Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 28
Kolmogorov's Theory Insights from Theory; Numerical simulations

From Kolmogorov's theory you can derive lot of properties of turbulent flow. I will sketch what can you see, therefore, example this k_η , we can get from Kolmogorov theory or in somewhere relation with small size versus big size, smallest grid. I will derive some of it, and you can find lot of it in my in the notes I have given you.

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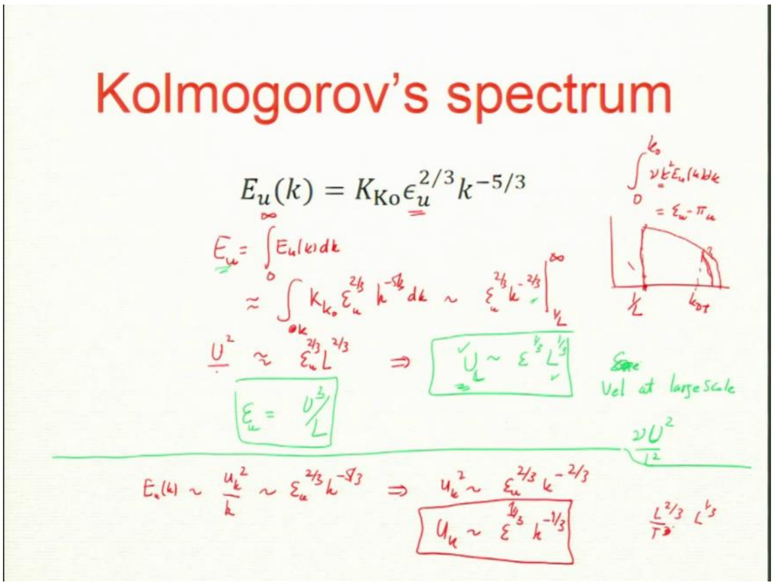

Now, so this is spectrum I derived you know. So, by the way Π_u and ϵ_u roughly equal. This slight difference and that difference comes from 0 to k_0 integral where the viscosities effects are small. That is why Kolmogorov's theories assume that your initial range is infinitely long.

Now, we can see something about what is the energy of the large scale flows. So, what is the energy of the full system? By the way this is let me also make this remark. So, my full spectrum, let me draw properly, a full spectrum will look something like this I am going to derive the right side, I will derive this region as well maybe in the next class. So, this region we can get under some more assumption, we can derive it. This region is also the theory but it depends on forcing, this region has no inertial theory forcing band, ok.

But total energy will be the integral over the full box, a full wavenumbers. So, total energy E, E_u is integral of $E_u(k)dk$, this is 0 to infinity but in physics we always make approximations. So, I say well I do not know what is this here, but I will say I am going to start not from here, but I start estimating energy from here to infinity. So, this will be approximately. Now, this side is some function know several that function energy is too tiny that I do not even worry about that part. So, I say I basically go up to here k_{Di} and integrate. So, this is Kolmogorov $\epsilon_u^{2/3}k^{-5/3}dk$.

So, I start from system size. So, this I am saying well, this is basically where it is like 1 by length, 1 by system size. So, Kolmogorov theory is applicable all the way up to system size. So, I am basically ignoring this part. I said this is my system wave number.

In turbulence flows, energy at large scale u is basically large scale you know this is this what is telling you, the energy is dominant by small k contribution. In this integral which is the most dominant part? Small scales are not dominant because they gives the 0, integral dominate from low end. So, these are called asymptotic methods. So, in integral which part is contributing most? And that is what I take into account.

So, here u is the full energy is dominant, dominantly coming from large scale or lower numbers. This is extremely useful result and this works for atmosphere for solar physics, you apply this some of these ideas in turbulent energy. So, these energy at large scale or velocity at large scale. So, if someone can estimate dissipation epsilon then we have U^3/L.

So, L is given to you as well, this sum this big thing is 100 kilo meter size and or maybe 1000 kilo meter size this eddies, and from that you can get how much dissipation is there we estimate it and you get how much should be moving or other way round. We can measure the velocity field from here by some measurements. So, U is known, L is known then you can compute ϵ. How do I invert it? By the way this is straightforward know. What is dissipation it? U^3/L.

So, given U and large scale velocity and large scale L, I can get ϵ. Now, this is in turbulent flow. Now, viscous flows I will not do it, but I will leave as exercise, it is $\nu U^2/L$, this so viscous flows, but not relevant for this discussion.

Now, this is velocity at large scales this one. What about velocity at intermediate scales or any scale L? Can we say something about that? In fact, Kolmogorov theory can immediately says something about it. So, let us estimate that here. $E_u(k)$ is basically energy at inside a shell of radius k. So, that should give you velocity field at a length scale $1/k$. So, this is now, let me make this another remark without proof, I can give my hand out. So, $E_u(k)$ is energy $u^2(k)$. So, $u(k)$ is $\epsilon^{1/3}k^{-1/3}$.

Now, we can do some more work, I can compute this k_η. There are several ways. So, we see that whatever energy is coming at large scale you know.

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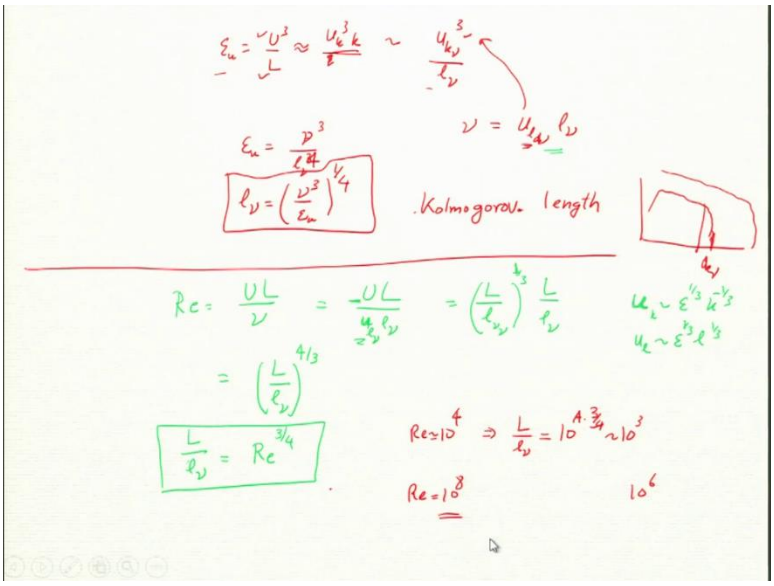

So, given viscosity and injection rate I can get l_ν. Now, is one funny thing in that this is funny formula, right because your dissipation length scale Kolmogorov depends on injection rate, is not mean free path length this tells you that. The mean free path length cannot depend on how I how much energy I am injecting. If I inject more energy, then I can push my l_ν smaller value?

The vortex at small scales will become, will be stronger when I push one energy. If I put in more energy than it is higher and it just go down furthermore. So, this is important I think, which is, this is kind of different then mean free path length, kinetic theory all that does not work here. Thermodynamics is not working here. So it can be thrown out, not required.

Now, one more formula we will derive is Reynolds number, is related with the large length scale and l_ν. So, that is the next formula. So, what is Reynolds number? Is $\frac{UL}{\nu}$ fine. For Kolmogorov's theory, what is the ratio of velocity field?

Now, turbulence typically. So, the smallest rate somebody need is something like few thousand. So, let us make for estimation, I will make $Re = 10^4$. For Re equal to 10 power 4 which is there in I mean like coffee if you just start this you can get up that order in somewhere.

If you want to simulate a turbulent flow of Reynolds number $Re = 10^4$, should be each direction must be $Re = 10^{10}$ minimum because I want to resolve the small scale, because simulation does not resolve the small scale or does not take into account the small scale is not a good simulation, not reliable simulation. So, what is the total number of mesh points? 1 billion. So, just Reynolds number. So, that is why it is a very difficult problem to simulate.

Now, if I want to simulate $Re = 10^8$, then what happens? It is just going to become 10^6. So, you get 1 million each direction, so that makes it $Re = 10^{18}$ which will fill up the largest supercomputer available on that. So, So, we cannot, we cannot do it but sometimes you do not really need to do for, I mean you have to get an idea of what is happening, you do not need to really simulate all of it. It turns out that we can get some idea about things by not stimulating this big Reynolds number, but you can do it lower.

So, there are some more predictions, but I think I will not get into all that and let me write one more remark for completeness, but this is again another field. So, viscosity here is nonzero, right viscosity is nonzero, but small. We want Re going to infinity limit, but Re is not infinity, viscosity is not 0.

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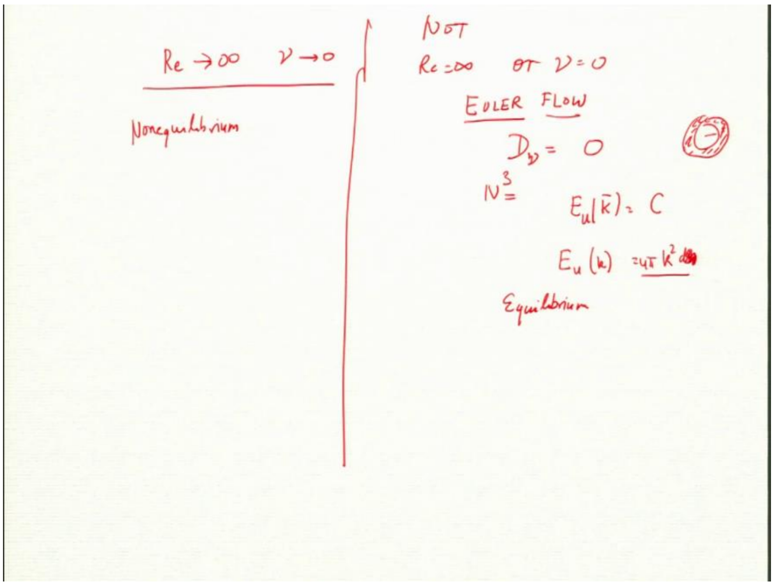


So, we assume that Re tends to 0, it tends to infinity or viscosity tends to 0, but it is not Re equal to infinity or ν equal to 0. This is called Euler flow, if viscosity is 0 is called Euler flow and turbulence is, viscosity tending to 0, I need a cascade, viscosity is not there then there is no dissipation of energy. If there is no dissipation there is no flux and nothing.

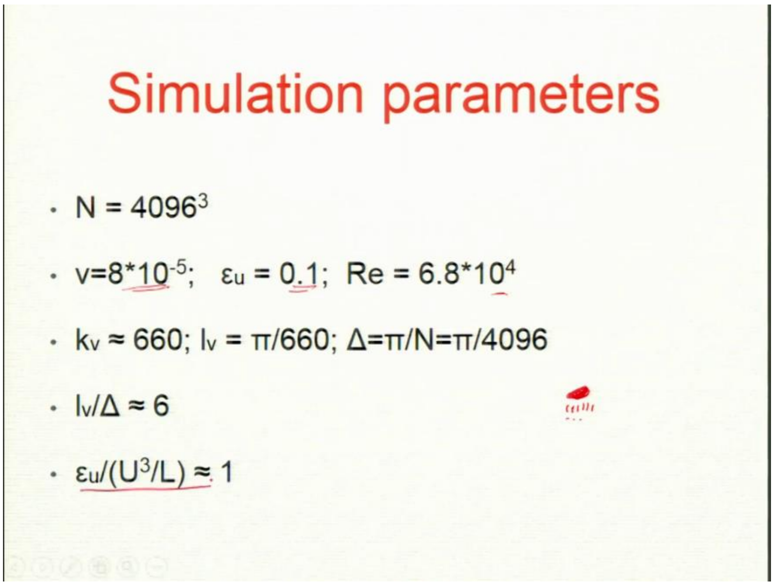
So in fact, if you recall in my flux lecture I said flux equal to 0 is also solution, and flux equals 0 that means, is equilibrium theory; that means, each mode is giving 0 energy to other mode, there may be fluctuation but it is not really giving any net energy to other mode. So, we can take lot of Fourier modes in Euler flow. So, I can have let us say N^3 Fourier modes but none of them are losing a giving energy. So, it is like a thermodynamic process.

So, thermodynamics if you, so this is each particle is, each wave number is like thermodynamic particle. So, all of them will have equal energy is associated made everybody has equal money. So, everybody equal money then what is the spectrum? So, constant know. So, Eu modal energy will be constant.

So, what about shell spectrum? In a radius k how many modes will be there? k^2. So, spectrum is $4\pi k^2$. It is not 5/3. So, for Euler flow we expect to be k^2 not five-third and this equilibrium theory and this is Kolmogorov is non-equilibrium. This is a net transfer of energy from large scale to small scale. So, though it is steady, but is non-equilibrium.

Equilibrium definition is not that it is constant energy, it means that there is no equilibrium implies, no exchange of energy from one scale to other scale. Equilibrium all main, all sense, nobody giving anything to anything is this on the average 0. So, it is an non-equilibrium phenomena where there is a cascade of energy.

We will quickly show that Kolmogorov predictions are seen in simulations, also in experiment, but I have not done experiments, I will not do it here, I will just show from numerical simulations. So, we will use our code, and this is 4000^3 simulation and is done by work by lot of people in the group.



It is not the largest, but is one of the largest. Now, largest is 8000^3 in the world, so we can do 4000^3 with our code which is very big this thing.

These are spectrum.

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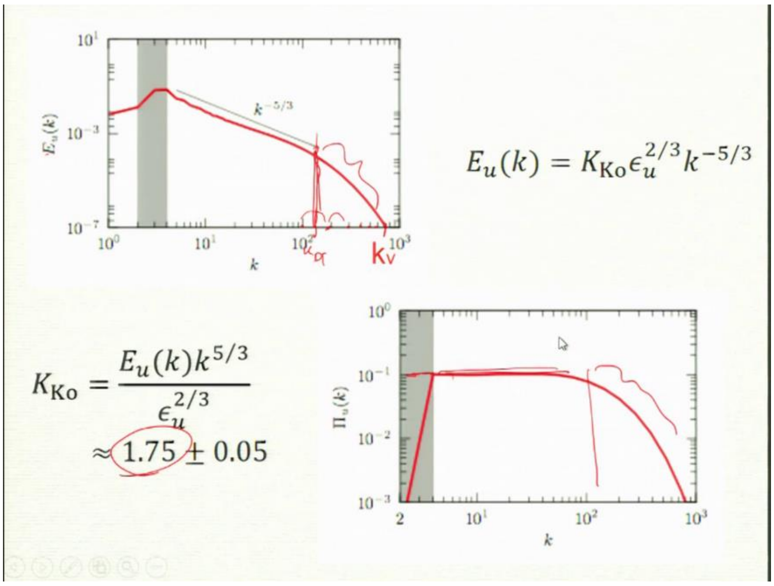

So, $u(k)$, so in fact, dk is 1 for us, in simulation of 2π box, so dk is 1. So, we can compute spectrum just by putting $u^2(k)$ at every k. So, I sum over all the modes in the shell of thickness 1. So, 50 to 51, 51 to 52 like that, so this is a number and k_ν is sitting here 660 know. So, everybody knows how to read the log scale. So, where is 200? Now, this is log scale know I mean some, some people do not know. So, these 100, this is 200, this is 300, 400, 500, 600, 700, so 660 is somewhere there. So, log scale between 100 to 200 is this, here to here is 300. So, it is getting small and small and equal to larger scale and it is a straight line in the log scale. In real space in, if I plot linear scale then I will not get a straight line is power law, ok. So, log scale is important.

Now, Kolmogorov constant we compute is 1.75 it is not 1.6, it is because its 4000^3. Now, you need bigger resolution to get. Now, there other effects, I mean 1.6 is hard to get, exactly. Now, you can look at the flux, flux is very flat here and it is close to ϵ_u, ϵ_u was 0.1, right here and it is in the same band where spectrum is five-third, it better be.

So, k_{Di}, where power law is not applicable anymore, so this is k_{Di} and this region we will do in the next class how to model that region, it is possible to model and this region also we can model. So, flux is constant. I hope you are happy with this, you know this like Kolmogorov theory, we can see in simulation in action.

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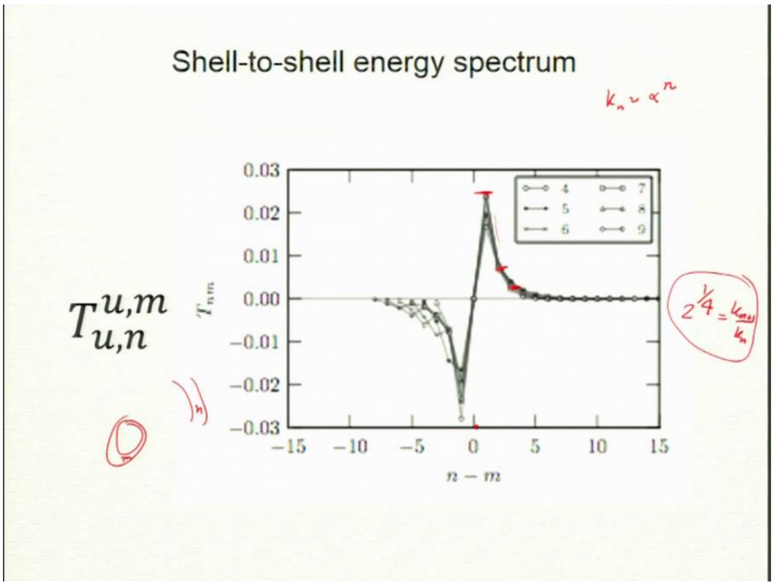

Now, we can do also shell to shell which we did in the class. So, we need to read this. So, $n - m$, so there are shells. So, this shell giver shell is m, this is receiver shell is n. So, in x axis I plot $n - m$. 0 means to itself, so 5 to 5 6 to 6; so, these are numbers, are 4 5 6. So, 5 to 5, 6 to 6. So, I do not give to myself, so 0. But this is $n - m$ is 1; that means, 5 to 6 is maximum you see this maximum. Then it decreases, so this is 5 to 7, it decreases quite rapidly, 5 to 8 is also tiny after that it become negligible. So, it is not only to next neighbour, but it is to two more neighbours. Now, here it depends on the width of the shell, I do not remember, right now, but is I think these are all power law. So, my shell radii are not linear. I do not bin like 2, 3, 4, 5, 6 for this computation, it is been like this.

Now, so this tells you that Kolmogorov's assumption, Kolmogorov made assumption that energy transfers local; that means, it goes from scale n to $n + 1$. Basically you interact in wave number space to the next shell or in scale by scale, it implies that one scale to the next scale in fact, you know one scale to next scale, you do not jump that scale; in hydrodynamic theory you do not. Other theories you could but in this theory you do not.

