Physics of Turbulence
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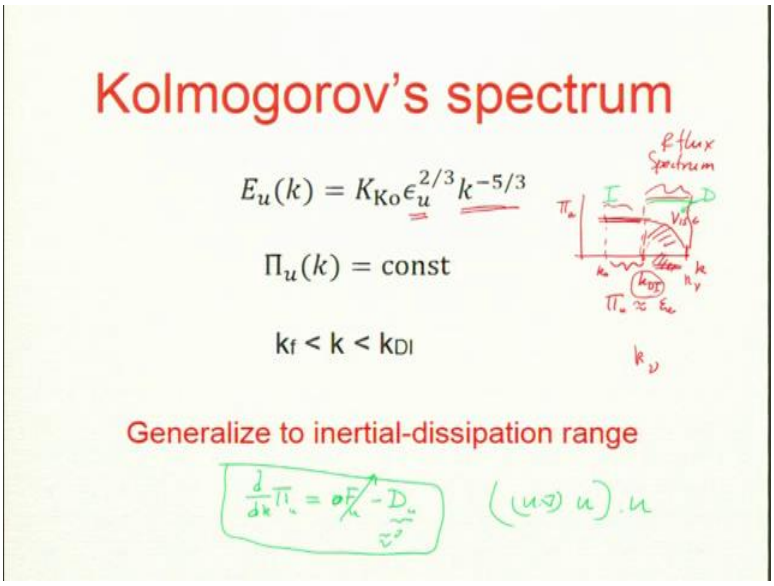
Lecture - 29
Kolmogorov's Theory
Spectrum & Flux in inertial-dissipation range

So, in the last class, we discussed in 3D hydrodynamic turbulence, what is the spectrum and flux? right, but that was for the inertial range. So, when the non-linear is very strong compared to viscous term. Viscous term is finite, but non zero then and Reynolds number will be finite. So, I have defined Reynolds number you know.

So, Reynolds number is large then we get a turbulent cascade. You know scale by scale transfer and that cascade leads to $k^{-5/3}$ spectrum; the question is; so this is what I summarize in my next slide.



So, the spectrum is $k^{-5/3}$ and the flux is constant. If you plot this function of wave number, y axis is flux. Flux is constant and flux decrease well I should not say this is a topic of today's lecture. So, this these where flux is constant and this flux equals the dissipation range that also I proved right.

If you see from that equation for the flux; you take this equation k_o as there will be a number k_o and compute flux energy in given by the external force matching the dissipation, you can match the two under steady state. The flux where you case approximately there is slight change is ε_u. The ε_u is slightly bigger than π_u right as I am discussed in the class or slightly different that depend but they are approximately equal, now this was the inertial range.

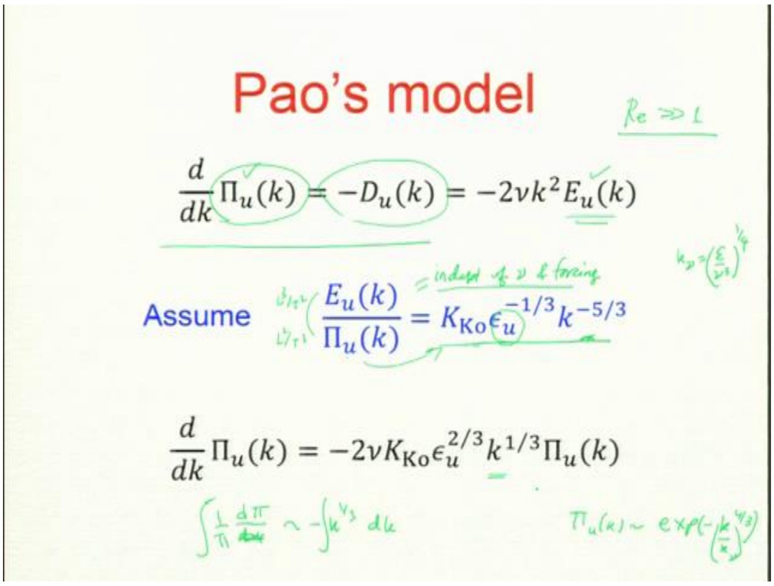
The viscous term is strong here. In this region the spectrum is $k^{-5/3}$ and flux is constant. Now question is "what is the flux or spectrum in this band"? We have in my book I make the notation k_{di}. The k_{di} is the wave number where dissipation is starting and the thing which I said $k^2 \nu$ or sometime I was writing as k_d.

