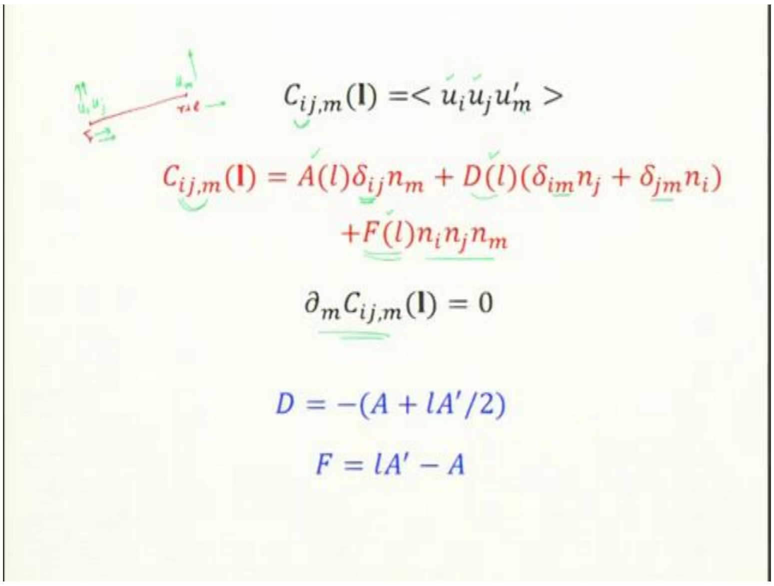
Physics of Turbulence

Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 31
Kolmogorov's 4/5 Law Isotropic Tensors & Correlations



So, we have three velocities, but we only have two positions, so not that the three different positions you have please recall we have **r** and (**r** + **l**). So, this first group you are using. So, there are three indices i, j, m. So, first two are at position **r** u_i, u_j and third one is at prime is a second point.

So, it is a coalition of three velocities, but at two different points important. Now, this comma separates this comma is separating the **r** with (**r** + **l**) ok. So, this is the notation. Now, prime is the second point. Now, again we can so, we discussed how to construct second order tensor, but now we can do third order tensor with this here only function of l. So, given l you can consider or **n** is non-dimensional, **n** is sorry **n** is a magnitude one unit vector.

So, you can easily make a guess n_i, n_j, n m right for l 1, l 2, l l i, l j, l m ok. Given the delta function how do I construct δ i, j, m. This is three indices right i, j, m, but here is the ij in the left, ij in the left ij has come here. Now, I can also have im; I can do this one im but then j will be here, but ij are symmetric right ij is i and j are the same platform, but m is a different thing. So, if I have im then I must also have jm. So, that will be jm, i n_i ok.

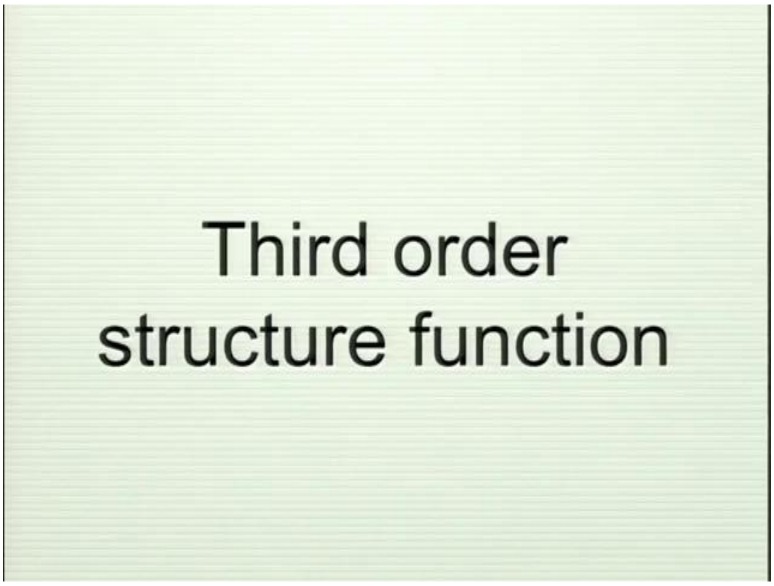
A(l) and D(l) are the coefficients which are and F(l) are function of only F(l). A(l) and D(l) are not equal, why? Because it is asymmetry in ij with m, is that ok? So, this is how you construct higher order tensors. So, this is the third order correlation function. Now, my unknowns are A(l), D(l) and F(l). Now, for second rank tensor I had this nice interpretation velocity; velocity like this. Now, third rank is not so.