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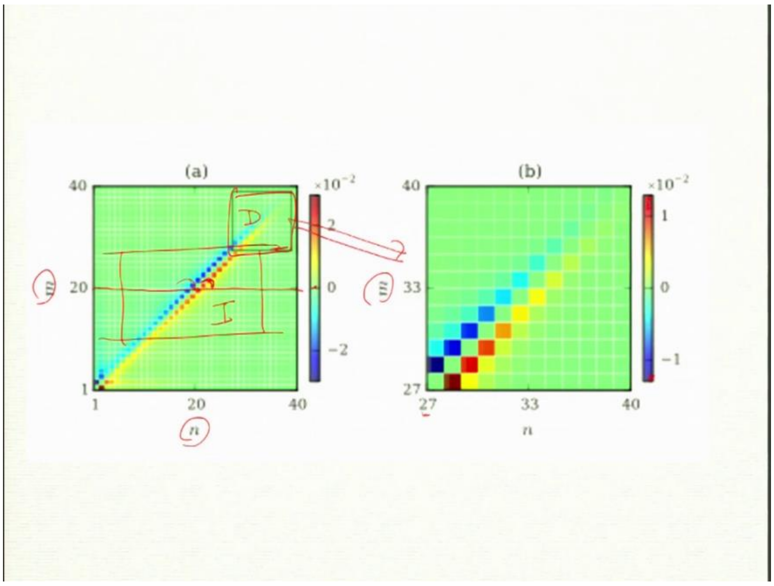

Now, we can also do this in a density plot. Now, this plot, so you should understand. Now, this color bar is here, blue is negative; red is positive and blue is negative. Now, this is giver, this is giver and this is receiver, ok; m is giver and so we should let us take one number 20. So, diagonal will be 20 to 20, that is 0, so it is green.

But 20 to 21 is red, 20 to 21 is red, is maximum in fact, to the right is strongest that mean it is giving highest energy the next shell and it fades very quickly. So, fading to yellow, then it becomes green. So, this is a region where it is giving and left is a region where this is receiving from. So, 20 receives from 19, 18 and 17. So, I am receiving and giving.

The initial range is somewhere here, this initial range and these are dissipation range and dissipation range is zoomed here, so this is zoomed picture. So, it start from 27, so 27 is sitting here, 27 to 40. So, this huge calculation, this really required a lot of time. And this again, in the dissipation range two energy transfer is local, is surprising, but it is also local and this is there in our same paper. These are basically verification of Kolmogorov theory.

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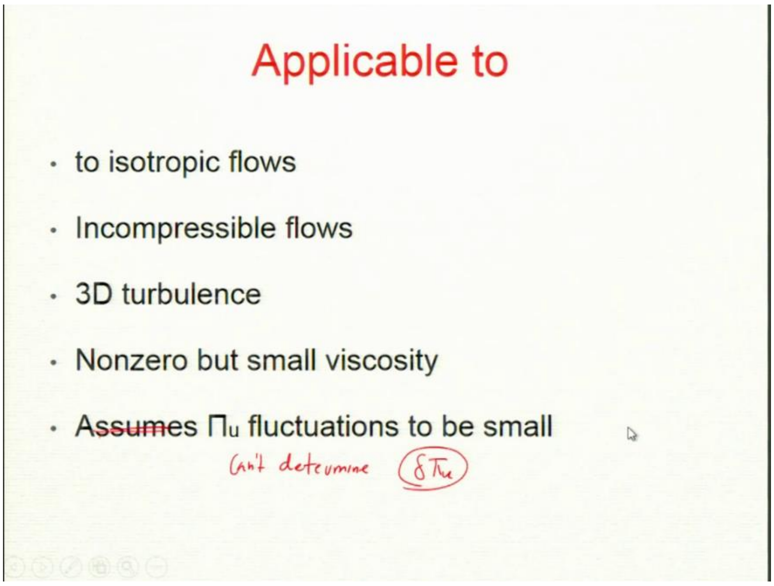

Now, some specifics I quickly want to say this, which we do it, is applicable to isotropic flows. So, what does it mean? If I look at small scale where five-third is applicable, if you look at any direction the flow loop similar is isotropic, but if I apply rotation then it is not true. Kolmogorov theory will not work under rotation; rotation makes it anisotropic and it does not work or if I apply magnetic field it does not work. So, compressible flows, highly compressible flows it does not work. It is for 3D, 2D turbulence has different theory.

Nonzero viscosity I mentioned, zero viscosity this will not work and we assume Kolmogorov theory, well Kolmogorov did not assume, but in the presentation I made the flux is not fluctuating, Π_u is we assume a mean flux. So, Kolmogorov theory cannot determine the fluctuations in flux. So, I should not really say that assumes, I cannot determine fluctuation $\delta\Pi_u$. So, that requires more sophisticated calculations and that is unsolved problem.

Thank you.

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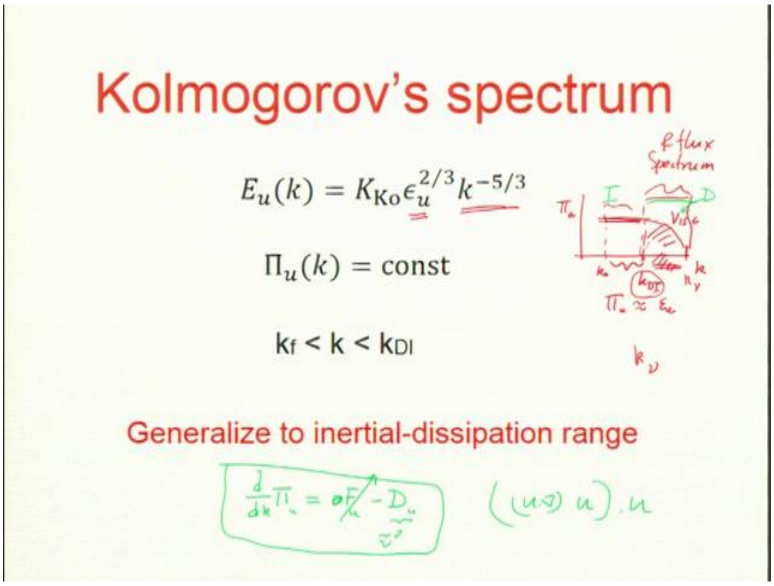
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 29
Kolmogorov's Theory
Spectrum & Flux in inertial-dissipation range

So, in the last class, we discussed in 3D hydrodynamic turbulence, what is the spectrum and flux? right, but that was for the inertial range. So, when the non-linear is very strong compared to viscous term. Viscous term is finite, but non zero then and Reynolds number will be finite. So, I have defined Reynolds number you know.

So, Reynolds number is large then we get a turbulent cascade. You know scale by scale transfer and that cascade leads to $k^{-5/3}$ spectrum; the question is; so this is what I summarize in my next slide.



So, the spectrum is $k^{-5/3}$ and the flux is constant. If you plot this function of wave number, y axis is flux. Flux is constant and flux decrease well I should not say this is a topic of today's lecture. So, this these where flux is constant and this flux equals the dissipation range that also I proved right.

If you see from that equation for the flux; you take this equation k_o as there will be a number k_o and compute flux energy in given by the external force matching the dissipation, you can match the two under steady state. The flux where you case approximately there is slight change is ε_u. The ε_u is slightly bigger than π_u right as I am discussed in the class or slightly different that depend but they are approximately equal, now this was the inertial range.

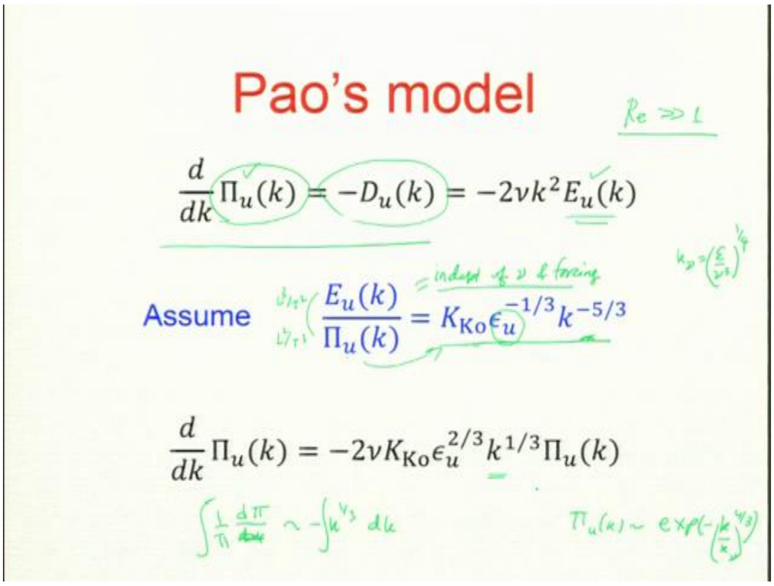
The viscous term is strong here. In this region the spectrum is $k^{-5/3}$ and flux is constant. Now question is "what is the flux or spectrum in this band"? We have in my book I make the notation k_{di}. The k_{di} is the wave number where dissipation is starting and the thing which I said k^2 ν or sometime I was writing as k_d.

So, k_{di} is where $k^{-5/3}$ spectrum is not valid anymore and is getting into some different form. In fact, we want to study what is the spectrum and flux in this region. Is that clear? This is inertial range and this is a dissipation range. This one, so that is a topic. Now, it is true in standard text books you will find that they do not talk about flux or spectrum in that range is argued, but not so much in standard books.

In fact, we say the flux is 0 if your non-linearity is weak. It turns out as long as there is non-linearity. There is energy transfer because it is a non-linearity. There is energy equation will nonlinear term, right. So, this is a non-linearity action k p, k=p+ q; so flux will be non zero. So, even for laminar flows where energy transfer is not equal to 0, but order 1; there to there is a flux is a misunderstanding that flux is 0 for laminar flows, as long as there is non-linearity there is a flux.

So, flux does not require the flow to be turbulent nah; flux is constant for turbulent flow turbulent hydrodynamic flows that I proved it know. So, please remember the how do you show that flux is constant? Is dπ/dk is 0 because this side was force minus dissipation [noise.] Now, we say that in inertial force is 0 and dissipation is weak. So, this was approximately 0, but we say well we ignore it. So, flux is constant π_u but in real flows of dissipation is not 0.

A laminar flow, so the flux will change ok. So, that is a; so these equation is for me is very-very powerful equation and that tells you about flux ok.



So, let us see; so we will work with first when Reynolds number is much larger than 1. So, this theory is for larger Reynolds number and I will discuss only Pao's model. There is one more, but I will not discuss all models ok. So, this course; so we discussed selected model which is kind of fits with the data better. So, this model be Pao which is very old model 1960s. So, this is for both inertial and dissipation this formula will work for inertial as well as dissipation approximately ok. It is not fully true; it only approximately valid, we know there are there are deviations from Pao's model in simulations.

So, we start with the same equation know. So, inertial range my force is 0, so $d\pi_u$ with a flux will change with k and how well flux will change because of dissipation right everybody is happy with this. So, this is one of the most important equation ok. $\frac{d\pi}{dk} = -2\nu k^2 E_u(k)$. Now the, so this is what we want to use; unfortunately there are two unknowns right π_u is also unknown and u is u is unknown and I have only one equation; so, you can solve this.

So, Pao suggested the less maker model where we assume that E_u by π_u this one; this ratio. So, I need one more equation according to him let us try to make $E_u(k)$ by π_u (k) independent of viscosity ν and forcing. So, when you make an assumption; so where let us assume that this is only depends on the flux and wave number flux is π , ε_u flux ε_u are same ok.

So, this is what let us assume it and if there is a case; if you assume I make this assumption then divisional analysis will tell you that it must be this ok. So, E_u by π_u has some dimension this is L cube by T squared and this is L squared by T cube. So, if you match the right hand side; you will get this in fact, I am just following the Kolmogorov's theory.

So, Kolmogorov's theory u(k) Kolmogorov $\varepsilon_u^{2/3}$ $k^{-3/5}$ that, divided that by ε_u you get $\varepsilon_u^{-1/3}$So, I am also putting the constant so in fact, powerful the constant this Pao's model not my model ok. So, let us; so we have now two equations and two unknowns now it is easy to solve right; it turns out its very trivial after this. So, once I substitute E_u from this equation. So, I just replace E_u by π_u multiplied by this; so, you will get this.

So, this is an ODE function half π_u is the unknown variable ok. So, here I had put ε_u by intention. So, the right hand side is only function of k in this part, k and $\varepsilon_{u.}$ The ε_u is the constant ok. So, this is just; so what is this form of. So, forget about all the constants; how will π_u look like its function of k?

So, what is the solution is a linear first order equation.

So, dπ/dk, please tell me what is the function.

What is the form of Pao's function?

Student: k to the power 4 by 3 and k to the power 4 by 3 and.

Student: Exponential.

exponential ok. So, dπ/π is this integral. So, I will give you log in the left hand side and in the side hand side it is $k^{4/3}$, so exponential (- $k^{4/3}$).

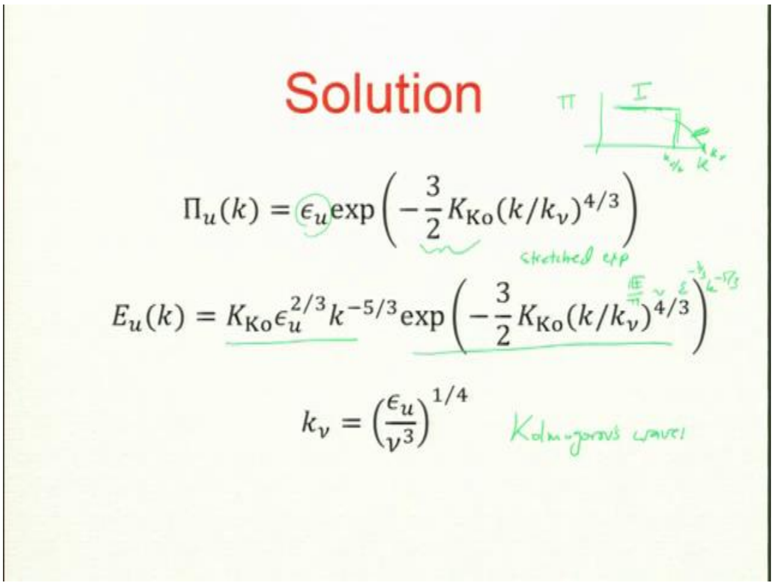
Student: Yeah.

So, the solution is you can immediately you can see apart from constant is exponential (-$k^{4/3}$). Now remaining part is just algebra you know, so we need to. So, please remember it cannot be $k^{4/3}$. It has to divide by l so k has dimension know. So, I cannot say sin of 1 meter. It does not make sense have you heard of somebody is saying sin of 1 meter.

Sin of 1, you can say. So, sin the argument of sin must be non dimensional number or exponential of 1; 1 second is not ever meaningful. So, you have to divide these by some quantity equity dimension with dimension of k. So, it turns out we get quantity k nu same as what we defined earlier. So, it is a Kolmogorov number k nu comes which is epsilon by nu q to the power 1 quarter ok.

So, this algebra I will not do it here; you can easily do, but the form is exponential; it is not k squared.

It is k. It is slightly I mean it comes from because the 5/3rd nah. So, origin of 4/3rd is coming from this 5/3rd will multiply by k^2. So, here remaining with $k^{1/3}$ integrate that you get 4/3rd.



So the solution, I have just say you, this is the full solution. So, $(k/k_v)^{4/3}$ and in front there is a 3/2 k Kolmogorov which is the order 1 and π_u has dimension ε_u. So, ε_u is coming here which is nice ok. So, what is it mean? If I plot this function of k is π_u; so it is constant these also called stretched exponential know is delayed exponential.

So, exponential drops quickly you know exponential would have dropped like this, but because of this 4/3, it drops bit slowly whereas, flux is always decreasing with k. This is telling you that flux is not constant flux. It is always decreasing, but decrease rapidly after some wave number.

So, k k_ν is somewhere here, but may be $k_\nu/2$ is somewhere k_ν /2. So, so in fact, would be interesting to see make it quantitative at what value; it has pi has become half [FL]. In fact, it is a good number to investigate I am not done it, but is good thing to play around ok; is that clear? So, flux always decreases because of dissipation.

So, even in the inertial range dissipation is important in real flows. So, flux is decreasing even in the inertial range, but it is less decrease. But then it rapidly decreases in the dissipation range and this exponential part will very important in the dissipation range. So, flux is non zero in the dissipation range as well ok.

Now, Kolmogorov assumes viscosity into 0 limit and that is what he is basically says well this k_ν is at infinity. So, he is a mathematician. He is ignoring it, so well, let us push it into infinity and so Kolmogorov is essentially gets a constant if k_ν is infinity; then this exponential part is 1. So, exponential power infinity is exponential part 0 sorry.

Exponential part is 0 is equal to 1 ok. So, that is why he gets, now what about the spectrum? Spectrum can also be solved. So, power told you what does power tell you? E by π is $\varepsilon^{1/3}$

Student: $k^{-5/3}$.

$k^{-5/3}$ ok, so I know the π now. So, I can just take it to the right hand side. So, you will you will basically at Kolmogorov multiplied by stretched exponential. So, the formula for s spectrum is this Kolmogorov part and I multiplied by this exponential part, stretched exponential ok; so this is the prediction of Pao. Now, k_ν I told you is; is this which is Kolmogorov number. So, these it has a name; so please remember Kolmogorov wave number.

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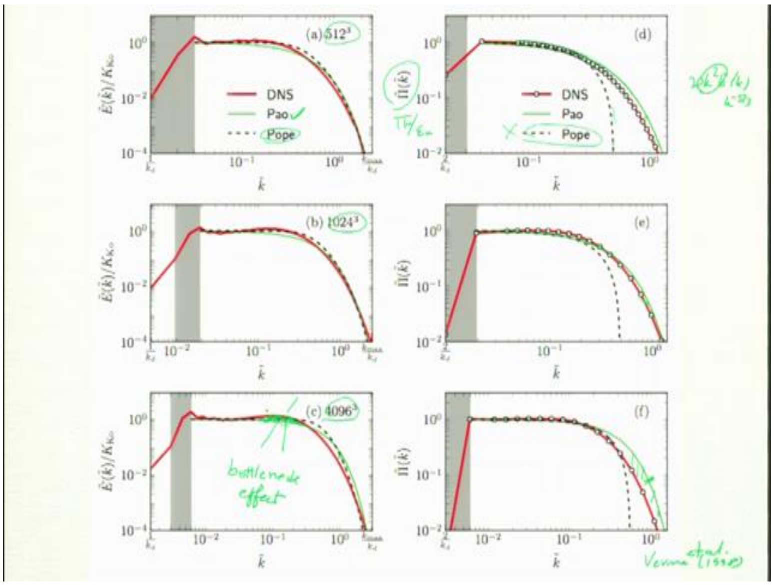


Now, let us see whether this fits with the stimulation.

