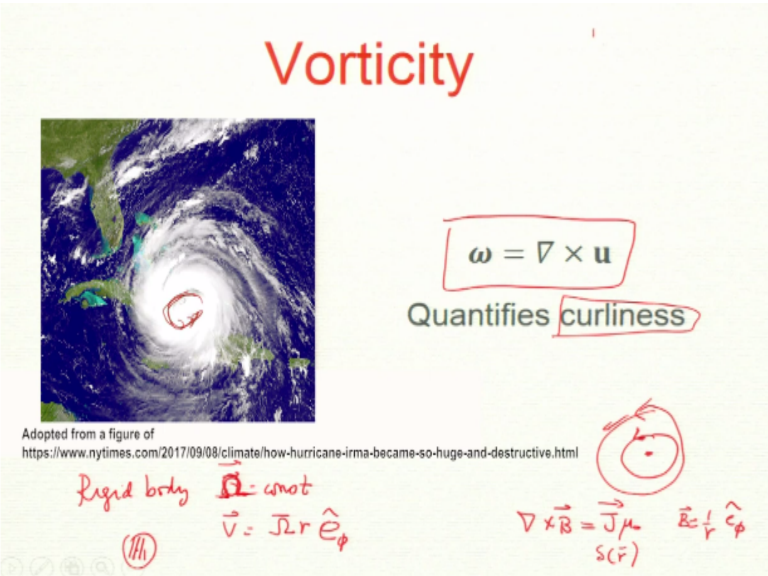
Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology – Kanpur

Lecture - 3
Basic Hydrodynamics: Vorticity

Okay, so now we go to the second part of start hydrodynamic description, so vorticity okay. So I am going to describe what is vorticity and evolution equation for vorticity, okay, and why it is important.

(Refer Slide Time: 00:30)



So this is the example of this big hurricane Irma which came last year I think okay in US. So this is the vortex, hurricane, big hurricane and it is going, you know it is going like this. In northern hemisphere, it will be anti-clockwise, okay, swirling in. Now, of course, we can work with velocity field, but it is also good idea to construct some more variables which are useful, I mean which are useful variables like energy. So given velocity field, I can also construct energy u squared, no.

Another quantity is vorticity. So vorticity just curl of **u** and curl if is nonzero then it gives you some kind of curliness, no, like magnetic field, now curl of **B** is current. So the curl is useful quantity which gives you, is there current then I can say I can get a magnetic field or divergence. So these are very useful construct for vector quantity. So, curl of **u** you see that is a very useful quantity is a vorticity and typically it is nonzero. We also have a curl-free situation, but typically in turbulence we will find the curl is nonzero okay.

So this is what I will describe what is the equation for $\boldsymbol{\omega}$. So this in some sense, this is what vector calculus people say that it gives you curliness. So this is curly, no, this is going around curly is curliness. Of course, we need to compute. So this curliness of velocity will be different than curliness of magnetic field, so you have to compute, then only we will know, okay, I want to make a caution. So if I have some just velocity going like this, does it mean that it has nonzero curl.

You know example where is going down like this, but it has zero curl except in the center. Now you know the example no, everybody knows example. What is the example where things are going around like this, what kind of vector field gives you zero curl everywhere except in the center. So curl of B, now just go back to your electrodynamics, curl of **B** is **J**, I am not putting **u** not **J** okay, let us put **u** not **J**. so if there is a current source here at the center only, line current, then **B** curl **B** will be zero everywhere except in the centre.