So, k_{di} is where $k^{-5/3}$ spectrum is not valid anymore and is getting into some different form. In fact, we want to study what is the spectrum and flux in this region. Is that clear? This is inertial range and this is a dissipation range. This one, so that is a topic. Now, it is true in standard text books you will find that they do not talk about flux or spectrum in that range is argued, but not so much in standard books.

In fact, we say the flux is 0 if your non-linearity is weak. It turns out as long as there is non-linearity. There is energy transfer because it is a non-linearity. There is energy equation will nonlinear term, right. So, this is a non-linearity action k p, k=p+ q; so flux will be non zero. So, even for laminar flows where energy transfer is not equal to 0, but order 1; there to there is a flux is a misunderstanding that flux is 0 for laminar flows, as long as there is non-linearity there is a flux.

So, flux does not require the flow to be turbulent nah; flux is constant for turbulent flow turbulent hydrodynamic flows that I proved it know. So, please remember the how do you show that flux is constant? Is $d\pi/dk$ is 0 because this side was force minus dissipation [noise.] Now, we say that in inertial force is 0 and dissipation is weak. So, this was approximately 0, but we say well we ignore it. So, flux is constant π_u but in real flows of dissipation is not 0.

A laminar flow, so the flux will change ok. So, that is a; so these equation is for me is very-very powerful equation and that tells you about flux ok.



So, let us see; so we will work with first when Reynolds number is much larger than 1. So, this theory is for larger Reynolds number and I will discuss only Pao's model. There is one more, but I will not discuss all models ok. So, this course; so we discussed selected model which is kind of fits with the data better. So, this model be Pao which is very old model 1960s. So, this is for both inertial and dissipation this formula will work for inertial as well as dissipation approximately ok. It is not fully true; it only approximately valid, we know there are there are deviations from Pao's model in simulations.

So, we start with the same equation know. So, inertial range my force is 0, so $d\pi_u$ with a flux will change with k and how well flux will change because of dissipation right everybody is happy with this. So, this is one of the most important equation ok. $\frac{d\pi}{dk} = -2\nu k^2 E_u(k)$. Now the, so this is what we want to use; unfortunately there are two unknowns right π_u is also unknown and u is u is unknown and I have only one equation; so, you can solve this.

So, Pao suggested the less maker model where we assume that E_u by π_u this one; this ratio. So, I need one more equation according to him let us try to make $E_u(k)$ by π_u (k) independent of viscosity ν and forcing. So, when you make an assumption; so where let us assume that this is only depends on the flux and wave number flux is π , ε_u flux ε_u are same ok.

So, this is what let us assume it and if there is a case; if you assume I make this assumption then divisional analysis will tell you that it must be this ok. So, E_u by π_u has some dimension this is L cube by T squared and this is L squared by T cube. So, if you match the right hand side; you will get this in fact, I am just following the Kolmogorov's theory.

So, Kolmogorov's theory u(k) Kolmogorov $\varepsilon_u^{2/3}$ $k^{-3/5}$ that, divided that by ε_u you get $\varepsilon_u^{-1/3}$So, I am also putting the constant so in fact, powerful the constant this Pao's model not my model ok. So, let us; so we have now two equations and two unknowns now it is easy to solve right; it turns out its very trivial after this. So, once I substitute E_u from this equation. So, I just replace E_u by π_u multiplied by this; so, you will get this.