High resolution simulation ok, so there are many-many plots. So let us try to understand. So, this is done by us; so this was this paper which is to be published Verma et al. So, there are many people; so this has for different resolution. So, 512 cube, 1000 cube and 4000 cube. So, let us look at and the 3 curves. So, rate is the DNS from simulation DNS means Direct Numerical Simulation ok. We do not make any assumption on viscosity or some model, periodic box $(2\pi)^3$ like what we describe; so the rate is there simulation.

Now, Pao is the green curve and the left and the Pope model which I am not describing here is also a model which is a dashed curve; dashed back. Now this is a flux, so we make a tilde know; so tilde is normalization. So, this one is π divided by ε is pi tilde, ε_u I am not writing this one ok. So, it should be 1 in the inertial range. Then it should decrease. So, let us look at the flux first. So, flux is decreasing; the simulation is this line, but on that the data points are all circuits ok.

So, red line with circles are the simulation data, now green is over shooting, but somewhat it fitting quite with the data fight will cube, but this black line is not fitting. So, Pao is not doing good for flux a sorry; Pope is not doing good Pao's doing quite well for fight well cube; here also for 1000 cube it is good.

So, this drop of flux is captured by Pao, but it is not doing well for 1000 cube; you see it is slightly quite off factor 2 ok, but flux is decreasing. If you look at these things flux is decreasing; even in the inertial range ok. Because viscosity is taking the energy out becomes stronger and stronger for large k because viscous term is $\nu k^2 E_u(k)$.

So, this k squared term and 5/3rd is here. So, k squared is getting more and more important for larger k ok. So, this is what is happening here; so this is increasing. So, the flux is decreasing so that one thing I want to emphasize. What happens to the spectrum? Spectrum the green line is below the red line, but somewhat fine is you almost looking quite good; except here again the green is above, but here green is below the red, can you see? The green is below the red. There is a gap.

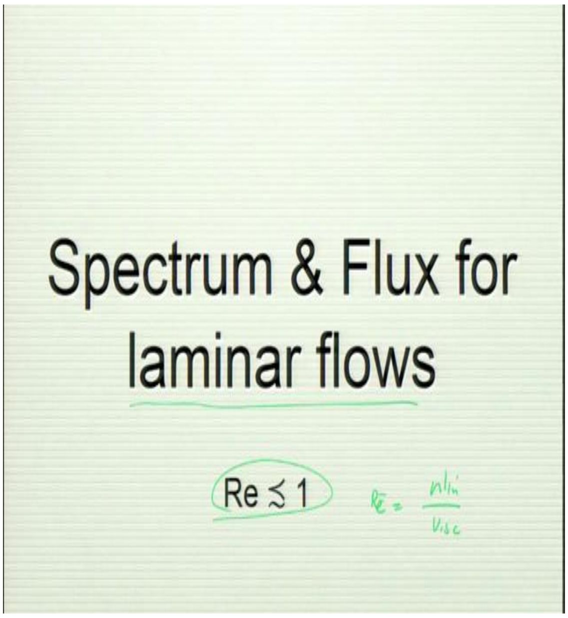
So, the red line is above green and this is called bottle neck effect. So, in crude sense, so well I this is still remain unsolved problem in some sense. People are not agreeing on one mechanism one universal one mechanism which everybody will say well this is the reason for bottle neck effect but is like you are running on a highway driving on a highway, but suddenly you have to go to city you know you take basically you take the detour rather you create off the highway.

So, they are normally congestion at the knee you know. So, the speed is coming down; so this is dissipation and as dissipation the speed of cascade is decreasing, so to increase the dissipation at that region. So, this is jam; the more cars at the turning point.

In some sense that is in picture I can explain why bottle neck is there because there is the mode of car which is travelling on the highway, as well as let us say you have to go to a smaller road, jam road or dissipative road; dissipative road the speed has decreased. So, basically there are more cars and that is what is kind of happening, but any 4 resolution I mean in its more arguments and so this is a bottle neck effect ok.

So, Pao’s model is not fully correct; Pao’s model assumes that it is a mean field theory. So, this is a constant flux and you just it is not taking care of subtleties of turbulence ok. So, we need to worry about this aspect, but is good know I mean to first order is a very good model and it seems to do well even for 2D, helical turbulence So, at least we have been quite successful in modeling lots of flows with Pao’s ideas; so this is about Reynolds number much greater than 1.

So, I give only one model but this model also useful for the dissipation range which we need it for many situations like rotating turbulence, where this is a strong I mean energy is quite small in the in lot of range of wave numbers, this model is doing very well ok. So this is what we find in recent work.

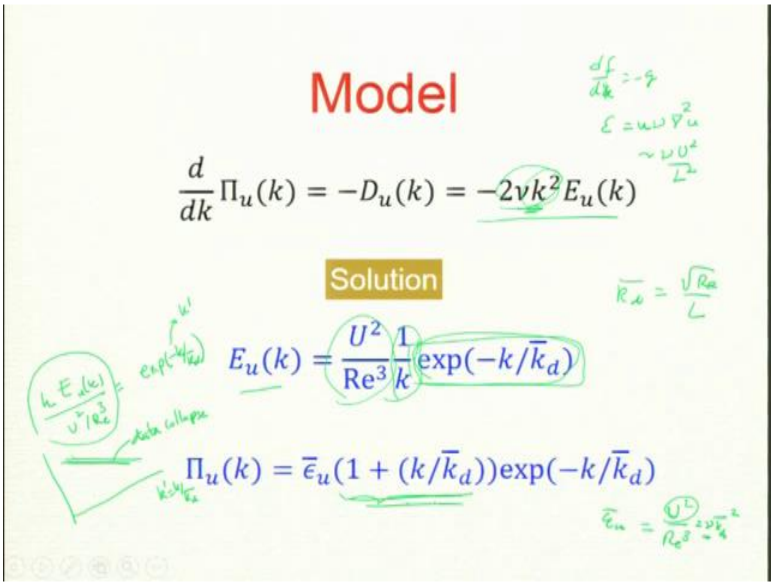


Now, let us look for the laminar flows. The laminar flow is what dissipative flows ok.

This name is different names are given by different people, but I just call it laminar; when Reynolds number is 1 or order 1, but it could be 10 as well. Turbulence flow required Reynolds number to be 10^4, 10000 or so or 10000 or bigger. But Reynolds number is when it is R 1, then non-linear term and viscous terms are same order you know, but the Reynolds number is ratio of nlin by viscous.

So, we say that viscous and non-linear terms are of the same order. So, flux constant is not true anymore, but what about the, what happens to the flux? And you find in many papers this say flux is 0; you drop the, we assume the non-linear term is negligible including Cambridge paper. So, I can research; so do not take all the experts you know with lot of faith.

So, you should trust your own ideas. So, Reynolds number is order 1 that mean non-linearity is important is as good as viscous terms; you can cannot just keep viscous term and ignore non-linear term. If you keep the non-linear term then there is a there is a flux from the argument which I was giving in the in the; in the first slide, in fact, it is straight forward.



So, we take the same equation dπ/d k minus D_u (k) and this is it. Now, you may say why not try the Pao's model? You know, so I can use the same idea Pao. In fact, same formula for Pao and try to fit in the numerical data is that k_v will be small, but terms out I will not show you here that Pao's model does not work for laminar flows. We had simulation data and we try to put Pao model is simply work it the lines were very-very different. I am not showing you here, but you can look at the paper we just say about to come.

So, idea is to look for some other formula; now the Pao idea that $E_u(k)/\pi_u(k)$ or $\pi_u(k)$ by $E_u(k)$ is independent of ν and viscosity. So, it is not that does not work very well, but what I thought was let us try something which is obvious function. Now, if we have something like this one dπ/dk is u(k), let us say you ignore this part well you cannot I guess you will see this let us drop this term, then what you expect what which function will satisfy this? So, here dF/dx is minus g which function would satisfy this?

What function will satisfy this very easily?

Student: Sin.

Sin no sin gives you cos. So, here in the right direction what you should; exponential?

So, just put E power minus k for both. So, you take the derivative E to the power minus k. So, exponential k is a good function pure exponential; no stress.

So, I said well, try exponential, but this k^2 will cause some you know some [FL]. So, there is k^2 will make some trouble, but that is you just have to solve it properly; so they are polynomial. So, it turns out v model $E_u(k)$ is well this algebra, I will again live it this is again fitting with the model know a fitting with simulation; this was not I cannot derive this fully from the first principle; we do fit from the simulation. So, exponentially

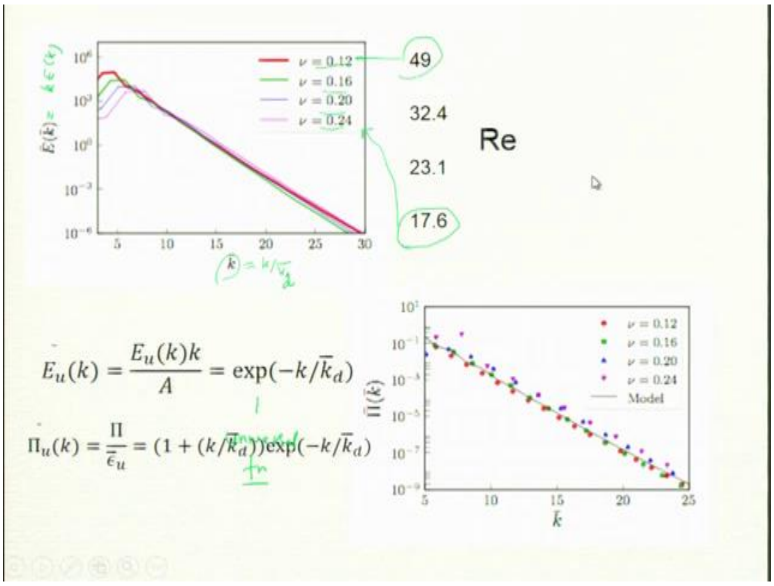
the leading factor and it turns out if you try to fit with the data which; which I have the data in the next slide, I think I have the data simulation.

In the simulation if I have divide by k it fits with the with the data ok. So, this is 1/k is additional thing we fit from the simulation ok. Now, I really welcome some of you to try out from the first principle try to derive Pao's model comes from the first principle; it does not require any one assumption.

But you can derive it after that, but this is one require simulation. In fact, this free factor is coming from simulation and k_d I did not try it in the slide is square root Reynolds number by l; this is again fit from the simulation. But the idea that you should be exponential is very intuitive idea, just guess what?

Now, because of k^2 this front part is polynomial 1 plus k. Now you can fit this and things will cancel nah a ε_u I did not try again in here, but is there in the notes $u^2/Re^{3,}$ 2 ν k_d^2. So, ε_u is for viscous term; it is for viscous term what is the dissipation by the way it is not E cube by l nah is; ν it is the viscous term nu Laplacian u u.

So, it is ν u^2/L^2. So, that is estimate for laminar flows. So, it is coming come from ν u^2, but there is some normalization which we, we get from simulation ok. So, this free factor is coming because of k^2 (1+k) and this model is also very good; when the flow is Reynolds number is out of 1, is excellent ok. So, there is an energy get different scales even from laminar flows.



So, spectrum; again I am not seen a paper published there is one paper came recently, one paper which came recently by Srinivasan's group we which has done the simulation other than us. So, this is normalized; so 1/k is here is of k $E_u(k)$ and this is a free factor know there is a u^2 by Re^2 and u^2 by Re^3. Now is there the free factor we also multiply the free factor; so that you get a universal function.

So, so this is important know. So, let me go back. So, we want this function to be compared if this is your conjecture then, what should you try to fit with various Reynolds number? So, E_u multiplied this is by k divided by u^2 by Re^3; what will that give you? Exponential minus k by k_d bar, so, x axis should be what? Should we put k or k what should I choose?

Student: k.

k/k_d bar. So, y axis will be just exponential of; so there is non-dimensional number k'.

So, plotting is also an art; you should really know how to plot. So, x axis should be k prime just k by k_d bar and y axis should be?

Student: E_u.

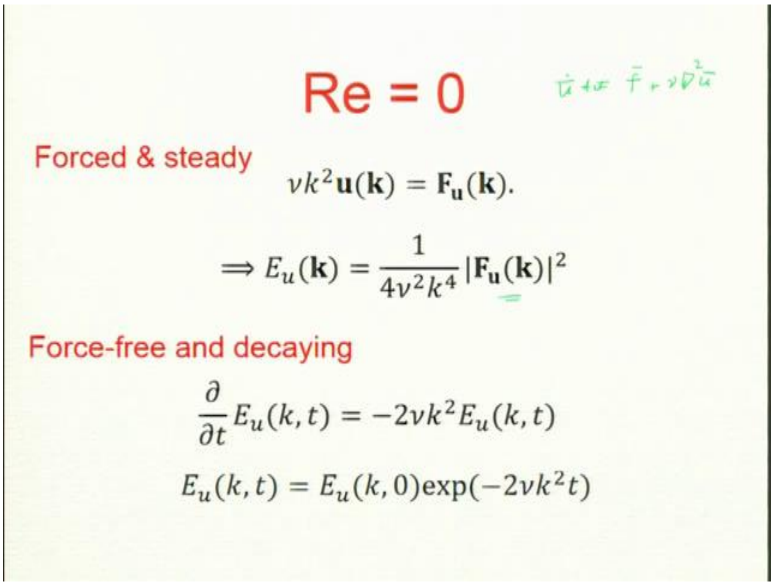
Not E u; y axis should plot this.

For different runs, so you have plot this then I will get one line for all of them same line. They all collapse this is called data collapse and we will collapse the exponential minus k; k prime minus k prime ok. So, this is what we did this from this is guess work. You know this is really requires doing back and forth guessing and same thing you do it for here itself exponential part divided by (1+k), so you have to just do it.

So, it is collapsing nicely you know; so x axis is k prime k bar which is k by k d bar. So, we put a k_d is the laminar; not k_d. So, k_d and k_d bar are different; k d is notation k d is same as k_v which is Kolmogorov number and this is flux where we divide by 1+k bar as well 1+ k'. So, you make the function which is universal; these are called universal function. Why universal? It is true for all data, true for all runs as long as Reynolds number is order 1.

So, our Reynolds number is here. So, viscosity is listed here and Reynolds number is 49 to 17.6. So, this corresponds to 17.6 and this corresponds to 49. So, Reynolds number is not too small nah; so our runs for not less than 1, so it fits with this. So, this requires some more work and this runs are very cheap we can do it on 64^3 and it is a 3 D runs and they are all steady state flows know. So, they were forcing; wait for steady state, look at the spectrum and that is what we get and this seems like a good model.

And laminar flows are also important though this force is on turbulence as a many flows which are laminar. So, like human body in this human body all the flows except near the heart; things are laminar, blood flow is laminar the fish wing in the ocean or river they are laminar. So, they are not very fast or also many hearing flows slow moving honey pipe and so on and so the earth ventral is slow; so there many many flows which are laminar ok.



So, that is what about Reynolds number is 0.

Reynolds number is 0 is trivial.

Student: Non-linear term.

Sorry.

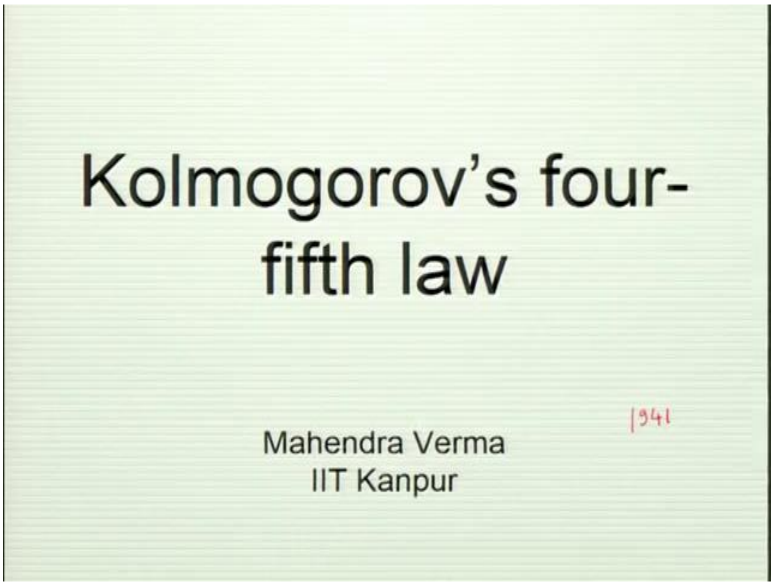
Yeah the non-linear term is dropped. So, for steady state d/d t also will dropped. So, if Re is 0 then if there is no forcing when we will basically get. So, there is no well basically if steady state the forcing must match the viscous term. So $\frac{\partial u}{\partial t} + (u.\nabla)u = 0$; so f plus ν k grad square u. So, viscous term must match, because what is the steady solution. So, that is what I wrote; equity phase that is what we get. So, a spectrum will be 1/ k^4 forcing mod square just; just straight forward. What if I say that where there is no forcing, but it is decaying?

So, spectrum will be this one; here there is no flux. So, there is no dπ/d k ok. So, flux is 0 for Re is 0 right, this is non-linearity if non-linearity is 0, so there cannot be any triad no flux. So, spectrum will be exponential it is a diffusion equation.