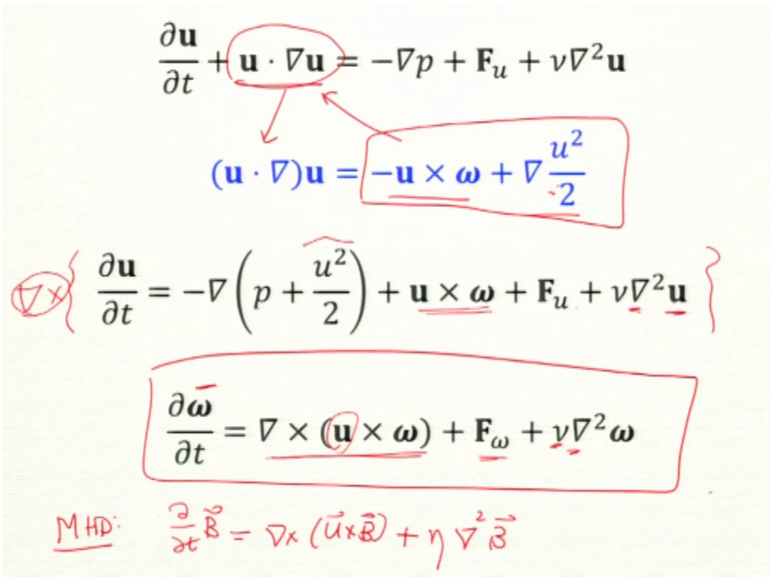
The source is in the centre, no, that is the delta function. For that particular case, delta of **r** is the centre. Now so what is the v? Now this has been solved no, you would have solved it. **B** is the tangential and it goes $1/r$, so if my **B** field, I am writing it here is $\frac{1}{r}\hat{e}_\phi$, so you can compute curl for this vector and the curl is 0 except in the centre where it is a delta function. So something curling around does not mean it has nonzero curl okay so, but I can change the function to let us say r, instead of $1/r$, r, then it is a nonzero curl everywhere okay.

So, I just caution if something curling around it is a good idea to check curl, maybe nonzero most of the time.

"Professor – student conversation starts" Doesn't rigid body rotation come under same effect? Rigid body rotation has, okay what is the rigid body rotation, what is the $\boldsymbol{\omega}$ for rigid body rotation. Rankine Vortex? no it is a constant $\boldsymbol{\omega}$ for all r okay. So it is this will be constant $\hat{e}_\phi$ right. Velocity field will be no no r, velocity field will be r times. So let me erase this. I should put this $1/r$. Rigid body rotation what is the, let us put it here, rigid body rotation. $\boldsymbol{\omega}$ is constant.

That $\boldsymbol{\omega}$ is angular speed, it is not curly speed, this is not, no no no so let us put Ω, capital Omega. So velocity field at any position will be Ω times r, Ωr and it is $\hat{e}_\phi$. I am assuming that there is only one axis to rotate. This one will have nonzero curl okay. So this is just the kinematics. Ya, this omega is not same as curl omega, so what Suprateek said is right. This omega is the rotation of the rigid body, so you can take disk, it is rotating okay okay. **"Professor –student conversation ends."**

(Refer Slide Time: 05:39)



So anyway, these are useful way to look at the vector fields and connections like this, vorticity is I tell you very strong connection with electrodynamics and we can do lot of interesting analogy, I will mention it in couple of minutes. So we start again with whatever we just leant, Navier-Stokes equation okay. So the same equation we just did, I drop primes and I don't want to keep primes. So this is one identity, this is a useful identify which we can, it is there in my notes, which I will give you.

So this nonlinear term can be written as in terms of what is $\boldsymbol{\omega}$ and this one in this okay. So if you substitute, what will I get? I just substitute replace this term by these, so it is quite easy to see. I am just going to click the next slide. So $\mathbf{u}\times\boldsymbol{\omega}$ in the left, $-\mathbf{u}\times\boldsymbol{\omega}$ will go to the right it comes to the plus sign, correct, agree, this was negative in the left so it comes, now this gradient of $\frac{u^2}{2}$ which is the plus sign, so it goes to right, so anyway this is the straightforward okay. So this term is this one and $\boldsymbol{u}\times\boldsymbol{\omega}$ is this.

