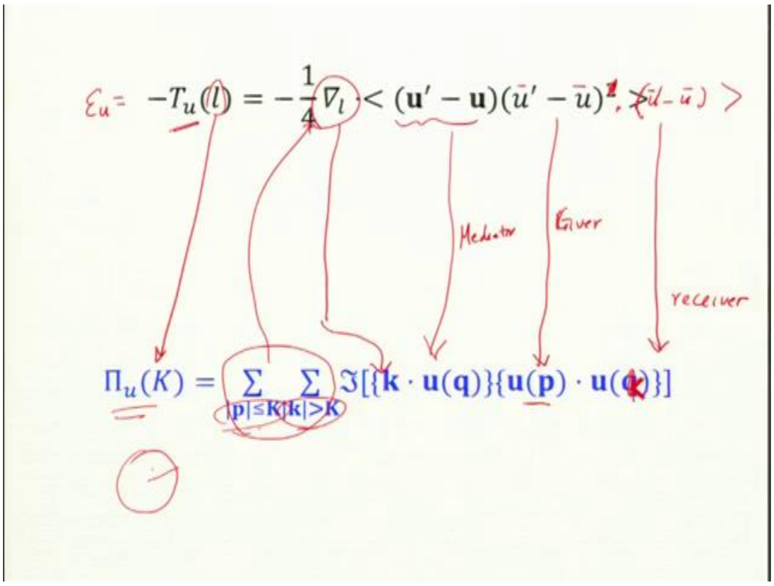
So, F_u equal to Fourier transform of this power spectrum E(k) dot i r, now this I force it small wave number. So, let us say I force it k naught, which is k naught being small. So, I put this stuff F($k_{o)}$ equal to F of minus k naught is ε/2.

So, the two wave numbers are forced, only two wave numbers not a shell. So, this will be ε/2 so, this is a constant. So, I get e to the power i k_o dot **r** fine and other guys also that ε/2 to the power i minus k_o dot r is there a minus k. Sum is only two wave numbers, sum it up what is ε/2, i k dot **r** plus 0 minus **k** dot **r**? Is cos k dot **r**; so, cos k_o dot **r** and epsilon the half will cancel with 2. Now, k naught go to goes to zero limit. What is k naught going to zero limit? Cos k_o will be 1.

So, this is approximately epsilon. So, these how you create ok, but it is kind of physically visible a force that large scale. So, it is going to there is visible at every scale ok. So, this is how we do the force space law. Now, just to complete the story; so, this is for the third order structure function ok.



I want to say how to relate this with the structural theory, our Fourier space theory. So, it is the connection is the following.

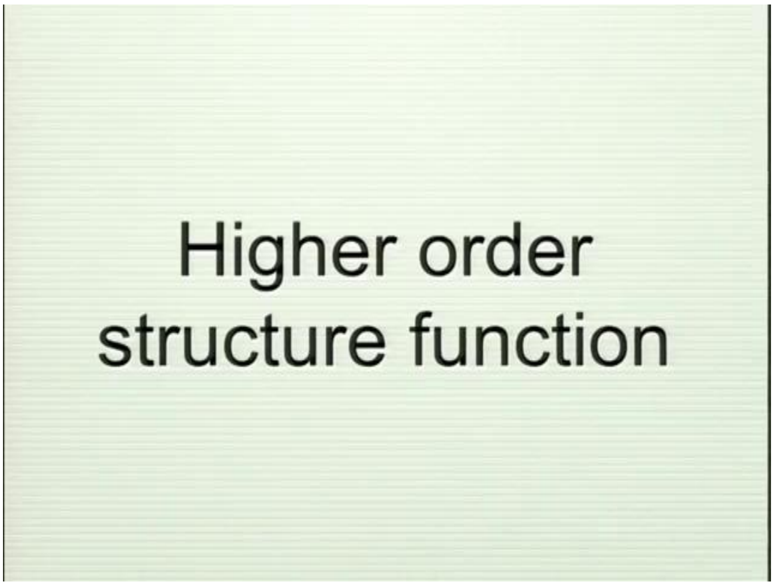


Now, remember I had the flux $\pi_u(K)$ is connected with the S^{uu} sum over. So, flux of wave number K is giver is inside and receiver is outside right; this all the stuff, giver more is inside the wave number sphere and receiver is outside when wave number sphere and this is equal to π_u (K) in the inertia range, this is ε_u.