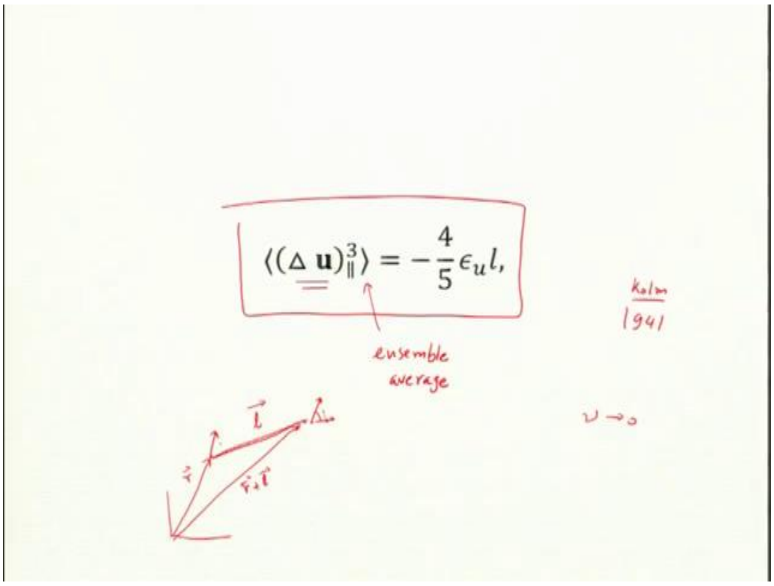
So, diffusion equation is term is wave number energy will drop its Gaussian E to the power minus k square t is a diffusion equation. So, finally, everything will go to 0 but it is a diffusion equation ok. So, I will stop.

Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 30
Kolmogorov's 4/5 law
Isotropic Tensors & Correlations



So today's lecture will be on Kolmogorov's four-fifth law. This law is in the real space. So, far we focused on furious space description, but now we will go to real space description and this was derived in 1941 ok. The Kolmogorov wrote three papers; the first two papers are related toward today. In fact, this is a second paper the dissipation. So, I gave you all the paper. So, everybody has those three papers, it is a second paper where he derives, what I am going to derive show today.



Of course those papers are very cryptic, he just writes one line, but to derive it takes quite a bit of effort, which I am going to do today ok. So, this is there in Kolmogorov's second paper 1941 and it is in a Russian journal, but it has been translated in English. So, what does it mean? So, we are take two points in real space. Now, this is in real space. So, we are going to going to real space which I separated by vector l ok. So, if I take a coordinate system, this is **r** and this is **r** +**l**. So, difference between two points will be **l**. So, there will be velocity at these two points V 1 and V 2.

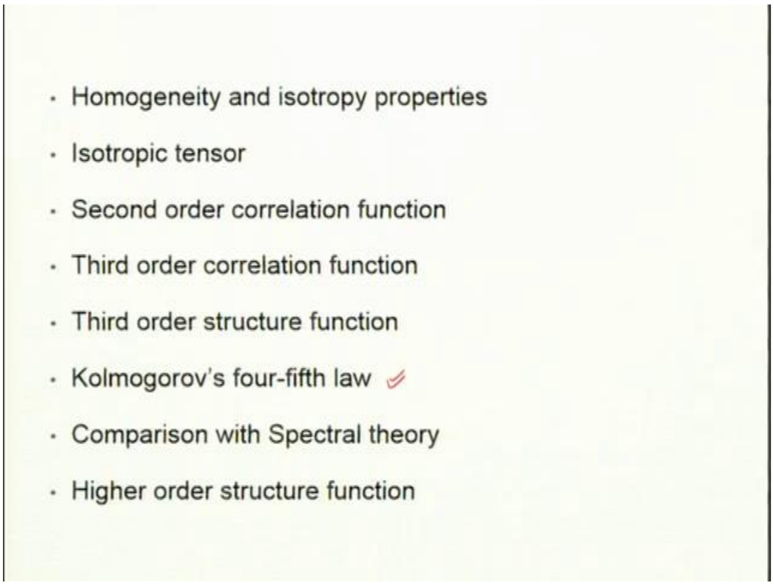
So, take the difference between the velocities at two points with Δ **u** take the projection of Δ **u** along **l**. So, that is what is meant by parallel. So, I take the projection. So, I take the projection like this till here and difference, perform average. Now, this average is supposed to be ensemble average means, I take many copies of turbulent flow started with similar initial condition, not same condition similar initial condition and then do this average over many copies this called ensemble average.

See, if I do this average then what I will get is -4/5 ε_u ,which is the dissipation rate times the distance between the two points, this l is the distance scalar of course, this is under certain assumptions like viscosity going to 0 limit steady state.

So, I will put similar assumption which I did before, same assumption, but this is a real space and this is connected with 5/3 which is easy to see, how is it connected?

So, one thing is, there are some details you know. So, assuming that it is a factor structure or. So, cube order is l. So, what about what do you expect about $(\Delta u)^2$. Now, of course it will be parallel only know, but a spectrum is connected with velocity difference between two points allow all directions. So, I think let us do the bit later ok. So, I think I do not want to get into so, this connected with five-third you can derive it ok.

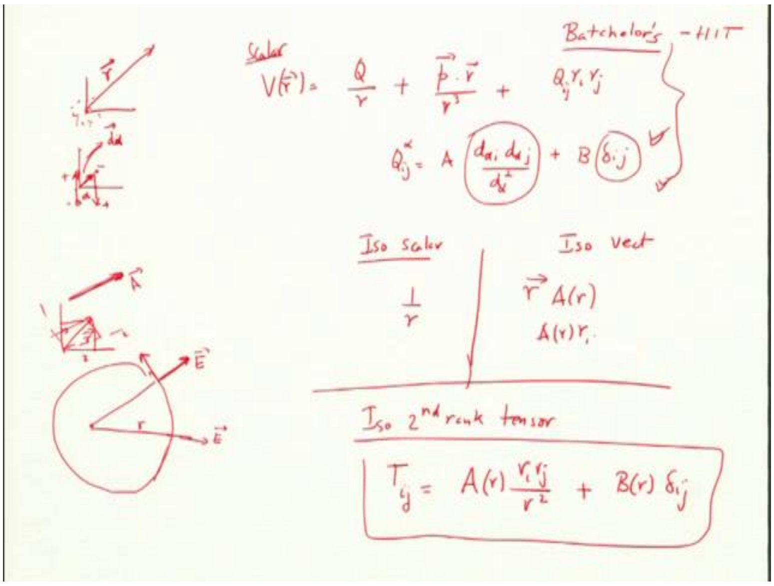
So, I will derive it may be in the next class ok. So, this is easily derivable. So, my objective today is to derive this ok. Now, I will skip some of it, I will give those notes where all the details are there. So, this bit of tensor algebra, but I will give the main steps because these steps are quite involved. So, I cannot do it on this computer ok.



So, the steps are first, I have to define what is homogeneity and isotropy, isotropic tensor, second order, third order correlation function, third order structure function. So, this all has to be done ok, then this law comes as four-fifth law and then I will compare with spectral theory and I will describe about the higher order structure function, which is generalization of Kolmogorov structure function.

So, let us start with what is homogeneity and isotropy, this assumed by Kolmogorov in his paper.

So, before going to isotropy and homogeneity, I have to talk about something about isotropic tensor, because we need this for the derivation. So, they are in fact, so it is a, so the two ways to look at it, but let me just give the simpler version. So, I think everybody has done electrodynamics in your undergraduate classes know it is so, I am just going to invoke that algebra. So, I have some charge, you know charge and I want to look for potential from this origin.



So, I make it this thing. So, charge they do not single charge, single charge I know the potential it is 1/r, but here many-many charges, then I want to find the potential. So, if you recall V(r) so, you expand for all charges. So, how does it work? First is; some of all the charges which would be $\frac{Q}{4\pi\varepsilon_{or}}$, I am not writing in CGS.

Now, second thing is well the total is Q, but they may not be a point source. So, this is scatter. So, imagine that they are two charges, which are so, one example you know that there is a plus and minus separate by distance d it is a dipole. So, dipole total charge will be 0, but I have a potential know. So, how do you write that part? So, there is a dipole electric dipole. So, which is **p**?

So, now I have to construct a scalar given **p**. So, **p** is a vector. So, how do I construct the scalar? So, now, they have another vector call **r** vector ok, I take dot-product this with r divide by some function of r. Now, we know that it goes r^2 right $1/r^2$. So, this one r dimension is already there. So, there will be r^3. Apart from free factors of coefficient like 2, 3 and so on and so forth. So, this is how we construct for dipole.

Now, third is, I am here plus minus, but net dipole I will not to make it 0. So, there is a minus here and plus here am I net I pull the 0, but still it has a dielectric field and potential. So, this is a quarter-pole. So, how do I make the quarter-pole. So, it is a higher order term. Now, that comes as $Q_{ij}(r)$. So, **r** is a vector which is component. So, r_i r_j so, this is Q now, Q_{ij}. Now, how do we construct Q_{ij}? Q_{ij} is second rank denser, it has two indices. So, because it is it is not a vector anymore like p is a vector know, dipole moment is a vector, it is not a vector, it is bigger object, because p is 0 for this. So, Q_{ij} is written as following.

So, there are so, we have basically for each charge there is a vector. So, he calls like, if you look at Griffins book, each charge is separated from the origin by distance d**l**, $\mathbf{d_i}$ vector. From the $\mathbf{d_i}$ vector I construct the following tensor ok. Now, I am not going to worry about the coefficients that you have to do more work, but it will be A. So, there is a vector **d,** this is vector **d** here you know. So, I am going to less. So, focus on d_α, I do not want to say α is that label of the charge particle α equal to first charge, second charge, third charge. So, $d_{\alpha i}$ $d_{\alpha j}$, this is a second rank. Now, they it requires i j contraction plus B δ_{ij}. So, delta function is when i equal to j it is 1 otherwise it is 0 ok.

Now, for convenience is also derived by d_α^2. So, this one and this one are the same dimension right, because this is dimensionless d d by d^2, it is also dimensionless. A and

B are some function of r, it is a number, it is a scalar quantity. So, this is a second rank tensor. Now, this for single charge, but you can do for more charges by summing up. Now, we will see that A B will be computed in books by like Griffins, but this is second rank, this is of this form. Apart from the numbers which is I think 2 by 3 minus 1 by 3 something like that apart from the numbers. This is the form. This has to come from mathematics ok. This also we will work for magnetic potential, I mean any of this.

I need, this is to be scalar and scalar will be expanded like that, this is I mean pure tensor algebra. Now, why is it call isotropic look? Now, this is a tricky part any vector let us say constant vector a under rotation satisfies certain property. Now, this you might have done right. You have done this now, what is reference of vector. So, in mathematics vector is not called with objects having length as at magnitude and directions. That is not a proper definition. So, what is proper definition of vector?

So, what is it?

It transforms exactly like a position vector **r**. So, if I rotate my coordinate system to like this, my A component of A vector will not be same as Ax Ay, it will become Ax', Ay', but that will transform exactly like, x comma y, because position vector also will change instead of x y now, it has become x' y' like this ok. Anyway, this I will not get into it ok. I mean this is this is big derivation. So, this how you define vector, Tensors also defined just like that, second rank, third rank, this is a definition, but what is the isotropic? Isotropic is that if I rotate my.

So, for charge particle point charge particle my vector field e vector, a different position not different right, it changes direction. So, I if take same r it changes direction **E,** but it is magnitude remain same and also it is direction is radialy outward. So, it is like isotropic, though it is not exactly same, because the vector has change here the a vector I need A vector under transformation, we will transform in particular way, but isotropic tensors have even more stronger constraints that this objects should have certain property ok.

Now, for isotropic vector **E**, Now, I am going to so, these one way to look at it these one part of it. So, the source, I would like you to see if you are interested is book on homogeneous isotropic turbulence homogeneous isotropic turbulence ok. Now, so, isotropic vector is easy, isotropic scalar is easy to visualize. Isotropic scalar is if I go around, I should get the same value, if I am. This is **r** away from the source.

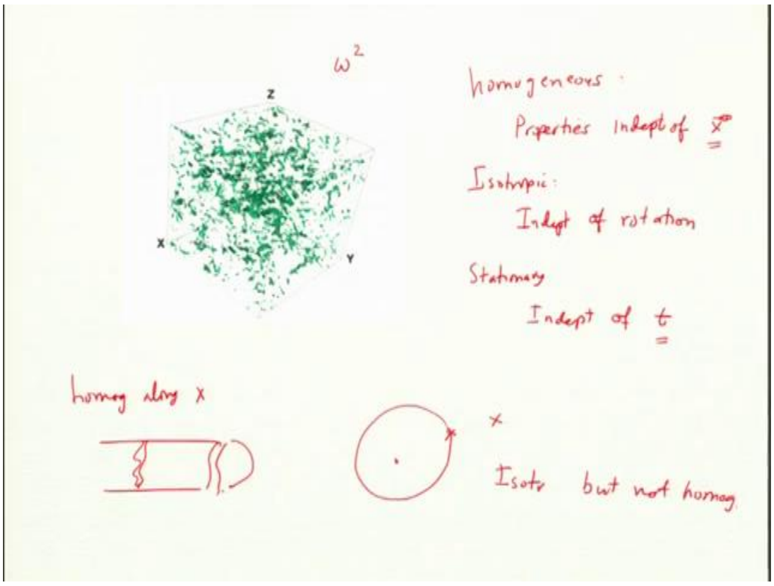
So, isotropic potential is 1/r is an isotropic scalar is that clear isotropic scalar. Scalar is by the way temperature is scalar field, because when I rotate my coordinate system, I get the same value at that position but temperature need not be constant or readily symmetric. This addition constraint from the center of the source a by temperature is same, it all points away from it, all points on a sphere of radius r. This is an additional constraint on this scalar, is that clear? and that temperature field which is only function of r is isotropic. For a point source that will be the well under condition, it possible that it may develop the temperature may spread, but it may spread symmetrically, if it is point source in the center.

Now, we can also think of isotropic vector. So, electric field, if I have many-many charges, electric field is have will have different and different directions right, but they still a vector, but isotropic vector is, if I go from one position to other position in this sphere, my magnitude remain same, but it is pointing outward ok. So, that will be isotropic vector and that will be **r** vector multiplied by A_r right this is the form. There is one more form which I will not discuss, it could be like tangential and we can construct like, but this is what you should think of today or A_{rri}.

So, these isotropic vectors, isotropic scalar, what about isotropic tensor second rank? Can you make a guess?

Now, this is not easy to visualize the vectors and scalars are easy to visualize, but second rank is not so easy to visualize ok. So, I can use this idea. So, construct a second rank tensor, which isotropic, only function of distance r. So, that will be T_{ij} is a of r_i r_j by r^2 let us keep that plus B(r) δ_{ij}. So, this is an isotropic second rank tensor motivated by this discussion on potential ok.

So, this I cannot visualize, second rank tensor I mean it has two components and rather two indices, not two components. It has nine components in 3 d, is that clear? So, I am going to use this result I need this. That is why I am giving you this background ok. So, everybody is happy with it. More discussion on bachelor book because I will not finish otherwise.

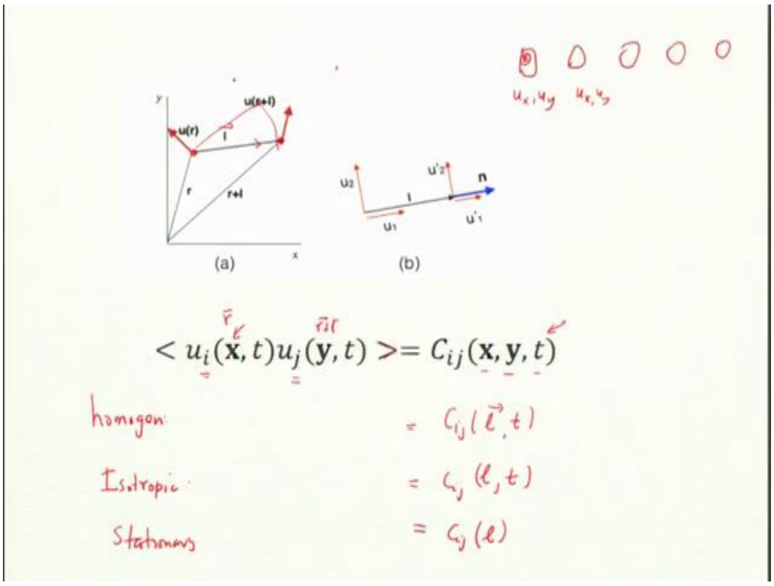


For homogeneity and isotropy of turbulence, this is from stimulation on a periodic box, which we did in our lab. So, this is verticity squared, it is a scalar quantity is magnitude a different positions are shown. Now, we can see that like is not coming from point source sitting in the middle. This is homogeneous in the sense no position is preferred.

So, homogeneity means, population is equally distributed on this planet, it is not, that is why it is not homogeneous, it is something is some position cannot be identified the unique position, then it is homogeneous ok. The inverse is supposed to be homogeneous ok, there is no heaven or something like that at least in visible inverse. So, this is homogeneous in the sense, there is no position which is unique position in this space. What about isotropy, if I rotate this box, then I get very similar structure.

So, that is an isotropy. So, we have homogeneous which is the properties are independent of x of position. I will be more specific in the next slide. Isotropic is independent rotation and we also need something called stationary that means, is independent of time. So, if I look at it now or later, it is looking approximately same, by the way this is our dynamic system.

So, you will not get exactly same phenomena or same distribution it will change, but it will randomly changed in this position ok. So, independent of time so, homogeneity means independent of space and a stationary means independent of time, but of course, I visually seen one thing, but now you will not quantify it and quantification is done by correlation tensor, velocity-velocity correlation is second order correlation ok.



So, this where we are now we are making good progress but if you are not getting my point please stop. Is that clear to everyone isotropy homogeneity?

Student: Sir, just give, if you have an example.

Yes, for example, potential of a charge particle, same a point charge is an isotropic but not homogeneous right, because we can let us go back. So, we have point source. So, go to any direction it will get a same potential, but potential here and here are different. It goes 1/r. So, these isotropic, but not homogeneous and their situation it is homogeneous, but not be isotropic. It could be like constant field, it is homogeneous along that direction, but it is not isotropic, because when you rotate, I will not get it ok.

Student: flowing sir, what that is anisotropic and inhomogeneous along both.

It is so, along the direction of flow.

It is homogeneous along that direction. So, this homogeneous I said about every direction, but you can say let say homogeneity along x. What is it mean? If I travel along x I get the same field. So, if I whatever get here is same as here ok. So, that is homogeneity along x but it is not homogeneity along y, because this parabolic profile so, I think change along y, but we will assume that we are homogeneous isotropic and which for turbulence and which is quite a good approximation away from the worked, without external field like gravity or magnetic field, if you are have if you do not have those fields, then you shake it up in periodic box it is homogeneous and isotropic ok.

So, now we are going to look at second order correlation first. So, I am going to focus on two points here **r** and (**r** + **l**). So, this is our notation. So, **r** is a vector (**r** + **l**) is vector the difference between two is also vector **l**. Now, I measure velocity at this two position **u**(**r**) and **u**(**r** + **l**). Now of course, both are vectors. So, we can have correlation which is product, but it is not really a distance, because this is a vector.