I mean of course, we have two velocities here and third velocity here two velocity is here and third velocity here like that. You can consider there are more combination known here I mean that has the only 3; well 6 components with 3 of them being 0. Why it is 6? Why I did not say 9?

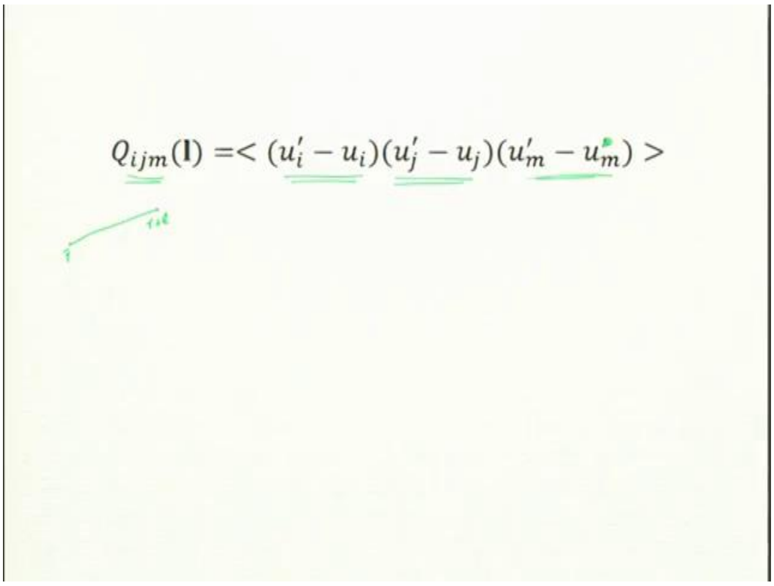
Student: Asymmetric.

Asymmetric ok, here I have 27 components some will be 0 and so on ok. So, we can construct. Now, this is I can also now do the derivative incompressible condition. So, incompressible condition will act on prime when you do well. So, that is why I have acting on m ok.

Now, that will once I apply the same in condition on this $C_{ij,m}$ do algebra then we will get some relations and these relations are the following it tells you how D and F are related A and A'. So, I can write this D and F in terms of A. So, instead of having more variables D and F, we can write it in a way here using this condition. So, basically A and A(l) so, the $C_{ij,m}$ can be written as the A m and A' which is quite convenient ok.

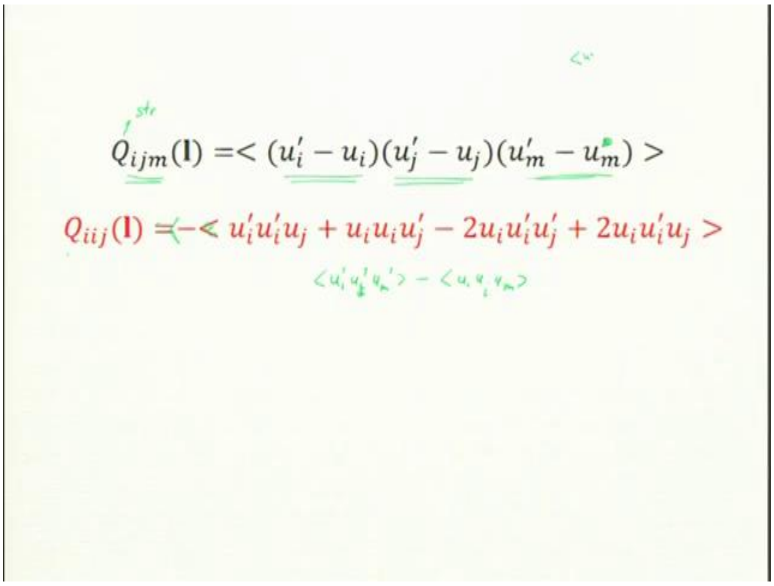


So, this is our third rank correlation (u, u, u). Now, I define third order structure function.