So, this is an ODE function half π_u is the unknown variable ok. So, here I had put ε_u by intention. So, the right hand side is only function of k in this part, k and $\varepsilon_{u.}$ The ε_u is the constant ok. So, this is just; so what is this form of. So, forget about all the constants; how will π_u look like its function of k?

So, what is the solution is a linear first order equation.

So, dπ/dk, please tell me what is the function.

What is the form of Pao's function?

Student: k to the power 4 by 3 and k to the power 4 by 3 and.

Student: Exponential.

exponential ok. So, dπ/π is this integral. So, I will give you log in the left hand side and in the side hand side it is $k^{4/3}$, so exponential (- $k^{4/3}$).

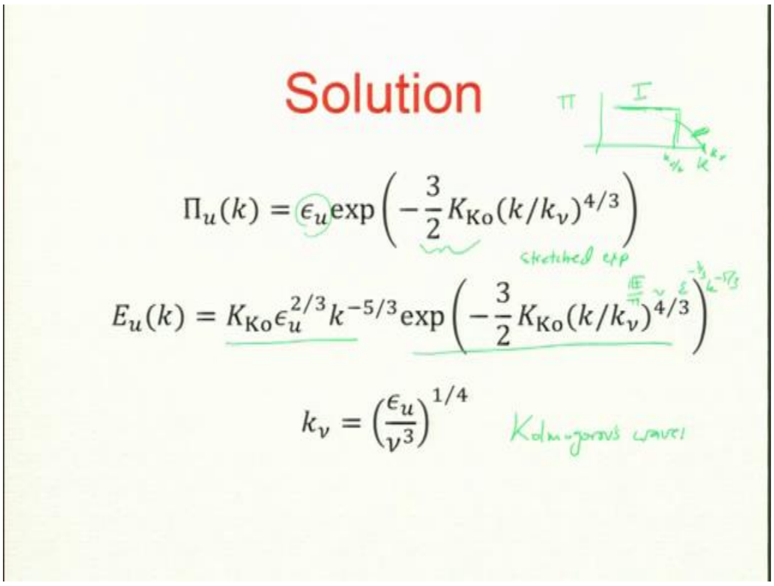
Student: Yeah.

So, the solution is you can immediately you can see apart from constant is exponential (-$k^{4/3}$). Now remaining part is just algebra you know, so we need to. So, please remember it cannot be $k^{4/3}$. It has to divide by l so k has dimension know. So, I cannot say sin of 1 meter. It does not make sense have you heard of somebody is saying sin of 1 meter.

Sin of 1, you can say. So, sin the argument of sin must be non dimensional number or exponential of 1; 1 second is not ever meaningful. So, you have to divide these by some quantity equity dimension with dimension of k. So, it turns out we get quantity k nu same as what we defined earlier. So, it is a Kolmogorov number k nu comes which is epsilon by nu q to the power 1 quarter ok.

So, this algebra I will not do it here; you can easily do, but the form is exponential; it is not k squared.

It is k. It is slightly I mean it comes from because the 5/3rd nah. So, origin of 4/3rd is coming from this 5/3rd will multiply by k^2. So, here remaining with $k^{1/3}$ integrate that you get 4/3rd.



So the solution, I have just say you, this is the full solution. So, $(k/k_\nu)^{4/3}$ and in front there is a 3/2 k Kolmogorov which is the order 1 and π_u has dimension ε_u. So, ε_u is coming here which is nice ok. So, what is it mean? If I plot this function of k is π_u; so it is constant these also called stretched exponential know is delayed exponential.

So, exponential drops quickly you know exponential would have dropped like this, but because of this 4/3, it drops bit slowly whereas, flux is always decreasing with k. This is telling you that flux is not constant flux. It is always decreasing, but decrease rapidly after some wave number.