So, you cannot has just multiply right. So, multiply x-component with x-component, x-component with y-component and x-component with z-component and so on. So, the short hand notation is u_i u_j, i takes value 1 2 3 and j takes value 1 2 3, but you see this is the first position x. So, I will call this is **r** and this is (**r** + **l**).

Of course, in general it will depends it is this correlation function, let us assume same time I am assuming right now same time. So, it will depend on x y and t right. These are the arguments. So, this is the general property of the tensor. Now, I am doing average velocity field fluctuates. So, I want to average it out. So, that is only function of (x y t) is not property of the sample right I mean. So, we wait for sometime average then you find its function of (x y t).

Now, I am going to in temperature average that is why the t is kept here is call ensemble average. If I average in the same box then t will go away. So, it takes many-many copies and at a given time the average, is that clear to everyone, I mean not clear.

I start with initial conditions, system one, system two, system three, system four, system five change the initial conditions ok, but total energy is the same, random start. So, this will give you some u_x and u_y, this will give you u_x u_y at a given time. So, all of them will give you velocity field. Now, take velocity fields u_x and u_x of **u** (**r** + **l**) for each box at given time, then average ok. So, this is at given time, but from different-different systems and this call ensemble. Ensemble means many-many systems, is that ok.

Now, if it is homogeneous then what do you think?

So, what we will depend on, it cannot depend on x and y together.

On both x and y, it will depend on the difference between the two, if this position were shifted by let us say I take it here, but I translate them just translate. So, translation is not changing the direction these two points I just translate it, in any direction, but do not change the orientation.

Student: Yes.

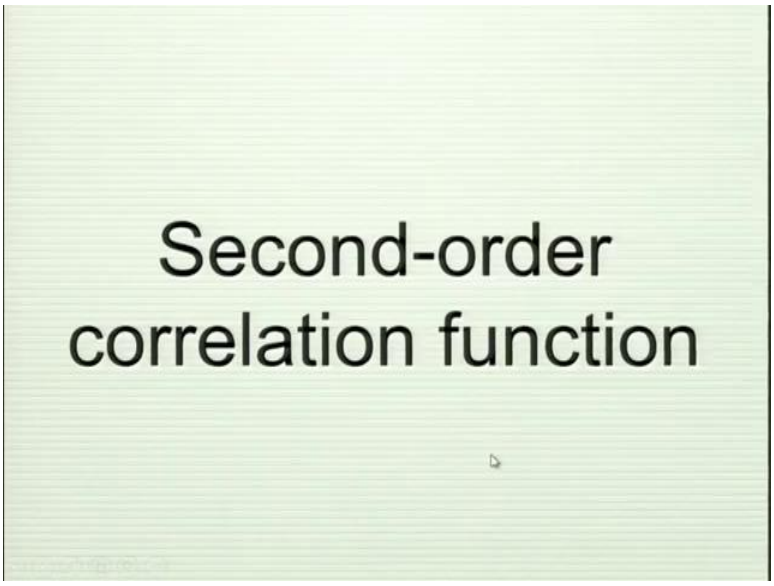
So, it will depend only on the difference between the two. So, this becomes for homogeneous, this is C_{ij}, it does not depend on x and y separately. It depends only on vector **l,** vector **l** nah it will depend l_x l_y l_z. So, this is homogeneous. So, first put homogeneity now, of then you put isotropic on top of it I want homogeneous and isotropic both. So, what do I do?

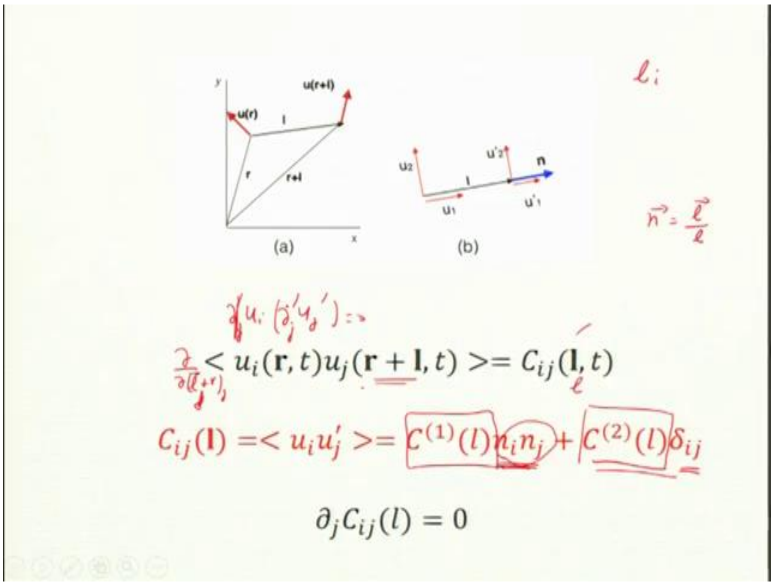
So, it says that not only this l but this l by keeping the distance equal if I rotate it so, this point is fixed but my second point is rotated here, I should get same correlation. Of course, it is a tensor. So, my components will change, but I should get well I mean, I am going to show you. Now, that it should be only function of l, it cannot depend on direction. I will be more specific in the next slide after this. So, magnitude of l and if it is stationary then at also will disappear it is only function of l. C_{ij} and l is magnitude of vector **l**.

Yes.

Student: If it is stationary, then we do not have to do all ensemble average right, we can take into.

Yes, stationary then you can take different times of the same system and that is what we typically do in simulations. You assume time average as equal ensemble average ok. So, this is what is assume by Kolmogorov ok.





So, I am putting the same picture again and this is same, but now, I put vector **l** here and isotropic means it becomes small l my isotropy remember, I said it is an isotropic. So, my second rank tensor has this is independent of direction of l.

So, if I rotate it I get the same thing so; that means, n what is vector **n**? Vector **n** is unit vector **l**/l this is my definition. So, short hand I do not want to write l_i l_j I write n_i n_j. So, this is what I motivated know so, isotropic tensor.

So, since it is does not depend on well basically, if I rotate I get the, my system should be the same so; that means, it is n_i n_j and δ_{ij} for vector, vector is easy to see it should be just l_i, because it depends only on **l** vector, but second rank tensor will be l_i l_j. This is coming from l_i l_j plus δ_{ij} ok. So, this is my motivation y z isotropic since C_{ij} is isotropic.

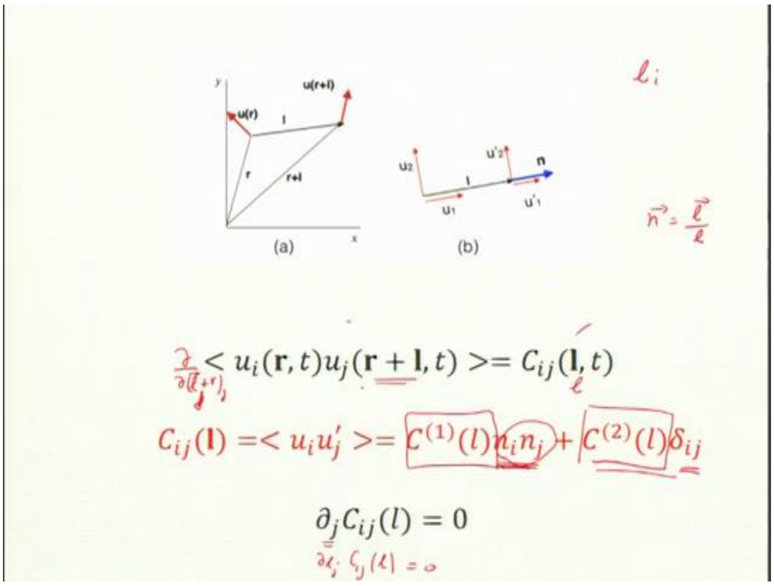
Student: Right.

Is not a same number for all i j; that means, it is scalar, but since it has a tensor it has to be of this form and this $C^{(1)}$ and $C^{(2)}$ is only function of magnitude l. If I know the magnitude, I know this.

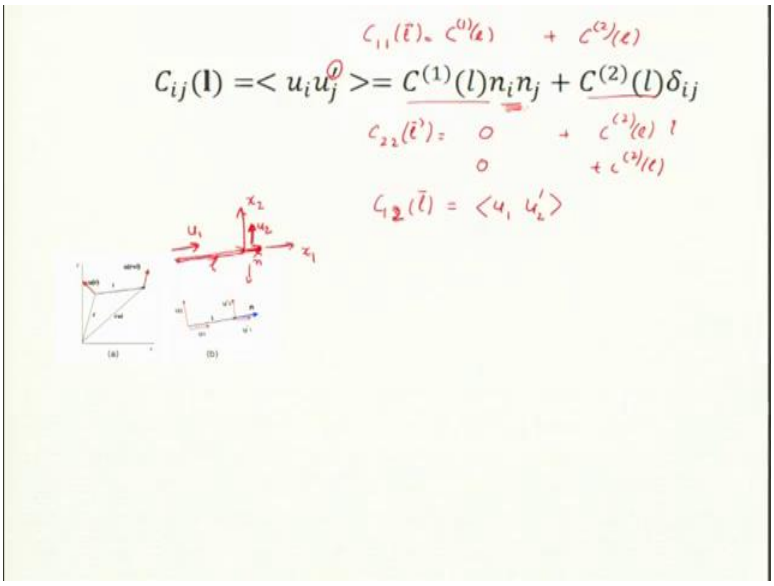
Now, I will make it more specific. So, this is a form and now you can see the beauty of this, this expression. So, we will go to the next slide ok. I one more point since **u** is incompressible. So, if I take the derivative of this with related to well I can take with related to l vector l_i l_j sorry, l_j. So, it will act on this one, it will not act on this right and since, I am doing over l_j summing over j. It should be at 0; yes or no?

So, these implies this equal to u_i u_i, I am going to call it prime because (**r** + **l**). So, this is a notation going to come very soon. So, (**r** + **l**) we will denote by prime we should writing (r + l) you write prime. Now, derivative with related to l_j is same as with related to r +l_j is a second variable which I am taking derivative∂_j'. So, u_i is a constant for it. So, it basically we acts on this is l_j and this is 0 ok.

So, incompressibility on the velocity one of the velocity, if I take the derivative divergence I should get 0. So, I am taking divergence on this if I do derivative with l now finally, by the way the point is when I do the ensemble average r disappears, r does not appear here. So, it is only function of l. So, I take the derivative with related to l, r is not there anymore.



So, that is why this is, when I say this I means ∂_{lj} $C_{ij}(l)$ this is 0 is that and I need this condition. So, this is helpful, I am going to only state the results what we derive from it, but we use this for our algebra ok.



Now, I made this picture smaller. So, these are my same equation C_{11} C_{21} n_i n_j. Now, I choose my coordinate system like this. So, my **n** vector of course, it is a long l right n is vector **l**/l. So, this small picture everybody see that is a two timing or.

Student: Ensemble average.

Visible, so, this is vector **l** and **n** vector will be unit vector along that fine. Now, choose my coordinate system, this is x axis x1, this is a y axis x2 and third one is towards me, above the page, the x1 x2 x3 is a three component. I choose local coordinate system with x1 x2 x3, x1 is along n.

Is that clear?

Student: Sir, C_{ij} is a function of l it is; that means, it is no longer isotropic right, because it is still a function of the vector **l**.

So, you are right. So, I can make it **l** small l.

Student: Sir, that is small l; Ok, but I by the way here these are function of l actually. So, it strictly speaking i should put keep l.

Student: No.

No, I should keep l actually. So, the l is coming here. So, it will become clear in a minute ok. So, I am taking the component in and see what happens.

So, x1 x2 x3 now, I am going to substitute that particular component. So, if I look for correlation C_{11}. So, actually let us let us do the algebra before I put my results C_{11} or vector l, what is it?

So, C_{11} is along x direction or along x1 direction. So, it will be C_{11} and what about n1 n1; what is the n 1 n 1.

What is n 1?

Student: 1.

1.

The n2 is 0 and n3 is 0, it has no n has n is a long vector.

Student: x.

x.

Student: Yeah.

So, x component, it has no y component, z component right. My **n** is along this direction. So, this become both of them, become 1, what about this one.

Student: Delta.

So, δ_{11}

Student: 1 is.

Is 1 ok? So, C_{11} is $C^{(1)} + C^{(2)}$, what about C_{22} ?, So, $C_{22}(l)$.

Student: $C^{(2)}$.

So, C_{22} what is this guy, n2, n2.

Student: 1, 1. n2, n2.

Student: 0, 0.

So, n2 is 0. So, this part is 0; what about this one?

Student: It is C_{12} there.

This is 1. So, $C^{(2)}$ will give you only $C^{(2)}$ of C_{13}.

Student: $C^{(2)}$.

So, C_{13} is coming towards u it is just, these again 0 and $C^{(2)}$ of l, because a isotropy.

The correlation so, it is a vector **l** isotropic means in the perpendicular plane, it should be same. So, along x2 and along x3 my correlations are the same. So, the two vectors, know this is vector here, velocity vector to velocity vector here. So, I am taking the component correlation my velocity like this that is C_{11}, my velocity like this it is C_{22}.

Student: Ok.

Velocity like this is C_{33} is that clear and C_{22} and C_{33} must be equal. What about C_{13} or C_{12}, what is this?

Student: 0.

Why is it 0?

From the formula it is 0, but can you argue why it is 0? It should be 0 for isotropic tensors.

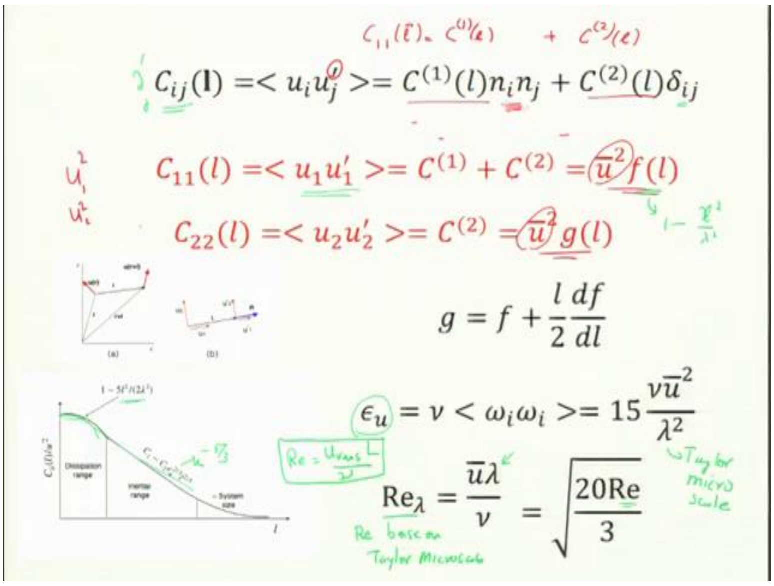
So, look this velocity along these direction **u1** velocity along these direction, I want u2 ok. These are correlation of C_{12} means u1 u2'.

Student: Ok.

Now, prime I have already introduce the notation nah prime means r + l, short hand otherwise it just too much to write. So, now, if u1 is fixed well at a given time what is the probability of finding u2 upward and u2 downward, half-half is so, it is random. So, sometimes u2 will be like this, sometimes u2 will be like this, that is why this u1 u2' prime is 0.

So in fact, any of these things, we have velocity like this either scalar or a vector in the perpendicular like scalar multiply, because that can take both positive and negative value you will get 0 or a vector component oppose in the perpendicular direction that can take this and this 0. So, all the cross terms are 0 is that ok. So, these are property of I mean these are all captured in my formula here, this formula by the way these true for not only this choice of coordinate system, but true for any coordinate system, I choose. I have to just transform by rotation, I could a chosen my x coordinate system to be like this like that in fine, just to do the transformation.

So, this n2 need not be 0, if I do like this n2 will not be 0 right ok.



So, this is what I have in my stuff. So, this $C^{(1)} + C^{(2)}$. So, I should so, this $C^{(2)}$ now, we write like this u^2 f(l) and u^2 g(l). So, u^2 is the u_1 u_1 multiplication at the same position ok, this is just u_1 u_1. So, $u_1{}^2$ it is $\overline{u^2}$. Should it be same as $u_2{}^2$?

Student: No.

Yes or no, why cannot I say, that is isotropic.

Student: Yes.

So, u_1^2 must be same as u_2^2. So, is isotropic. So, take component of velocity magnitude in that direction.

Or y direction or z direction should be same. So, that is why I have the same thing sitting here and sitting there fine. Now, so, this one if I remove this magnitude u^2 then this function will be dimensionless and for l equal to 0, it becomes 1 right, because u_1 equal to u_1'. So, f (l) is one for l equal to 0 and it decreases. In fact, correlation will decrease when you increase l right. You go away from the same point correlation expected to decrease.

Student: No.

Is highest correlation? So, it is maxima at l equal to 0 ok. Now, this part I am skipping, this derivation I am skipping. This one function you can expand it. It is $1 - \frac{l^2}{\lambda^2}$, it goes this form and g also goes with this form. Now, are f and g, are they related to each other?

Now, they related, because they are constrained that $\partial_j' C_{ij}$ is 0, if I take the derivative of this with related l I should get incompressible condition k can and I will give it a relation, which is there in the notes, I will skipped. It is quite a bit of algebra and this gives a relation g and f are related like this.

So, take the l derivative. So, this is a derivative of l and we can plot both f and g, but I plot it together. So, C_{ii} is the contract it, it should be so, these becomes 3 right n_i n_j sorry, n_i n_i is what 1.

Student: 1.

So, n_i n_i is 1 and this gives you 3 δ_{11} δ_{22} δ_{33} ok. So, this part I will skip please, look at the notes and do it. So, this is C_{ii} by u^2 and it has this part is parabolic $1 - \frac{51^2}{2\lambda^2}$ it comes by algebra.

Now, this part goes $l^{2/3}$, it is from Kolmogorov theory, because it is five-third will if you do the Fourier transform you get two-third. This I will do in the next class and it vanishes finally, for very large l. So, there are two important things is parabolic and this is a power law.

These from Kolmogorov k minus five-third and this is by maxima at l equal to 0. Now, this lambda is very specific it has specific name is what Taylor micro scale ok. This is coming from this definition. So, lambda is called Taylor micro scale and that is connected with the dissipation rate.