Now, what is it? Is a difference in the velocities? Now, please remember I have this two position **r** and (**r** + **l**).

So, I have differential in velocities. So, this is along i component, along j component, along m component I made one mistake there is no prime here fine. So, is the difference of the velocity, but I am taking the components along i. So, i is 1 means x component, 2 is y component. So, it is again three indices object, but now it is there is no comma there is a difference is same for all of them there is no comma and I am my notation is the Q and that was C; Q is structure function. Now, I can expand this in terms of correlation function. So, what will I get? I get prime, prime, prime. So, I think I am not going to write.



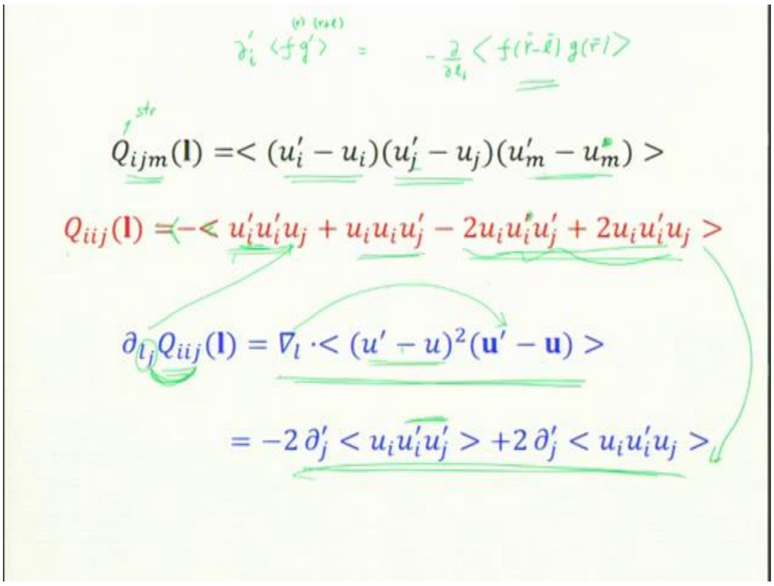
So, this is how the four terms are. So, how do I get this term two primes and third is?

Not prime here or two here and one prime ok. So, I made one mistake this correlation should be outside this one should be outside. So, we can get this stuff. So, you see 2 primes then third are un-prime no. So, I made one more change j equal to i, I make j equal to i if i. So, I make j equal to i and expand. So, it is iij. So, it will have all the combination three prime two primes and one un-prime, all no prime. So, this is how we get now this you can see it yourself I will not go deep with the details; except that two terms all the three prime terms is not there right.

So, I do not have the term of the form (u_i’ u_j’ u_i ‘), where i equal to j and minus u_i, u_j u_m. Now, since i equal to j, ii. These are not present in this. Why is it not present?

Is 0 because of homogeneity because if I do the triple product and that should be same with two different positions like you know average. But what is the average of u? u_i average.

What is the average u_i? 0 is a random with no mean flow average velocity 0 at the position and this is this should not be 0 is 0 for Gaussian probability, but need this will not be 0 triple correlation, but they cancel because of homogeneity ok. Now, this is what we are left with this average ok.



Now, we will do one more step, I take the derivative of this with respect to l_j. So, what do I do when I take derivative what happens with this term? This term goes to 0 because I take the incompressible condition and I forgot to tell you that this also will be 0. There is a proof and the proof is kind of nice proof. So, I have f and g, g prime. I take this with relate to prime.

Let us call it ∂_j derivative ok; f g' this one. It is a function of the function of l. So, I take on this now you can write this since it is homogeneous I can write this is f(r), f(r + l) I do not know green is not visible properly or.