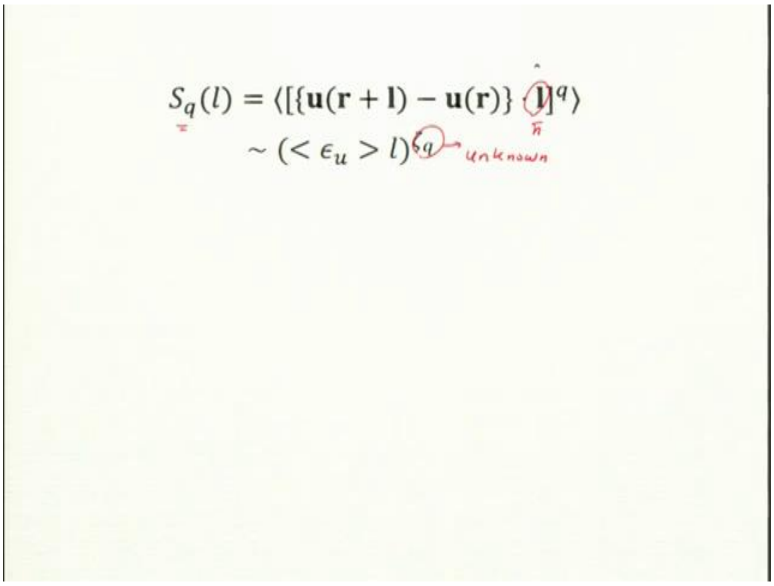
So, what is the connection? So, I write this u squared is a product dot product of u prime minus u vector. Now, beautifully this is like a mediator and this is mediator which is connected with that, we did with u^3. Now, this is product of the same function we said ∇u know. So, one of them is a giver and one of them is a receiver, giver receiver.

So, this giver sorry giver I made a mistake no, this k, receiver and this is the mediator u prime in fact, is mediating and there is a divergence is coming as this k vector of course, this is summing over all; this is in real space I am summing really because all the Fourier modes. So, is implicit that this sum is sitting here, take divergence basically we are doing some kind of sum and this l is connected with 1/ l, here is connected with K 1 by l is K.

So, these are connections in fact. It is in fact, is quite nice that there is, it must be related and this relation is visible ok. Now, this is for third order structure function. So, epsilon e is connected with u delta u delta u delta u, in this object. What about higher order?

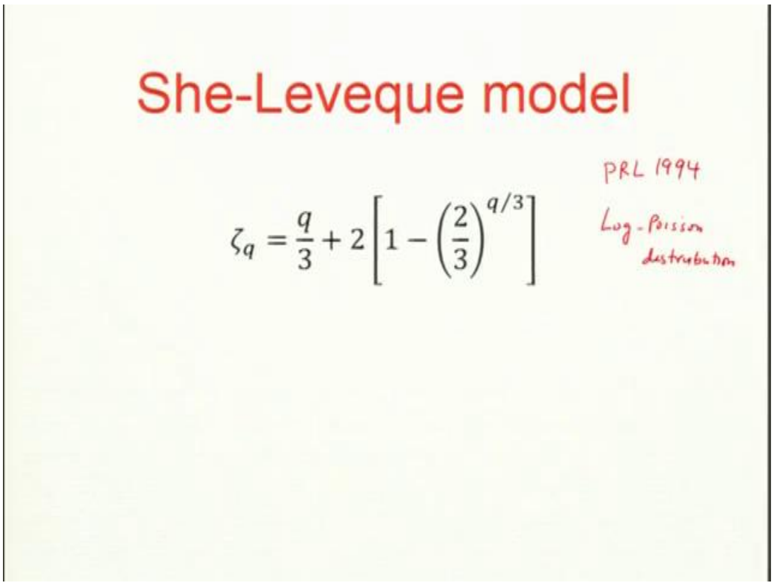


So, this has been a puzzle for in physics for many in fact, 50 years or 60 years.



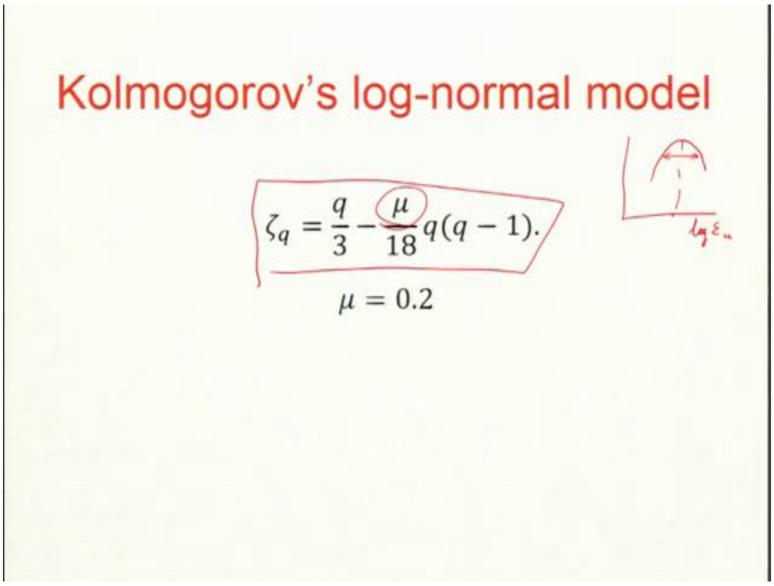
So, we can take the projection along again. So, l cap is along l direction or in fact, is as **n** is same as **n** ok. So, take the cube qth order structure function, S_q is q order. So, $(\nabla u)^q$, q equal to 3 is what Kolmogorov showed, but if q is let us say 4 or 5. What happens? So, it turns out these nobody has even to get a close form formula from the first principle. Start from Navier-Strokes equation we cannot get 2, q not equal to 3, q equal to 4, nobody can do it; nobody has been able to do it ok.