Now, this is also quite a bit of algebra. So, I am connecting ε_u by vorticity-vorticity given correlation function, I can compute vorticity correlations and that gives you this it is doable, but you have to just dig in and you have to do the algebra and once, I know lambda from dissipation rate and viscosity and u again computed Reynolds number fine and Reynolds number that is called so, this is lambda is called Taylor micro scale and this is called Reynolds number based on Taylor micro scale. So, this is multiplied by lambda$\frac{\bar{u}\lambda}{\nu}$. So, instead of system size that is Reynolds number.

So, Re, remember what is Re? R e is$\frac{u_{rms}}{\nu} l$ there is definition of Re yes or no, but Re based on Taylor micro scale is $\bar{u}$ which is not u_{rms} $\bar{u}$ and this $\bar{u}$ and u_{rms} are different ok.

Now, this I will again ask you to look at the notes. So, $\bar{u}$ is just a pre factor of λ/ν and that is connected with Re_λ by this is $\sqrt{\frac{20Re}{3}}$ ok. This can be shown. This is there in my notes. Now, this is about second order correlation function. Now, we are long way to go on, I mean there is we just do is that any question on second order correlation function.

So, we have f and g now, C_{11} C_{22} and so, their second order correlations we can compute given velocity field. It is isotropic. So, it depends only on l but remember, it is not a number. It is a tensor.

So, we have C_{11} and C_{22} they are different right, they are different here f and g are different functions. So, it is not a single number, but these are beauty of this tensor which captures all of it in one formula ok, respecting isotropy. So, it tells you the $C^{(1)}$ $C^{(2)}$ $C^{(1)}$ $C^{(1)}$ comma C_{12} is 0 C_{11} is number, C_{22} is same as C_{33} all that follows from here.

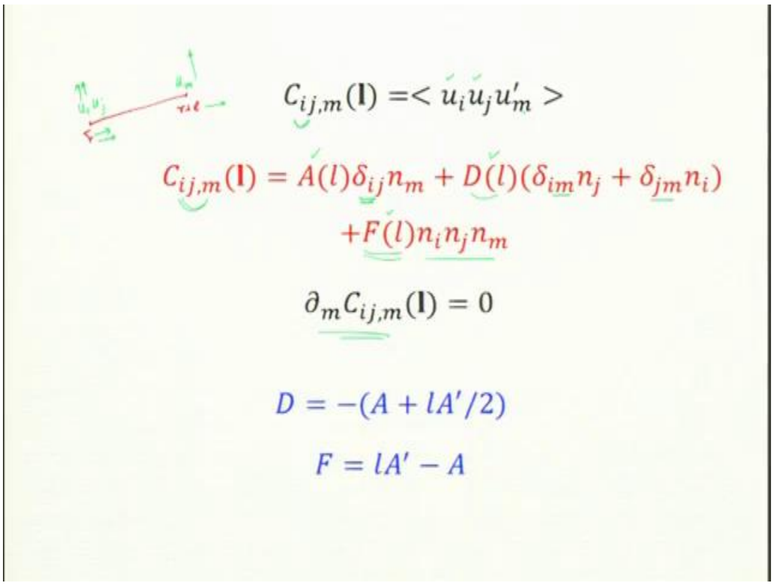
Student: Sir, this correlation is always 0, only is this applicable in homogeneous isotropic.

Homogeneous isotropic, anisotropy not all; some of it will go through but not all ok. So, with magnetic field this will not work; ok so.

Thank you.

Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 31
Kolmogorov's 4/5 Law Isotropic Tensors & Correlations



So, we have three velocities, but we only have two positions, so not that the three different positions you have please recall we have **r** and (**r** + **l**). So, this first group you are using. So, there are three indices i, j, m. So, first two are at position **r** u_i, u_j and third one is at prime is a second point.

So, it is a coalition of three velocities, but at two different points important. Now, this comma separates this comma is separating the **r** with (**r** + **l**) ok. So, this is the notation. Now, prime is the second point. Now, again we can so, we discussed how to construct second order tensor, but now we can do third order tensor with this here only function of l. So, given l you can consider or **n** is non-dimensional, **n** is sorry **n** is a magnitude one unit vector.

So, you can easily make a guess n_i, n_j, n m right for l 1, l 2, l l i, l j, l m ok. Given the delta function how do I construct δ i, j, m. This is three indices right i, j, m, but here is the ij in the left, ij in the left ij has come here. Now, I can also have im; I can do this one im but then j will be here, but ij are symmetric right ij is i and j are the same platform, but m is a different thing. So, if I have im then I must also have jm. So, that will be jm, i n_i ok.

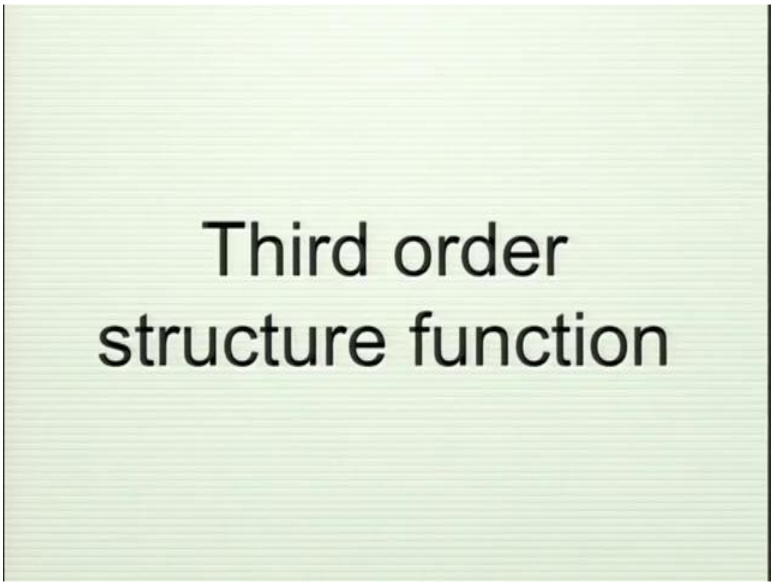
A(l) and D(l) are the coefficients which are and F(l) are function of only F(l). A(l) and D(l) are not equal, why? Because it is asymmetry in ij with m, is that ok? So, this is how you construct higher order tensors. So, this is the third order correlation function. Now, my unknowns are A(l), D(l) and F(l). Now, for second rank tensor I had this nice interpretation velocity; velocity like this. Now, third rank is not so.

I mean of course, we have two velocities here and third velocity here two velocity is here and third velocity here like that. You can consider there are more combination known here I mean that has the only 3; well 6 components with 3 of them being 0. Why it is 6? Why I did not say 9?

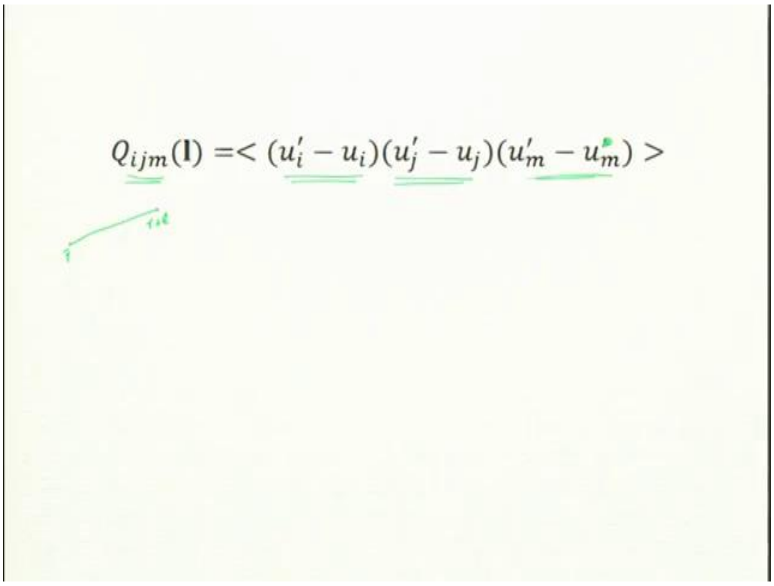
Student: Asymmetric.

Asymmetric ok, here I have 27 components some will be 0 and so on ok. So, we can construct. Now, this is I can also now do the derivative incompressible condition. So, incompressible condition will act on prime when you do well. So, that is why I have acting on m ok.

Now, that will once I apply the same in condition on this $C_{ij,m}$ do algebra then we will get some relations and these relations are the following it tells you how D and F are related A and A'. So, I can write this D and F in terms of A. So, instead of having more variables D and F, we can write it in a way here using this condition. So, basically A and A(l) so, the $C_{ij,m}$ can be written as the A m and A' which is quite convenient ok.

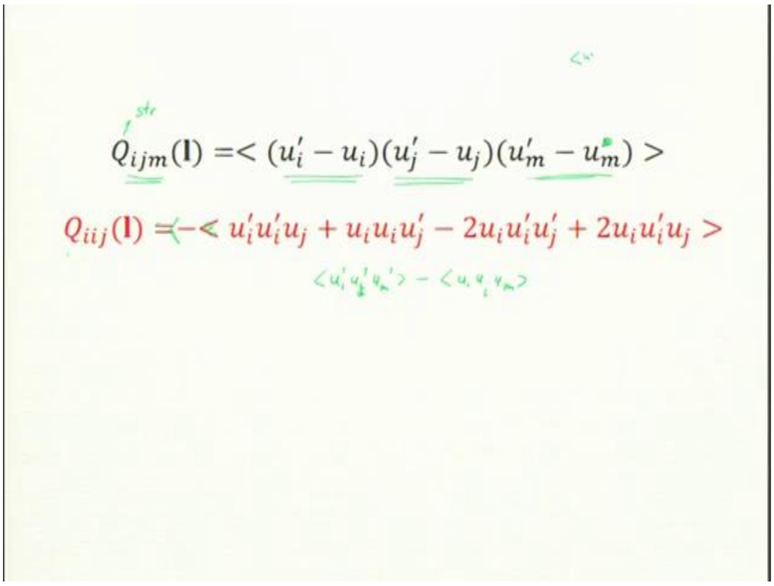


So, this is our third rank correlation (u, u, u). Now, I define third order structure function.



Now, what is it? Is a difference in the velocities? Now, please remember I have this two position **r** and (**r** + **l**).

So, I have differential in velocities. So, this is along i component, along j component, along m component I made one mistake there is no prime here fine. So, is the difference of the velocity, but I am taking the components along i. So, i is 1 means x component, 2 is y component. So, it is again three indices object, but now it is there is no comma there is a difference is same for all of them there is no comma and I am my notation is the Q and that was C; Q is structure function. Now, I can expand this in terms of correlation function. So, what will I get? I get prime, prime, prime. So, I think I am not going to write.



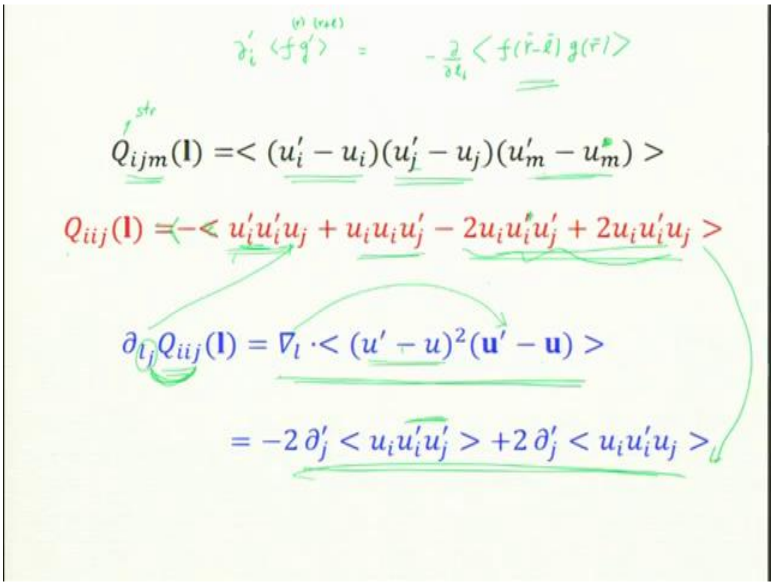
So, this is how the four terms are. So, how do I get this term two primes and third is?

Not prime here or two here and one prime ok. So, I made one mistake this correlation should be outside this one should be outside. So, we can get this stuff. So, you see 2 primes then third are un-prime no. So, I made one more change j equal to i, I make j equal to i if i. So, I make j equal to i and expand. So, it is iij. So, it will have all the combination three prime two primes and one un-prime, all no prime. So, this is how we get now this you can see it yourself I will not go deep with the details; except that two terms all the three prime terms is not there right.

So, I do not have the term of the form $(u_i' u_j' u_i')$, where i equal to j and minus u_i, u_j u_m. Now, since i equal to j, ii. These are not present in this. Why is it not present?

Is 0 because of homogeneity because if I do the triple product and that should be same with two different positions like you know average. But what is the average of u? u_i average.

What is the average u_i? 0 is a random with no mean flow average velocity 0 at the position and this is this should not be 0 is 0 for Gaussian probability, but need this will not be 0 triple correlation, but they cancel because of homogeneity ok. Now, this is what we are left with this average ok.



Now, we will do one more step, I take the derivative of this with respect to l_j. So, what do I do when I take derivative what happens with this term? This term goes to 0 because I take the incompressible condition and I forgot to tell you that this also will be 0. There is a proof and the proof is kind of nice proof. So, I have f and g, g prime. I take this with relate to prime.

Let us call it ∂_j derivative ok; f g' this one. It is a function of the function of l. So, I take on this now you can write this since it is homogeneous I can write this is f(r), f(r + l) I do not know green is not visible properly or.

It is ok? Because there are too many red I have written. So, I would not do. So, this can be written as (**r** – **l**) into g f(r) I shift left. So, this is r I just shift left by **l** and now I am taking derivative with related to l. So, this is (**r** – **l**). So, I cannot take derivative with g now, I take related to the f. So, it will be minus sign with l. So, minus sign comes because it is shifted to the left. So, if I take the derivative the first object and put a minus sign. Now, here I want to take with l_j.

So, that means, I want to take derivative on this one right there is j that one. So, this is at I make this minus sign. So, the minus sign comes here this minus, minus becomes plus and this goes to 0. So, so this is properties also used quite often. You want to shift the point then you are to use the minus sign and finish it off. Now, we left basically with these two terms and these two terms equal to that ok. Now, this proof I will again well. No I will.

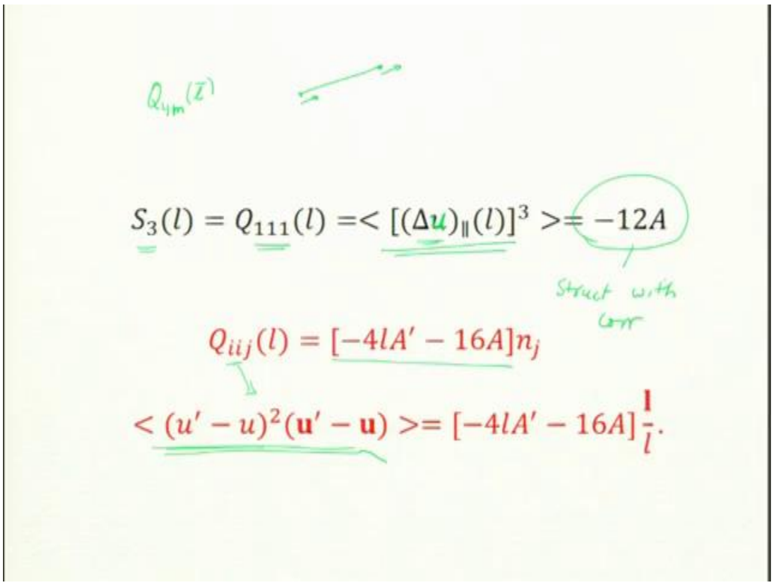
This so, these two are gone because of the incompressible condition you are left only with this and this is what is written here, is it ok? So, the derivative has come here I put a prime put a prime put a prime this is plus prime, prime and this should be this is this prime is acting on these objects is acting on these objects by the way please, note that this j prime is this prime was not there it will be 0.

If I take the derivative if this prime was not there, it would have been 0; this prime is there which is making it nonzero this part. There is some subtleties which you will do it yourself then you will find that Kolmogorov was indeed a very clever and he could do all this stuff.

Now, so, what is definition of Q_{ii}? Q_{ii} is j equal to i here j equal to i. So, that makes it square know? I am summing over i. So, this becomes $u_i' u_i^2$ that is $(u'-u)(u'- u)^2$; repeated indices are to be summed. So, this is a ∂u_i delta u i sum. So, this is coming from here and this is u' m u' m but I am now taking the derivative with related to same object; that means, it is a divergence.

So, that is why I put $\nabla_l(u'- u)$. So, this divergence acting on this, but it is acting on because this is a prime, here it acts on all three. So, this l is acting on all three not only one but I am just writing a shorthand, it is a nice vector notation. So, because of this was I am taking the derivative of u_j ∂j this one so, I write like that ok. So, this is shorthand notation which is convenient notation ok.

So, what I described very quickly third order correlation function and third order structure function and this object will appear in now Kolmogorov four-fifth law with this you have to remember this one.



Now, I need one more step which I will quickly go through S3. So, we are all describing only S3 but third order structure function, but this is very specific. So, this was that was Q notation I had put $Q_{ij,m}$ (l), but now I am I am defining S3. So, then sorry I must say is a new function I am defining S3 which is i equals to j equal to m equal to 1. So, do this one. So, 1 means along the **l** direction. So, I am choosing the same notation x axis is along.

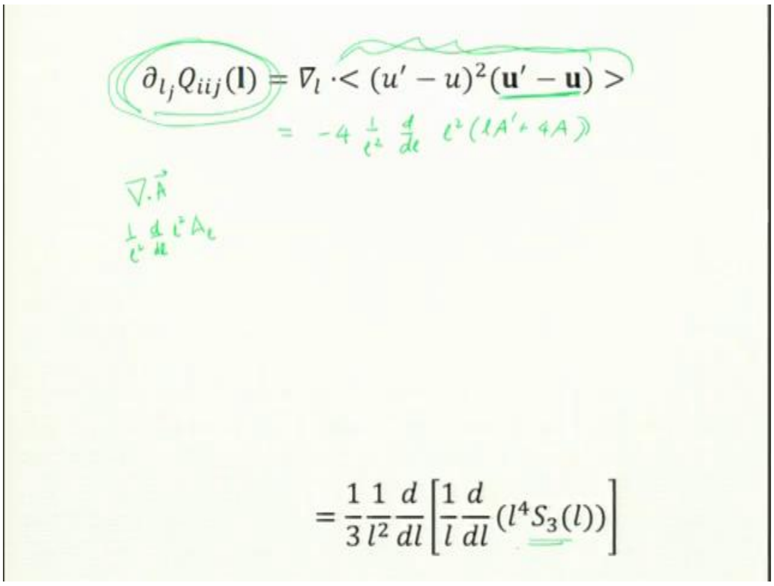
Student: **l**.

l direction. So, this will be Δ **u** I delta I should really put a vector **u** Δ **u** component along parallel direction is that fine. So, I have these two positions. So, velocity component along that duration is difference Q bit. So, this is what remember I started my first slide Kolmogorov basically is relating this object with the dissipation rate. So, we can define using the third order structure function but you see it is very specific my components are 111.