So, k k_ν is somewhere here, but may be $k_\nu/2$ is somewhere k_ν /2. So, so in fact, would be interesting to see make it quantitative at what value; it has pi has become half [FL]. In fact, it is a good number to investigate I am not done it, but is good thing to play around ok; is that clear? So, flux always decreases because of dissipation.

So, even in the inertial range dissipation is important in real flows. So, flux is decreasing even in the inertial range, but it is less decrease. But then it rapidly decreases in the dissipation range and this exponential part will very important in the dissipation range. So, flux is non zero in the dissipation range as well ok.

Now, Kolmogorov assumes viscosity into 0 limit and that is what he is basically says well this k_ν is at infinity. So, he is a mathematician. He is ignoring it, so well, let us push it into infinity and so Kolmogorov is essentially gets a constant if k_ν is infinity; then this exponential part is 1. So, exponential power infinity is exponential part 0 sorry.

Exponential part is 0 is equal to 1 ok. So, that is why he gets, now what about the spectrum? Spectrum can also be solved. So, power told you what does power tell you? E by π is $\varepsilon^{1/3}$

Student: $k^{-5/3}$.

$k^{-5/3}$ ok, so I know the π now. So, I can just take it to the right hand side. So, you will you will basically at Kolmogorov multiplied by stretched exponential. So, the formula for s spectrum is this Kolmogorov part and I multiplied by this exponential part, stretched exponential ok; so this is the prediction of Pao. Now, k_ν I told you is; is this which is Kolmogorov number. So, these it has a name; so please remember Kolmogorov wave number.

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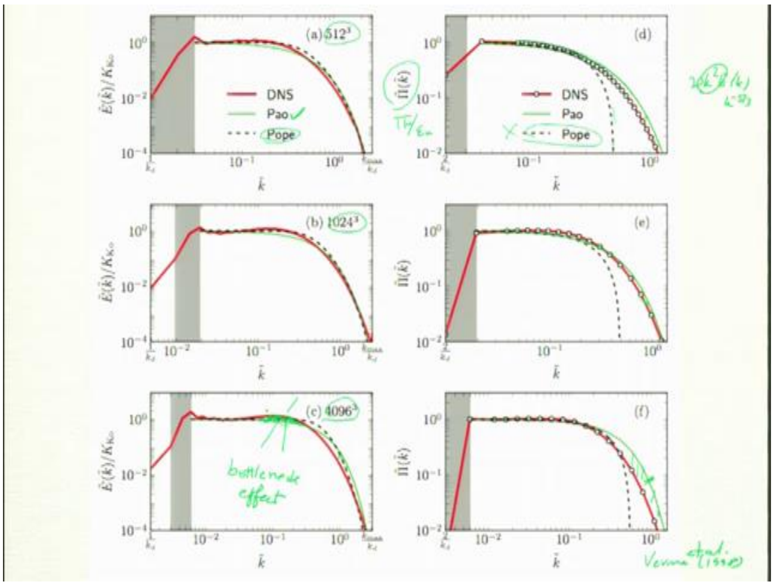


Now, let us see whether this fits with the stimulation.

High resolution simulation ok, so there are many-many plots. So let us try to understand. So, this is done by us; so this was this paper which is to be published Verma et al. So, there are many people; so this has for different resolution. So, 512 cube, 1000 cube and 4000 cube. So, let us look at and the 3 curves. So, rate is the DNS from simulation DNS means Direct Numerical Simulation ok. We do not make any assumption on viscosity or some model, periodic box $(2\pi)^3$ like what we describe; so the rate is there simulation.

Now, Pao is the green curve and the left and the Pope model which I am not describing here is also a model which is a dashed curve; dashed back. Now this is a flux, so we make a tilde know; so tilde is normalization. So, this one is π divided by ε is pi tilde, ε_u I am not writing this one ok. So, it should be 1 in the inertial range. Then it should decrease. So, let us look at the flux first. So, flux is decreasing; the simulation is this line, but on that the data points are all circuits ok.