It is ok? Because there are too many red I have written. So, I would not do. So, this can be written as (**r** – **l**) into g f(r) I shift left. So, this is r I just shift left by **l** and now I am taking derivative with related to l. So, this is (**r** – **l**). So, I cannot take derivative with g now, I take related to the f. So, it will be minus sign with l. So, minus sign comes because it is shifted to the left. So, if I take the derivative the first object and put a minus sign. Now, here I want to take with l_j.

So, that means, I want to take derivative on this one right there is j that one. So, this is at I make this minus sign. So, the minus sign comes here this minus, minus becomes plus and this goes to 0. So, so this is properties also used quite often. You want to shift the point then you are to use the minus sign and finish it off. Now, we left basically with these two terms and these two terms equal to that ok. Now, this proof I will again well. No I will.

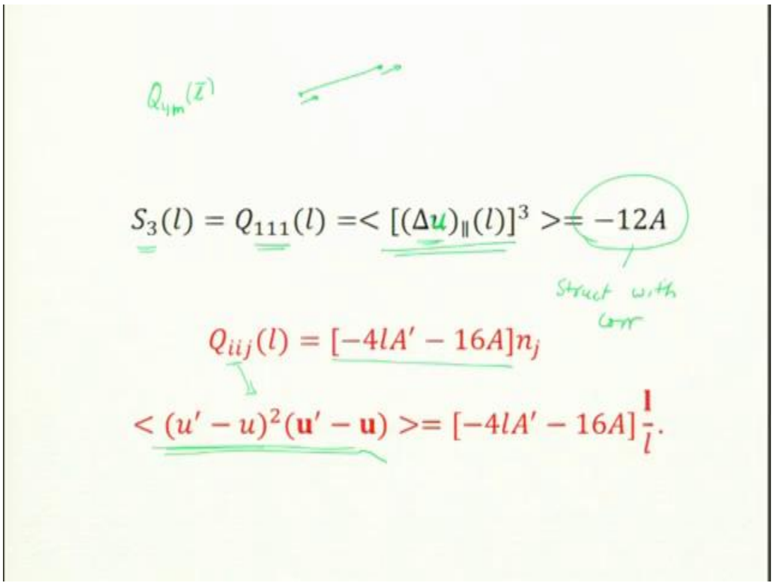
This so, these two are gone because of the incompressible condition you are left only with this and this is what is written here, is it ok? So, the derivative has come here I put a prime put a prime put a prime this is plus prime, prime and this should be this is this prime is acting on these objects is acting on these objects by the way please, note that this j prime is this prime was not there it will be 0.

If I take the derivative if this prime was not there, it would have been 0; this prime is there which is making it nonzero this part. There is some subtleties which you will do it yourself then you will find that Kolmogorov was indeed a very clever and he could do all this stuff.

Now, so, what is definition of Q_{ii}? Q_{ii} is j equal to i here j equal to i. So, that makes it square know? I am summing over i. So, this becomes u_i' u_i^2 that is $(u'-u)(u'- u)^2$; repeated indices are to be summed. So, this is a ∂u_i delta u i sum. So, this is coming from here and this is u' m u' m but I am now taking the derivative with related to same object; that means, it is a divergence.

So, that is why I put $\nabla_l(u'- u)$. So, this divergence acting on this, but it is acting on because this is a prime, here it acts on all three. So, this l is acting on all three not only one but I am just writing a shorthand, it is a nice vector notation. So, because of this was I am taking the derivative of u_j ∂j this one so, I write like that ok. So, this is shorthand notation which is convenient notation ok.

So, what I described very quickly third order correlation function and third order structure function and this object will appear in now Kolmogorov four-fifth law with this you have to remember this one.



Now, I need one more step which I will quickly go through S3. So, we are all describing only S3 but third order structure function, but this is very specific. So, this was that was Q notation I had put $Q_{ij,m}$ (l), but now I am I am defining S3. So, then sorry I must say is a new function I am defining S3 which is i equals to j equal to m equal to 1. So, do this one. So, 1 means along the **l** direction. So, I am choosing the same notation x axis is along.