Now, it is connected with A, A is remember third order structure function; third order correlation function at ADF. Now, this I will not prove it but this is equal to minus 12A because see the connection. Structure function is connected with the correlation function and correlation function is function of A. So, that is how it comes is minus 12A. These are connection of structure with correlation ok. This I will not be able to do it I mean this too much is required here.

Now, Q_{iij} is again related with A; an A prime it is like this. So, you have to plug it in because and replace d and f by the formula and you will get this ok. Now, I am just giving you the steps you have to work it out yourself. So, this is one is basically this object know Q_{iij} is basically is just that I just showed you. So this is square coming from ii in this.

Now, remember S3 is well not S3 then in the earlier slide there is a divergence of this of this object, yes. So, you have just seen the divergence of this. So, if it takes the divergence of that which is very useful quantity which I am going to come to it with later.



So, I take the divergence of Q_{iij}, now I think I wanted to do it, but I will skip the algebra if you do the algebra dot, dot, dot, because this is only bit of algebra and you will get this one. So, this one is related with S3 S not I mean look this is function of is quite I can just tell you $-4\frac{1}{l^2}\frac{d}{dl}l^2(lA' + 4A)$ ok.

So, this is coming from the previous slide you know I am guess feel I am sure you people are tired and you can see that this object is $(-4lA' - 16A)\frac{1}{l}$. So, I have in the earlier slide I have showed.

So, this is these I computed from this is third order structure function ok. Take the divergence of it. So, divergence will be in spherical polar coordinate. Now, this I want to see emphasize is spherical symmetry. So, what is divergences spherical coordinate, divergence of vector A?

It is $(1/l^2)$ (d/dl) l^2 $(A'l+4A)$, Al component. It is only Al component ok. So, that is why this is a along l direction yes or no? u', u is along l direction, isotropic ok. So, this is what has you we get I think I am losing I mean everybody is losing interest I think. Anyway let us finish it off. So, this is a connection of this object which I need in my derivation to this structure third order structure function ok.

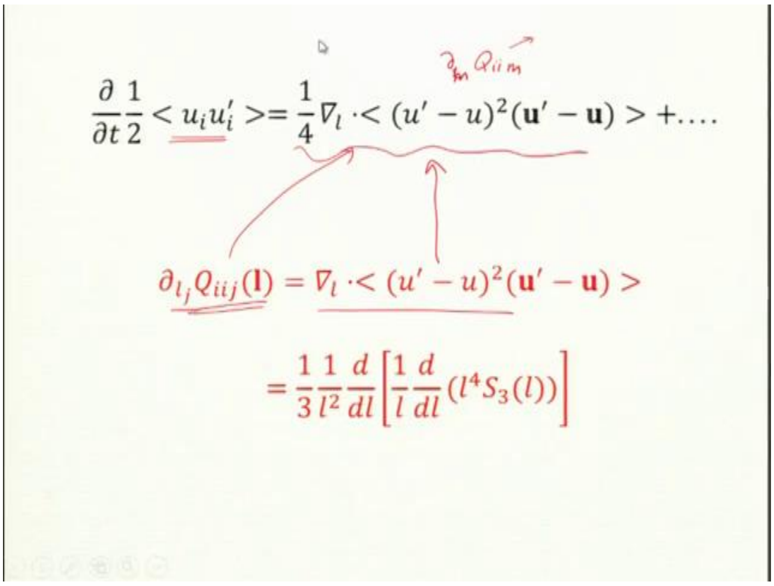
So, I stop with this and now we quickly go to the last lab ok.

Thank you.

Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 32
Kolmogorov's 4/5 Law
4/5 Law: The Final Step

Ok. So now, we are at the last step. We developed all the tools, second order and third order correlation function, a structure function. So, what I am going to do is the following. I am going to show the following.

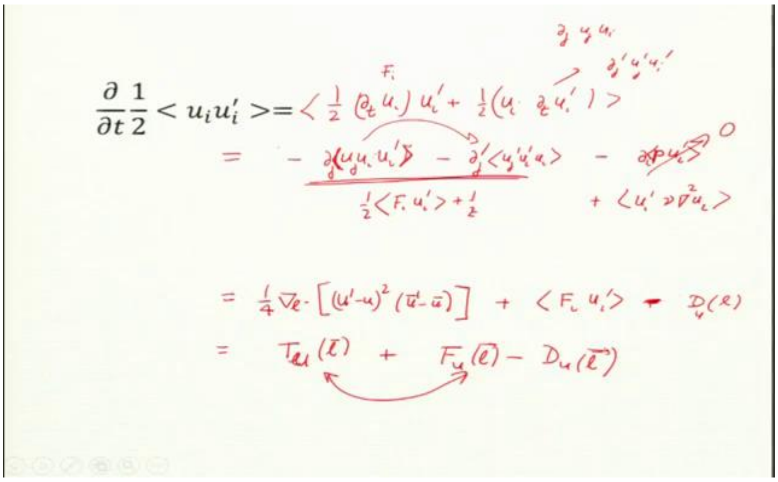


So, these are in layout. So, I am going to, now go through dynamics. So, write down the velocity from the Navier-Strokes equation using that time derivative of the second order correlation function u_i u_i'; earlier you remember that you are doing u_i u_i total energy or total. So, it was not at different point, it was same point but now I am taking two different points.

Now, this can be shown to be this is why we need all this stuff, the right hand side the non-linear term appears that like that u dot u term appear says in fact, structure function. This is $Q_{ij,m}$ ∇_l del m right is which I derived, this was there in my old slide and this is that stuff.

So, this can be shown, now this is connected with a third order structure function and that is the connection ok. So, this one is $Q_{ii,m}$ will which I had done this. This is equal to this one.

Except factor 1/4 but this was one-third this object. So, I will just map it to that and relate; so, is a dissipation rate and we are home ok. Now, let us fill up the gaps.



So, how do I write this time derivative? So, half so, $(\frac{\partial u_i}{\partial t}u_i' + u_i\frac{\partial u_i'}{\partial t})$. Yes, so it is a product rule. Now, what is $\frac{\partial u_i}{\partial t}$? It is $\partial_j(u_j u_i)$ right. Yes, in a ∂_j yeah what?

So, that is a non-linear term this is of course, pressure term. So, first term is that with a minus sign.

Student: Minus sign.

Minus sign, so one term, so, I am just going to just sketch it ok. So, u_j $\partial_j(u_j u_i)$, but u_i, so, this is operated ∂_j will not act on u_i'. So, I can push it inside right, where this operator it is only on **r** position not on **r** pressure position. Now, I am doing averages, so, this average will come here. So, this is the third order correlation function has come now ok, now that I relate to that order structure function which by the all the derivatives which have done. Now, the second term we just follow the same step. Now, derivative with here we just follow the same steps prime, but this will be on prime variable.

So, this will be ∂_j' because this is the second point u_j' u_i', now u_i will go inside because this thing. So, this is u_j' u_i' u_i' . What about pressure? So, d/dt is the minus ∇P so, minus ∂_jP, now u_i' can be pushed in because derivative acts on P not on P prime variable. So, it is u_j', same thing you derive for the other one. So, they have come in pair with prime switched and then the force as well. So, d/dt is right hand side is force F_i. So, F_i u_i' averaged half and half from there. So, plus half like that and this is the viscous term.

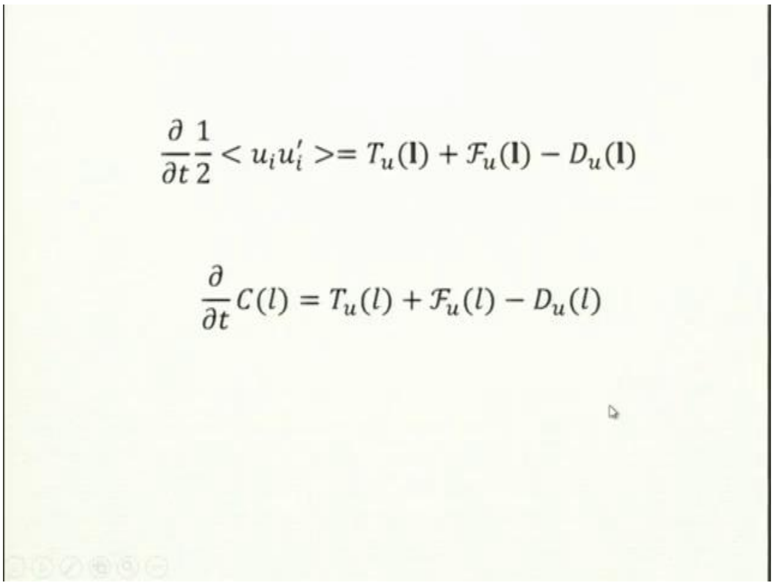
So, this is u_i' ν ∇^2u u_i average ok, is that fine? I mean these how; so, we write that it is a straightforward no problem. What about this one? Pressure with u_j' , this one.

Student: 0.

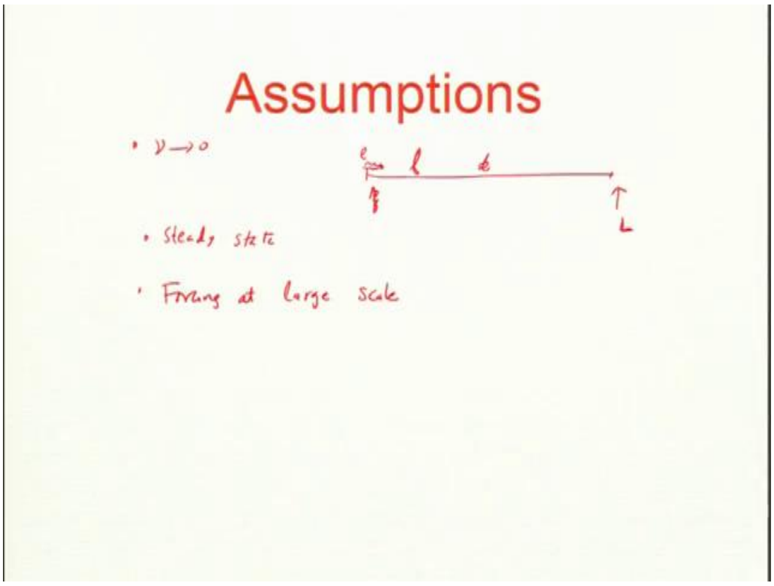
0. So, for given pressure my velocity will fluctuate plus minus plus minus it assigned. So, this goes to 0. So, pressure term does not contribute like our earlier derivation in incompressible flows pressure go with drops out gone. Now, these two by that logic that I can switch these derivatives these two in fact, can be connected together. I will skip this algebra ok. So, I will just write down the final step. So, you can relate this prime, these are exactly remember the third order correlation function and I connected the structure function.

So, you follow the step and this is $\frac{1}{4}\nabla_l[(u'-u)^2(\boldsymbol{u'}-\boldsymbol{u})]$. This is this non-linear term ok, this is just very few steps which I have in my notes and the second term is the half and half, a force in that equal F_i u_i' average. So, there are two forces coming they are equal and the viscous term. Let us assume viscosity going to 0. The viscous term is small in the regime of my interest, but right now, let us keep this viscous term; I will call it dissipation D(l).

I put a minus sign here following the notation. Now, this one is called $T_u(l)$ is a non-linear transfer term and this is coming as force and this is a $D_u(l)$. So, these how we connected the second order correlation, time derivative with the non-linear transfer term. This is that s if we call S^{uu} (kpq) and this is what is exactly similar to what we did for Fourier transforms. Now, we are almost there; so, we will make some assumptions and connect these two ok. So, I make one more step. So, these are in fact, function of vector l, since it is isotropic it becomes function of l only magnitude l. So, that is my next slide.

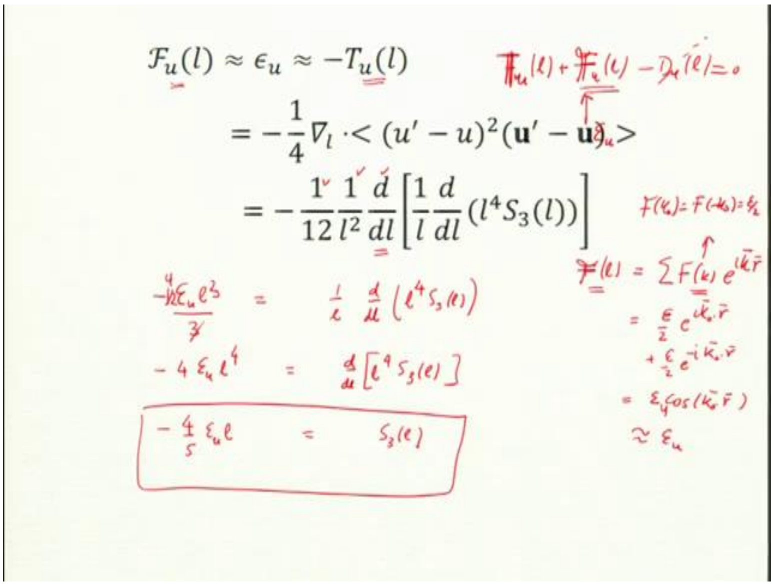


So, this is magnitude l. So, instead of writing vector l I just write l.



Now, the next step let us make assumptions. So, same assumption like before we make viscosity tending to 0 ok. So, we are making the flow very turbulent, but we have very strong a very large inertia range. So, I have forcing somewhere here and I have l here that forcing is large scale no; so, we are looking into it. So, if l is increasing I have l here and viscosity is very small l. These I forced my I have to flip it, because the fourier space I forced at small scale but I am forcing it very large scale.

Steady state what happened to a steady state; d/dt is gone. So, set d/dt to 0 and assume the forcing at large scale forcing at large scale, the force set very large scale ok. So, these are assumptions which we made already before I make the same assumptions, the same theory what is in real space ok.



If we make that assumption then d/dt term is gone. So, if we remember so, d /dt is gone. So, T_u (l) plus F_u(l) math cal actually because, this is force times velocity minus D_u(l). I am interested in the inertia range which is bigger than whose length scale is bigger than the dissipation range. So, viscosity is very small-scale dissipation range. So, go to inertia range because intermediate scale. So, this term is negligible, dissipation is effective only it small scales. So, this term is gone. So, we get you relate these two will with the minus sign.

Now, what about this F_u(l)? What is it equal to? This is force injection net any position, if you look at it. These equal to energy supply rate is same at all positions approximately. Now how do I see this? I have argument but we can write this as I think I have to skip the argument. You just look at the proof. I think I do not want to make you more tired. So, this is equal to is constant everywhere approximately and equal to energy supply rate. So, this is a constant which is a ε_u (l), because the energy supply rate equal to dissipation rate so that means, minus T_u must be ε_u this equal to 0.

So, this must be equal to it right. So, this is how we connect dissipation rate with the energy term non-linear transfer rate. Now, you can see the proof is almost there. Now, T_u(l) is connected with third order structure function. Now, we can we can just finish off the proof. So, this is T_u (l) is minus one quarter of this one and this was I proved this one;

one I taken the already the divergence in the spherical polar coordinate. Now, how do I do it? They just straight forward, so, multiply with take that $(12\ l)^2$ to the left. So, let me do the last bit of algebra now minus ε_u $(12\ l)^2$ equal d/dl. Shall I do the integral again?

Student: Absolute.

Integrate the two d b/dl. So, so let us do the first integral. So, this will be l^3 / 3. So, I have done this one, this one and this one; this is the operations $(12\ l)^2$ went to the left and the d by dl integral is (1/l)(d/dl) l^4 S_3(l). So, what is this? This 4 right; now the one l is sitting here, again take it to the let because minus 4 ε_u l^4 is (d /dl) l^4 S_3 (l). Integrate this 4 by 5 right.

So, 4 by 5 ε_u l^5 is l^4 S_3(l) and now divide by l^4. So, this goes and ok. So, you can clap. So, these how we get S_3 (l), S minus Fourier 5 ε_u (l). So, I have say few more words, but if you want to ask some questions you can do it now.

Student: Personally at large scale but in absence.

So, let us see. So, so I can I can I can do it now. So, large scale forcing means you are forcing everywhere. So, what does it mean? Forcing everywhere; so, if I take this bucket and stir this stuff is forcing everywhere.

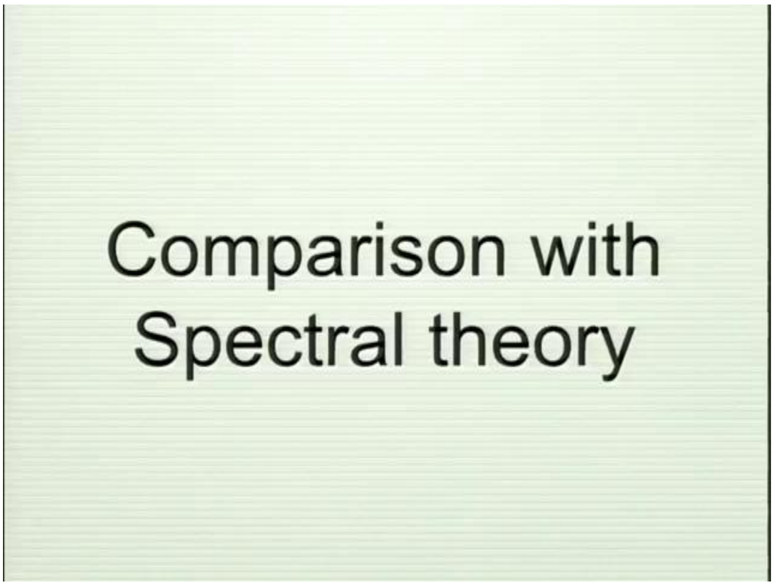
Student: In real space.