So, red line with circles are the simulation data, now green is over shooting, but somewhat it fitting quite with the data fight will cube, but this black line is not fitting. So, Pao is not doing good for flux a sorry; Pope is not doing good Pao's doing quite well for fight well cube; here also for 1000 cube it is good.

So, this drop of flux is captured by Pao, but it is not doing well for 1000 cube; you see it is slightly quite off factor 2 ok, but flux is decreasing. If you look at these things flux is decreasing; even in the inertial range ok. Because viscosity is taking the energy out becomes stronger and stronger for large k because viscous term is $\nu k^2 E_u(k)$.

So, this k squared term and 5/3rd is here. So, k squared is getting more and more important for larger k ok. So, this is what is happening here; so this is increasing. So, the flux is decreasing so that one thing I want to emphasize. What happens to the spectrum? Spectrum the green line is below the red line, but somewhat fine is you almost looking quite good; except here again the green is above, but here green is below the red, can you see? The green is below the red. There is a gap.

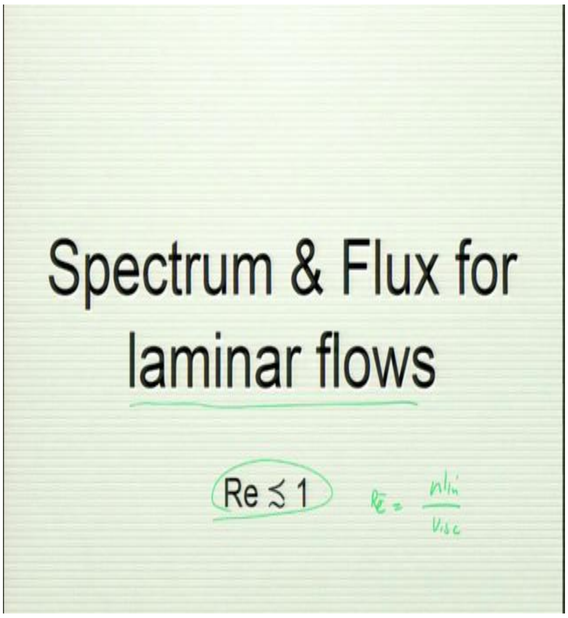
So, the red line is above green and this is called bottle neck effect. So, in crude sense, so well I this is still remain unsolved problem in some sense. People are not agreeing on one mechanism one universal one mechanism which everybody will say well this is the reason for bottle neck effect but is like you are running on a highway driving on a highway, but suddenly you have to go to city you know you take basically you take the detour rather you create off the highway.

So, they are normally congestion at the knee you know. So, the speed is coming down; so this is dissipation and as dissipation the speed of cascade is decreasing, so to increase the dissipation at that region. So, this is jam; the more cars at the turning point.

In some sense that is in picture I can explain why bottle neck is there because there is the mode of car which is travelling on the highway, as well as let us say you have to go to a smaller road, jam road or dissipative road; dissipative road the speed has decreased. So, basically there are more cars and that is what is kind of happening, but any 4 resolution I mean in its more arguments and so this is a bottle neck effect ok.

So, Pao's model is not fully correct; Pao's model assumes that it is a mean field theory. So, this is a constant flux and you just it is not taking care of subtleties of turbulence ok. So, we need to worry about this aspect, but is good know I mean to first order is a very good model and it seems to do well even for 2D, helical turbulence So, at least we have been quite successful in modeling lots of flows with Pao's ideas; so this is about Reynolds number much greater than 1.

So, I give only one model but this model also useful for the dissipation range which we need it for many situations like rotating turbulence, where this is a strong I mean energy is quite small in the in lot of range of wave numbers, this model is doing very well ok. So this is what we find in recent work.