So, this is by dimensional arguments it is ε_u (l) zeta q, this is unknown and this is called exponent of the structure function. Now, I put average because ε_u is fluctuating ok; what Kolmogorov says is a kind of average epsilon u.



So, we use a epsilon u and idea is to estimate or obtain ζq. Now, I will not discuss in this lectures but ζq was proven by She-Leveque by some assuming that this dissipation rate is fluctuating and it has a log Poisson distribution ok. Now, I will not discuss this, but you can look at the original paper, distribution is physical regulators 1994 ok. Now, if you make certain assumptions, you find s function is a function of q.

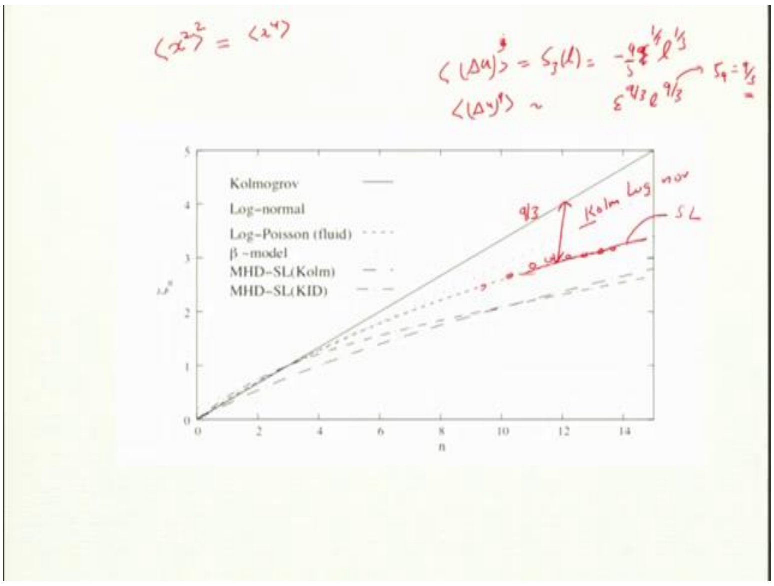
Now, it turns out this fits with the data quite well but it is not starting from first principle or you cannot prove it from this is analytically proven. It makes certain assumptions of this distribution of dissipation rate and it seems to work quite well.



Kolmogorov himself had given a theory that the dissipation itself log normal. So, normal distribution is you know, I mean there is a peak. So, normal distribution will be ε_u, so, this is probability y axis is; so, this one I can explain a bit more ε_u. So, it peaks at ε_u with certain distribution; had it been Gaussian if it is on linear scale it has a certain width. It is not a given number, but it has certain width right, there is Gaussian like heights of human beings are not same height. It is a average and this is the distribution is supposed to be Gaussian, but log normal means if I take the log in the x axis; x axis or y axis?

Student: x.

x axis know so, x axis log ε_u then it is normal. So, instead of taking ε_u you take logε_u in terms of it is Gaussian ok. So, there is a log of normal distribution and Kolmogorov argued why it is so and he postulated that the formula well from this ζq comes up this form and the data fits with nu equal to 0.2. So, mu is a free parameter here.



Let us look at how these things where, I forgot to say that if Kolmogorov's five-third theory or that $S_3(q)$ is εl know; what did I write S_3 (l) is equal to $-\frac{4}{5}\varepsilon l$. So, this is Δu(q), but if I make a extrapolation Δ u(q) what will that be? Just my extrapolation, so, divide by so, take one-third order. So, one-third if I just like that, take $\varepsilon^{q/3}$ $l^{q/3}$, so, ζq will be q/3.

You got in to Kolmogorov formula, but these are makes lot of assumption; if you averaging you cannot take power in and out right, you cannot do that. I mean that is not for x^2 average and x^4, average this is not square of this right. We know this, this is this is you cannot say this is like that. So, this is not proper, this is only proper for some distribution, but not for all distribution. So, well we make some assumption we said. So, this line is q/3 line linear line, She-Leveque is this line, this line this is She-Leveque; a log normal Kolmogorov log normal if this line.

The data appears to be closer to this line but you can see that it deviates very strongly from this q/3. So, by the time it reaches 10, the difference is around 10 percent ok. So, for q equal to 10 this will be 3.3, but answer is around 2.4 or so ok. So, the difference is in experiment; so, these are experimental data and numerical simulation which is also there is also done some of it. So, we can do this data and we find that you are away from the linear line and this theory is still not done yet. We do not know how to compute ζ_q from some theoretical framework ok.

So, this is what I had to say about structure function. I will stop ok.

Thank you.