So this straight from Navier-Stokes equation. I can curl of it, I take curl of this equation, I want vorticity no, so curl of this equation. So $\partial/\partial t$ and curl commute, commute means I can change the order, so this becomes $\partial\boldsymbol{\omega}/\partial t$, and what about curl of a gradient? 0. So curl of a gradient is 0. So pressure is gone, thus a huge advantage in vorticity formulism, pressure disappears. Now, this other term will be curl of $\boldsymbol{u} \times \boldsymbol{\omega}$ and then we get curl of, so in fact I have this equation in the next line, this here, so let us erase this part.

So pressure disappears and we get pressure and u^2 both disappear, so we get this, this term, and this is curl of $\mathbf{F}_u$. So this object, this is curl of $\mathbf{F}_u$ vector okay. Now by the way this notation bold means vector. So when I say bold, this means vector, I am not putting arrow, so these are standard notation okay. Now curl and Laplacian also commute, that means this will become curl of $\mathbf{u}$ is $\boldsymbol{\omega}$. So this is equation for vorticity and this we will use it in future okay. So we can also time step it, but is also tricky so you can't simply time step $\boldsymbol{\omega}$ because you need $\mathbf{u}$.

So if you are, well I will not discuss it here, but it is found that $\boldsymbol{\omega}$ replaces $\mathbf{u}$ completely and you can just work with $\boldsymbol{\omega}$, you need to compute $\mathbf{u}$, in fact that will require again Poisson equation okay. So, I will not delve into it for this equation which is for $\boldsymbol{\omega}$. Now, some people who are who know MHD, magnetohydrodynamics, I just want to make a remark that omega has very similar structure as magnetic field okay. So in fact, it has exactly same structure. So *d/dt* of $\mathbf{B}$, MHD okay, so let met jus write magnetohydrodynamics which we will do later.

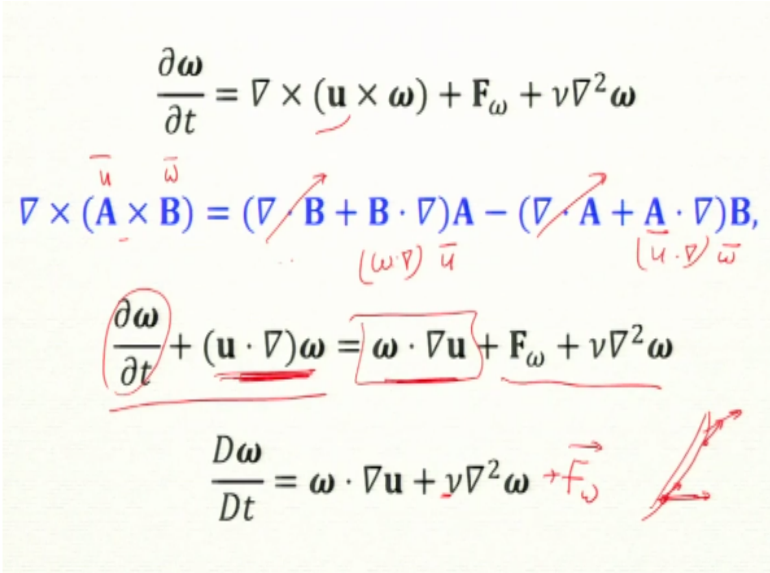
It is written as curl of $\mathbf{u} \times \mathbf{B}$, I will put force 0 okay and $\nu\nabla^2\mathbf{B}$, this equation for magnetic field under MHD approximation. So $\boldsymbol{\omega}$ and $\mathbf{B}$ you can see the analogy, except the analogy is not one-to-one analogy, I also want to caution, in fact Bachelor who is a very big shot, he makes this analogy and he made some deductions and they are incorrect okay, so this is not fully correct, there is a reason why it might not exactly one to one. So $\boldsymbol{\omega}$ is curl of $\mathbf{u}$, but $\mathbf{B}$ is not, $\mathbf{B}$ is independent variable, but $\boldsymbol{\omega}$ is a dependent variable okay.