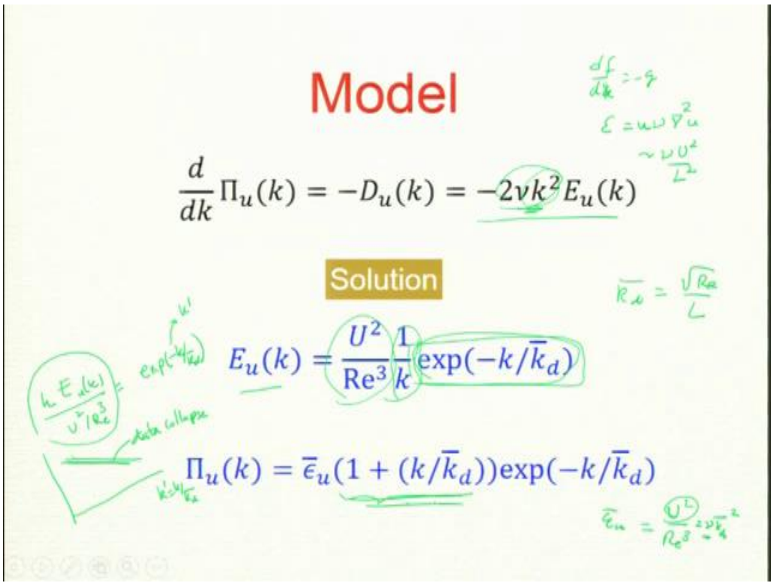


Now, let us look for the laminar flows. The laminar flow is what dissipative flows ok.

This name is different names are given by different people, but I just call it laminar; when Reynolds number is 1 or order 1, but it could be 10 as well. Turbulence flow required Reynolds number to be 10^4, 10000 or so or 10000 or bigger. But Reynolds number is when it is R 1, then non-linear term and viscous terms are same order you know, but the Reynolds number is ratio of nlin by viscous.

So, we say that viscous and non-linear terms are of the same order. So, flux constant is not true anymore, but what about the, what happens to the flux? And you find in many papers this say flux is 0; you drop the, we assume the non-linear term is negligible including Cambridge paper. So, I can research; so do not take all the experts you know with lot of faith.

So, you should trust your own ideas. So, Reynolds number is order 1 that mean non-linearity is important is as good as viscous terms; you can cannot just keep viscous term and ignore non-linear term. If you keep the non-linear term then there is a there is a flux from the argument which I was giving in the in the; in the first slide, in fact, it is straight forward.



So, we take the same equation dπ/d k minus D_u (k) and this is it. Now, you may say why not try the Pao's model? You know, so I can use the same idea Pao. In fact, same formula for Pao and try to fit in the numerical data is that k_v will be small, but terms out I will not show you here that Pao's model does not work for laminar flows. We had simulation data and we try to put Pao model is simply work it the lines were very-very different. I am not showing you here, but you can look at the paper we just say about to come.

So, idea is to look for some other formula; now the Pao idea that $E_u(k)/\pi_u(k)$ or $\pi_u(k)$ by $E_u(k)$ is independent of ν and viscosity. So, it is not that does not work very well, but what I thought was let us try something which is obvious function. Now, if we have something like this one dπ/dk is u(k), let us say you ignore this part well you cannot I guess you will see this let us drop this term, then what you expect what which function will satisfy this? So, here dF/dx is minus g which function would satisfy this?

What function will satisfy this very easily?

Student: Sin.

Sin no sin gives you cos. So, here in the right direction what you should; exponential?

So, just put E power minus k for both. So, you take the derivative E to the power minus k. So, exponential k is a good function pure exponential; no stress.