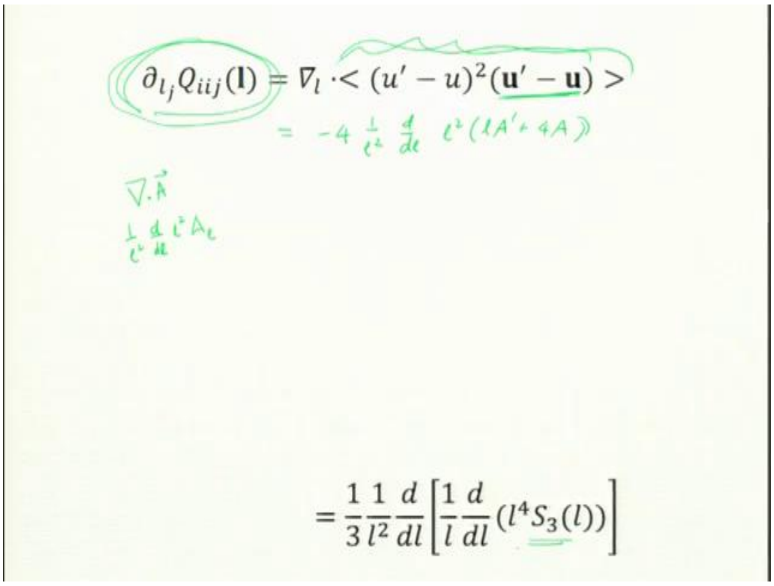
Student: **l**.

l direction. So, this will be Δ **u** I delta I should really put a vector **u** Δ **u** component along parallel direction is that fine. So, I have these two positions. So, velocity component along that duration is difference Q bit. So, this is what remember I started my first slide Kolmogorov basically is relating this object with the dissipation rate. So, we can define using the third order structure function but you see it is very specific my components are 111.

Now, it is connected with A, A is remember third order structure function; third order correlation function at ADF. Now, this I will not prove it but this is equal to minus 12A because see the connection. Structure function is connected with the correlation function and correlation function is function of A. So, that is how it comes is minus 12A. These are connection of structure with correlation ok. This I will not be able to do it I mean this too much is required here.

Now, Q$_{iij}$ is again related with A; an A prime it is like this. So, you have to plug it in because and replace d and f by the formula and you will get this ok. Now, I am just giving you the steps you have to work it out yourself. So, this is one is basically this object know Q$_{iij}$ is basically is just that I just showed you. So this is square coming from ii in this.

Now, remember S3 is well not S3 then in the earlier slide there is a divergence of this of this object, yes. So, you have just seen the divergence of this. So, if it takes the divergence of that which is very useful quantity which I am going to come to it with later.



So, I take the divergence of Q$_{iij}$, now I think I wanted to do it, but I will skip the algebra if you do the algebra dot, dot, dot, because this is only bit of algebra and you will get this one. So, this one is related with S3 S not I mean look this is function of is quite I can just tell you $-4\frac{1}{l^2}\frac{d}{dl}l^2(lA' + 4A)$ ok.

So, this is coming from the previous slide you know I am guess feel I am sure you people are tired and you can see that this object is$(-4lA' - 16A)\frac{1}{l}$. So, I have in the earlier slide I have showed.

So, this is these I computed from this is third order structure function ok. Take the divergence of it. So, divergence will be in spherical polar coordinate. Now, this I want to see emphasize is spherical symmetry. So, what is divergences spherical coordinate, divergence of vector A?

It is $(1/l^2)$ (d/dl) l^2 (A'l+4A), Al component. It is only Al component ok. So, that is why this is a along l direction yes or no? u', u is along l direction, isotropic ok. So, this is what has you we get I think I am losing I mean everybody is losing interest I think. Anyway let us finish it off. So, this is a connection of this object which I need in my derivation to this structure third order structure function ok.

So, I stop with this and now we quickly go to the last lab ok.

Thank you.