So, it is useful analogy and I will show you that $\mathbf{B}$ and $\boldsymbol{\omega}$ have very similar dynamics as well, but you can't simply derive everything from equation for $\boldsymbol{\omega}$. So we will discuss that in this course.

"Professor – student conversation starts" Cannot the vector potential of $\mathbf{B}$ be interesting then. Yes, yes, we will do that later, that will be also tricky again, but $\boldsymbol{\omega}$ and $\mathbf{B}$ are related, I am going to show you in the next slide itself, some connection. Is that ν in the second equation be the same as kinematics response? Yes it is same because curl, this curl acts from this, MHD equation. No, no, no, that is diffusion, oh sorry, you are right, thank you, so this is η, magnetic diffusivity, but is a diffusion term, is similar form, but is not same ν, it is a different parameter.

"Professor – student conversation ends."

(Refer Slide Time: 11:09)



Next one, so we can also simplify this equation, another form of this equation. So I will use another identify. So there are lots of vector identities which I don't remember myself, but we will. So curl of $\mathbf{A} \times \mathbf{B}$. Now this is, this is interesting to remember. So what is $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$? So this I like $\mathbf{C} \times \mathbf{A} \times \mathbf{B}$, so this $\mathbf{A}(\mathbf{C} \cdot \mathbf{B})$, so $\mathbf{A}$ has come here and $\mathbf{C} \cdot \mathbf{B}$, so $\mathbf{C}$ is curl, so it is $\mathbf{C} \cdot \mathbf{B}$. You have to take divergence twice because divergence acts on variables okay.

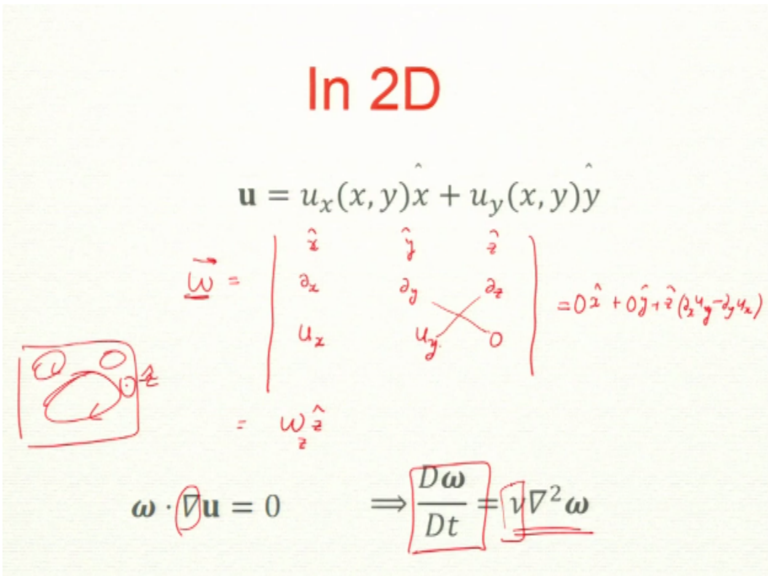
So this is how we can remember if you like, but that is aside, and this is our identity which you need. Now $\mathbf{u}$ is $\mathbf{A}$ and $\mathbf{B}$ is $\boldsymbol{\omega}$ right from here. I want to replace this one. So what is $\nabla \cdot \mathbf{u}$? 0, so this one is 0. $\nabla \cdot \mathbf{B}$, it is $\nabla \cdot \boldsymbol{\omega}$, 0. Divergence of a curl is 0, this is 0, so we get 2 terms. So $\mathbf{u} \cdot \nabla\boldsymbol{\omega}$ and this is $\mathbf{B}$, $\mathbf{B}$ is $\boldsymbol{\omega}$ sorry, I erase this one. $\boldsymbol{\omega} \cdot \nabla\mathbf{u}$ and this is $\mathbf{u} \cdot \nabla\boldsymbol{\omega}$. So shall I replace these 2 terms here, so what we get is this.