So, I said well, try exponential, but this k^2 will cause some you know some [FL]. So, there is k^2 will make some trouble, but that is you just have to solve it properly; so they are polynomial. So, it turns out v model $E_u(k)$ is well this algebra, I will again live it this is again fitting with the model know a fitting with simulation; this was not I cannot derive this fully from the first principle; we do fit from the simulation. So, exponentially

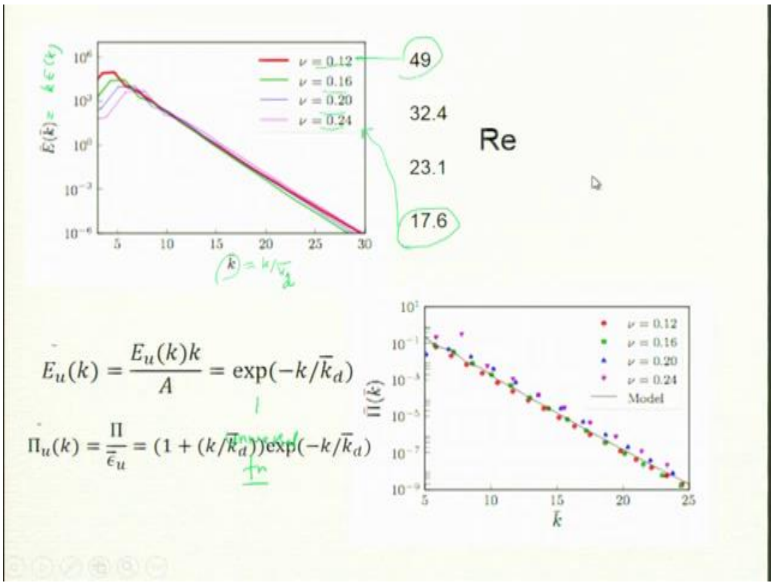
the leading factor and it turns out if you try to fit with the data which; which I have the data in the next slide, I think I have the data simulation.

In the simulation if I have divide by k it fits with the with the data ok. So, this is 1/k is additional thing we fit from the simulation ok. Now, I really welcome some of you to try out from the first principle try to derive Pao's model comes from the first principle; it does not require any one assumption.

But you can derive it after that, but this is one require simulation. In fact, this free factor is coming from simulation and k_d I did not try it in the slide is square root Reynolds number by l; this is again fit from the simulation. But the idea that you should be exponential is very intuitive idea, just guess what?

Now, because of k^2 this front part is polynomial 1 plus k. Now you can fit this and things will cancel nah a ε_u I did not try again in here, but is there in the notes $u^2/Re^{3,}$ 2 ν k_d^2. So, ε_u is for viscous term; it is for viscous term what is the dissipation by the way it is not E cube by l nah is; ν it is the viscous term nu Laplacian u u.

So, it is ν u^2/L^2. So, that is estimate for laminar flows. So, it is coming come from ν u^2, but there is some normalization which we, we get from simulation ok. So, this free factor is coming because of k^2 (1+k) and this model is also very good; when the flow is Reynolds number is out of 1, is excellent ok. So, there is an energy get different scales even from laminar flows.



So, spectrum; again I am not seen a paper published there is one paper came recently, one paper which came recently by Srinivasan's group we which has done the simulation other than us. So, this is normalized; so 1/k is here is of k $E_u(k)$ and this is a free factor know there is a u^2 by Re^2 and u^2 by Re^3. Now is there the free factor we also multiply the free factor; so that you get a universal function.

So, so this is important know. So, let me go back. So, we want this function to be compared if this is your conjecture then, what should you try to fit with various Reynolds number? So, E_u multiplied this is by k divided by u^2 by Re^3; what will that give you? Exponential minus k by k_d bar, so, x axis should be what? Should we put k or k what should I choose?

Student: k.

k/k_d bar. So, y axis will be just exponential of; so there is non-dimensional number k'.

So, plotting is also an art; you should really know how to plot. So, x axis should be k prime just k by k_d bar and y axis should be?

Student: E_u.