So this one is negative, it comes to the left, is positive this one and so these 2 terms are sitting here, this one and that one, and this is usual as before okay. So this is another form for $\boldsymbol{\omega}$ dynamics. So this is the total $\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla\boldsymbol{\omega}$ as I said is total derivative or convective derivative, material derivative. So the $\frac{D\omega}{Dt}$ is equal to this. So this again is MHD equation, no, same equation as B except for course this Roshan pointed out, I drop $\mathbf{F}_\omega$, if you like $\mathbf{F}_\omega$ you can keep here okay. So this is 3D.

I want to make this remark okay. So this term is advecting the vorticity, no, it is advecting. So in fact is same as advecting temperature, it carries the temperature or also vorticity is the vector, so advect the vorticity, but this term, what does this term mean? This term is stretching the vorticity okay, which will show is stretching. So this is interpretation. You take this vorticity field, the velocity field will stretch this, so vorticity can decrease or increase because of stretching.

Exactly same analogy which is there for MHD, magnetic field term, magnetic field will change because of velocity field will, so if the velocity field at two ends are different and these are tube, vorticity tube, you know, so this velocity here and velocity here, so it will stretch okay. So this is the interpretation of term, this term is called stretching of vorticity and in 3D of course this is nonzero and 2D I will show you that is zero okay. So this is like one-to-one analogy of vorticity and magnetic field.

(Refer Slide Time: 14:52)



Okay so in 2D, so let us also talk about 2D which is also very useful. So 2D means there is no z component. So my notation will be the motion is only in xy. So you can just think of a full motion on the surface of this floor surface. Real example is soap film. So people do experiment. In fact in IIT, there is a nice experiment. So there, Sanjay he is in aeronautical department, he does experiments on soap films you know. So this experiment is 2D. So there is no z component, it is basically 2D.

Now what is vorticity of this. I can compute vorticity easily, so omega. Let us do it because this is important, so I want to do it once. So x, y, z, dx, dy, dz, u_x, u_y, and 0. So what happened to x component. So by the way u_x and u_y only is function of x and y, not function of z. So, what happens to x component. This minus this, so $\partial_z u_y = 0$, no, because u_y is not a function of z. So, x component is 0. What about y component, $\partial_z u_x - \partial_x 0$? 0 of y component.

What about z component, it is $\partial_x u_y - \partial_y u_x$, it is nonzero, but $\boldsymbol{\omega}$ has component only along z direction. So this is what I got in my next equation. So this is $\boldsymbol{\omega}$, I can call it scalar, ya so this is somewhat, so this is the vector and this is scalar and I may call it ω_z if you like. It is only one component, so my velocity fields are in this plane you know, things may be going like this and like this or it could be you know this is various motion, but my vorticity field is only along $\hat{z}$ vertical direction.

So what happens to $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ term? 0, because $\boldsymbol{\omega}$ is along z and ∇ is only active along x and y, so is naturally 0. So in the previous slide, so there are 2 terms, one is $\mathbf{u} \cdot \nabla \boldsymbol{\omega}$, other is $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$, $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ term is 0. So what we get is this, I put force to be zero. So force is not, so $\frac{d\omega}{dt}$ is here, it is which will be dissipated by viscous term. It has no source, stretching is 0 for 2D.

So, in some cases you see this vortex tubes are infinite, they do not change with z and there is no velocity in z direction, so nothing to stretch it, vorticity is not along, so these horizontal guys can't do anything, it can only advect this pen, you know that is vorticity. So vorticity does not, so material derivative of $\boldsymbol{\omega}$ is 0, viscosity is 0. So, I must say what fluid can do, it can only kill the vorticity by dissipation. It cannot increase it, it cannot change it other than viscosity. So this is what happens with vorticity, so this is what I wanted to communicate for vorticity. Thank you.