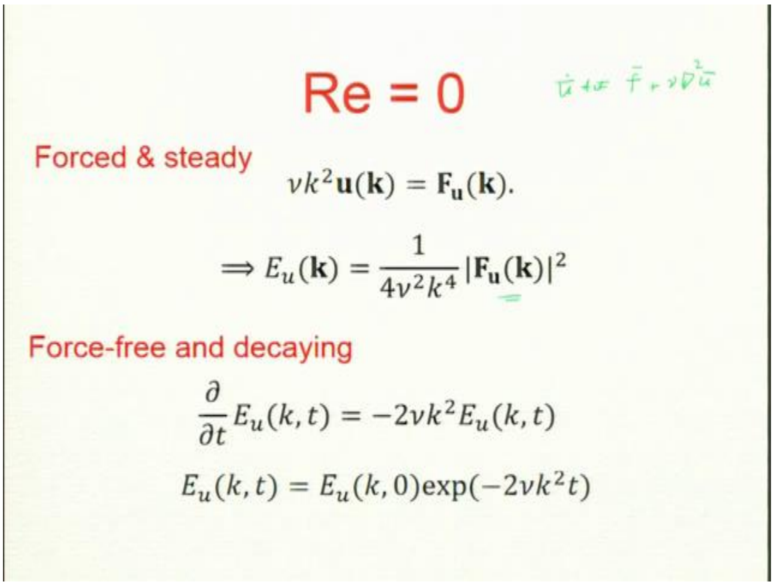
Not E u; y axis should plot this.

For different runs, so you have plot this then I will get one line for all of them same line. They all collapse this is called data collapse and we will collapse the exponential minus k; k prime minus k prime ok. So, this is what we did this from this is guess work. You know this is really requires doing back and forth guessing and same thing you do it for here itself exponential part divided by (1+k), so you have to just do it.

So, it is collapsing nicely you know; so x axis is k prime k bar which is k by k d bar. So, we put a k_d is the laminar; not k_d. So, k_d and k_d bar are different; k d is notation k d is same as k_v which is Kolmogorov number and this is flux where we divide by 1+k bar as well 1+ k'. So, you make the function which is universal; these are called universal function. Why universal? It is true for all data, true for all runs as long as Reynolds number is order 1.

So, our Reynolds number is here. So, viscosity is listed here and Reynolds number is 49 to 17.6. So, this corresponds to 17.6 and this corresponds to 49. So, Reynolds number is not too small nah; so our runs for not less than 1, so it fits with this. So, this requires some more work and this runs are very cheap we can do it on 64^3 and it is a 3 D runs and they are all steady state flows know. So, they were forcing; wait for steady state, look at the spectrum and that is what we get and this seems like a good model.

And laminar flows are also important though this force is on turbulence as a many flows which are laminar. So, like human body in this human body all the flows except near the heart; things are laminar, blood flow is laminar the fish wing in the ocean or river they are laminar. So, they are not very fast or also many hearing flows slow moving honey pipe and so on and so the earth ventral is slow; so there many many flows which are laminar ok.



So, that is what about Reynolds number is 0.

Reynolds number is 0 is trivial.

Student: Non-linear term.

Sorry.

Yeah the non-linear term is dropped. So, for steady state d/d t also will dropped. So, if Re is 0 then if there is no forcing when we will basically get. So, there is no well basically if steady state the forcing must match the viscous term. So $\frac{\partial u}{\partial t} + (u.\nabla)u = 0$; so f plus ν k grad square u. So, viscous term must match, because what is the steady solution. So, that is what I wrote; equity phase that is what we get. So, a spectrum will be 1/ k^4 forcing mod square just; just straight forward. What if I say that where there is no forcing, but it is decaying?

So, spectrum will be this one; here there is no flux. So, there is no dπ/d k ok. So, flux is 0 for Re is 0 right, this is non-linearity if non-linearity is 0, so there cannot be any triad no flux. So, spectrum will be exponential it is a diffusion equation.

So, diffusion equation is term is wave number energy will drop its Gaussian E to the power minus k square t is a diffusion equation. So, finally, everything will go to 0 but it is a diffusion equation ok. So, I will stop.