In real space, the Fourier space it is banned is small wave number, but is forced everywhere. So, if I multiply the velocity everywhere. So, large scale velocity versus large scale force it is affects everything, is like in the central government know; something happens it takes up is affects everything, in real space. A Fourier space is only the low band. Now, if you want to prove then it is also easy. So, let us let us do quickly F (l) math l is F (k) Fourier. So, there is a theorem that a real space correlation is Fourier transform of the spectrum, now this you have to believe me. So, this is the theorem you can relate. So, this is the correlation a two point.

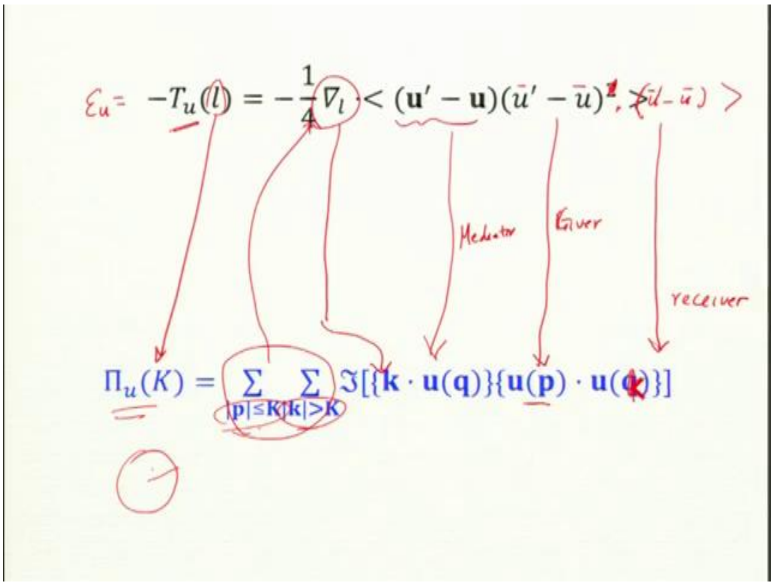
So, F_u equal to Fourier transform of this power spectrum E(k) dot i r, now this I force it small wave number. So, let us say I force it k naught, which is k naught being small. So, I put this stuff F($k_{o)}$ equal to F of minus k naught is ε/2.

So, the two wave numbers are forced, only two wave numbers not a shell. So, this will be ε/2 so, this is a constant. So, I get e to the power i k_o dot **r** fine and other guys also that ε/ 2 to the power i minus k_o dot r is there a minus k. Sum is only two wave numbers, sum it up what is ε/2, i k dot **r** plus 0 minus **k** dot **r**? Is cos k dot **r**; so, cos k_o dot **r** and epsilon the half will cancel with 2. Now, k naught go to goes to zero limit. What is k naught going to zero limit? Cos k_o will be 1.

So, this is approximately epsilon. So, these how you create ok, but it is kind of physically visible a force that large scale. So, it is going to there is visible at every scale ok. So, this is how we do the force space law. Now, just to complete the story; so, this is for the third order structure function ok.



I want to say how to relate this with the structural theory, our Fourier space theory. So, it is the connection is the following.

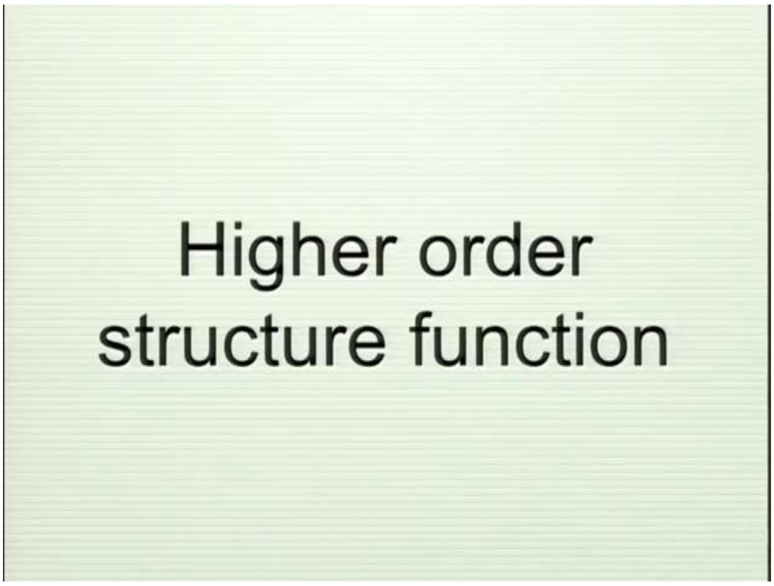


Now, remember I had the flux $\pi_u(K)$ is connected with the S^{uu} sum over. So, flux of wave number K is giver is inside and receiver is outside right; this all the stuff, giver more is inside the wave number sphere and receiver is outside when wave number sphere and this is equal to π_u (K) in the inertia range, this is ε_u.

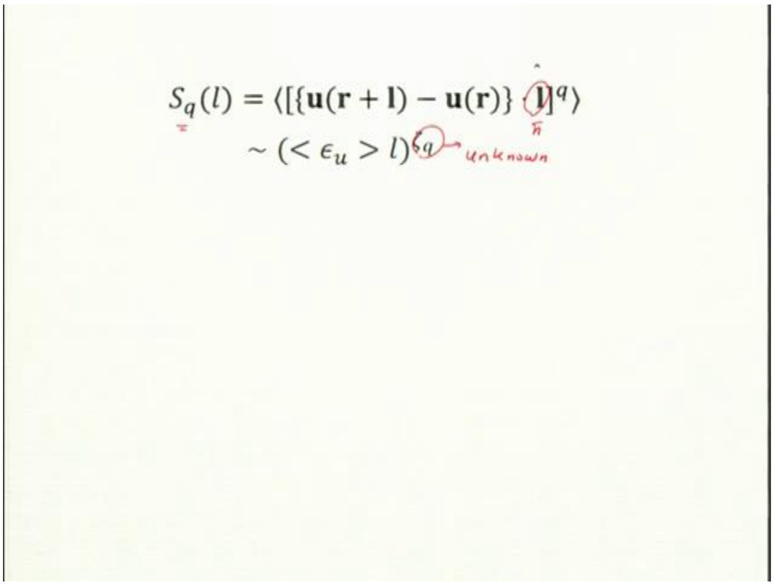
So, what is the connection? So, I write this u squared is a product dot product of u prime minus u vector. Now, beautifully this is like a mediator and this is mediator which is connected with that, we did with u^3. Now, this is product of the same function we said ∇u know. So, one of them is a giver and one of them is a receiver, giver receiver.

So, this giver sorry giver I made a mistake no, this k, receiver and this is the mediator u prime in fact, is mediating and there is a divergence is coming as this k vector of course, this is summing over all; this is in real space I am summing really because all the Fourier modes. So, is implicit that this sum is sitting here, take divergence basically we are doing some kind of sum and this l is connected with 1/ l, here is connected with K 1 by l is K.

So, these are connections in fact. It is in fact, is quite nice that there is, it must be related and this relation is visible ok. Now, this is for third order structure function. So, epsilon e is connected with u delta u delta u delta u, in this object. What about higher order?

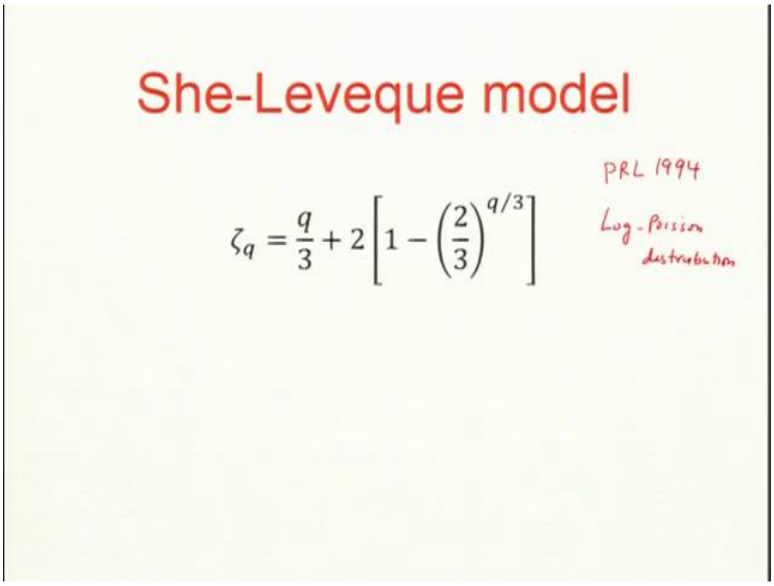


So, this has been a puzzle for in physics for many in fact, 50 years or 60 years.



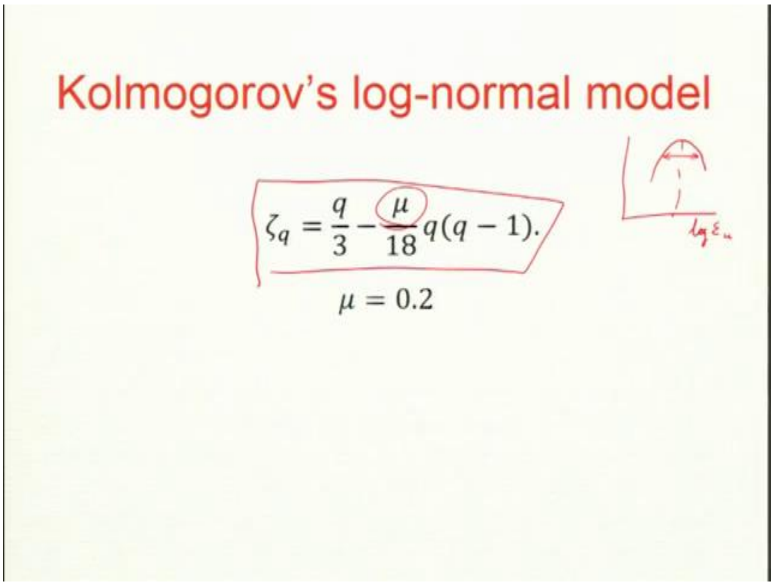
So, we can take the projection along again. So, l cap is along l direction or in fact, is as **n** is same as **n** ok. So, take the cube qth order structure function, S_q is q order. So, $(\nabla u)^q$, q equal to 3 is what Kolmogorov showed, but if q is let us say 4 or 5. What happens? So, it turns out these nobody has even to get a close form formula from the first principle. Start from Navier-Strokes equation we cannot get 2, q not equal to 3, q equal to 4, nobody can do it; nobody has been able to do it ok.

So, this is by dimensional arguments it is ε_u (l) zeta q, this is unknown and this is called exponent of the structure function. Now, I put average because ε_u is fluctuating ok; what Kolmogorov says is a kind of average epsilon u.



So, we use a epsilon u and idea is to estimate or obtain ζq. Now, I will not discuss in this lectures but ζq was proven by She-Leveque by some assuming that this dissipation rate is fluctuating and it has a log Poisson distribution ok. Now, I will not discuss this, but you can look at the original paper, distribution is physical regulators 1994 ok. Now, if you make certain assumptions, you find s function is a function of q.

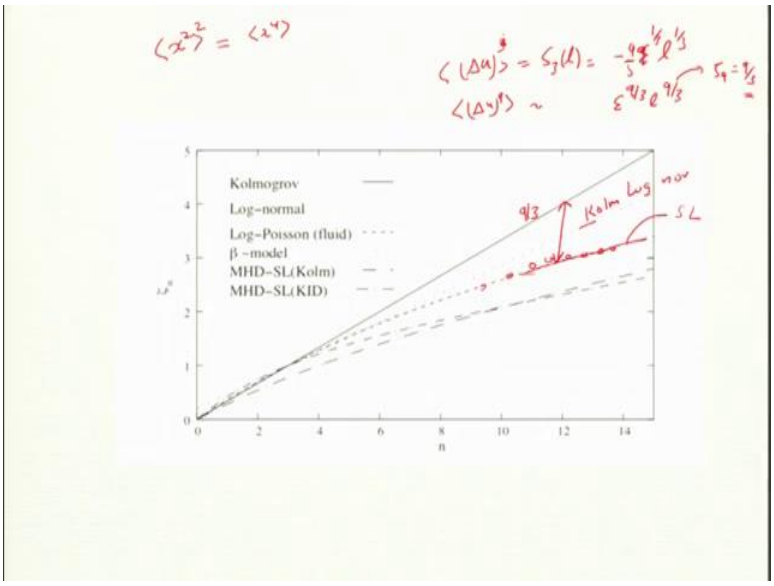
Now, it turns out this fits with the data quite well but it is not starting from first principle or you cannot prove it from this is analytically proven. It makes certain assumptions of this distribution of dissipation rate and it seems to work quite well.



Kolmogorov himself had given a theory that the dissipation itself log normal. So, normal distribution is you know, I mean there is a peak. So, normal distribution will be ε_u, so, this is probability y axis is; so, this one I can explain a bit more ε_u. So, it peaks at ε_u with certain distribution; had it been Gaussian if it is on linear scale it has a certain width. It is not a given number, but it has certain width right, there is Gaussian like heights of human beings are not same height. It is a average and this is the distribution is supposed to be Gaussian, but log normal means if I take the log in the x axis; x axis or y axis?

Student: x.

x axis know so, x axis log ε_u then it is normal. So, instead of taking ε_u you take $\log\varepsilon_u$ in terms of it is Gaussian ok. So, there is a log of normal distribution and Kolmogorov argued why it is so and he postulated that the formula well from this ζq comes up this form and the data fits with nu equal to 0.2. So, mu is a free parameter here.



Let us look at how these things where, I forgot to say that if Kolmogorov's five-third theory or that $S_3(q)$ is εl know; what did I write S_3 (l) is equal to$-\frac{4}{5}\varepsilon l$. So, this is $\Delta u(q)$, but if I make a extrapolation Δ u(q) what will that be? Just my extrapolation, so, divide by so, take one-third order. So, one-third if I just like that, take $\varepsilon^{q/3}$ $l^{q/3}$, so, ζq will be q/3.

You got in to Kolmogorov formula, but these are makes lot of assumption; if you averaging you cannot take power in and out right, you cannot do that. I mean that is not for x^2 average and x^4, average this is not square of this right. We know this, this is this is you cannot say this is like that. So, this is not proper, this is only proper for some distribution, but not for all distribution. So, well we make some assumption we said. So, this line is q/3 line linear line, She-Leveque is this line, this line this is She-Leveque; a log normal Kolmogorov log normal if this line.

The data appears to be closer to this line but you can see that it deviates very strongly from this q/3. So, by the time it reaches 10, the difference is around 10 percent ok. So, for q equal to 10 this will be 3.3, but answer is around 2.4 or so ok. So, the difference is in experiment; so, these are experimental data and numerical simulation which is also there is also done some of it. So, we can do this data and we find that you are away from the linear line and this theory is still not done yet. We do not know how to compute ζ_q from some theoretical framework ok.

So, this is what I had to say about structure function. I will stop ok.

Thank you.

Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 33
Enstrophy Spectrum and Flux

So now we start. So it is 3D hydro only, but we will look at Enstrophy. Now, it is a useful quantity again not heavily emphasized vorticity is of course heavily emphasized but not as much as it. So, I will tell you few things which are new ok; you will not find anywhere else. So, enstrophy flux is new enstrophy spectrum is in fact trivial.

So, what is enstrophy I defined it before is $\frac{\omega^2}{2}$, $\boldsymbol{\omega}$ is vorticity. So ω^2.

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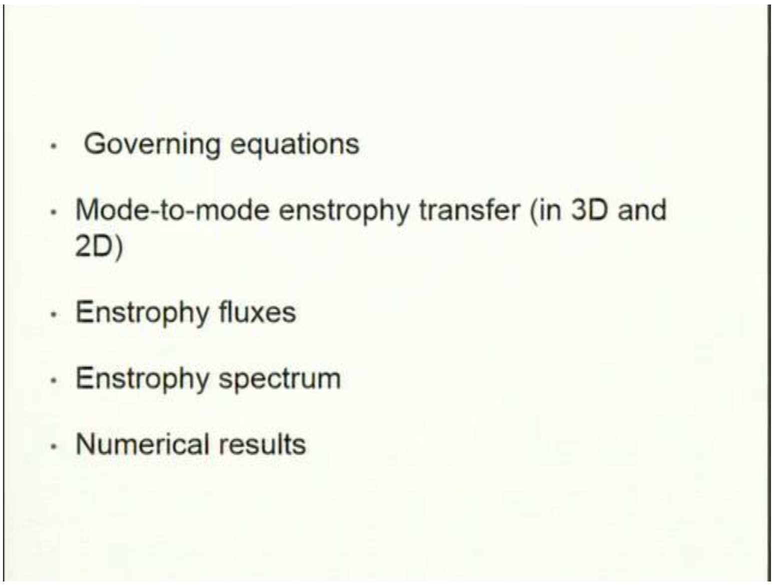

So, we will cover governing equations, mode to mode enstrophy transfer. So, you can generalize what happens for kinetic energy enstrophy transfers; it will fluxes spectrum is trivial which I will tell you and then some numerical results.

So, governing equation I will recap this has been done in earlier lectures.

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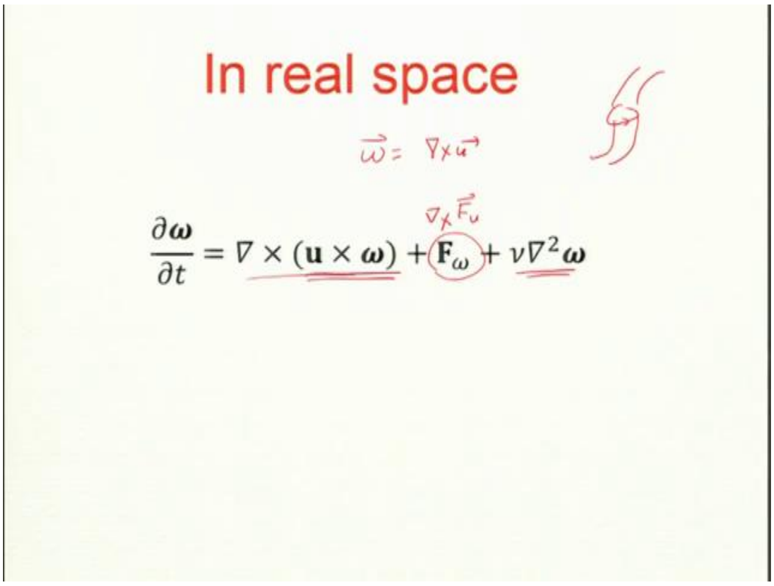

So, real space, vorticity is curl of $\mathbf{u}$ is a vector, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, we are working with 3D. we will come to 2D later, but vorticity in it is like a vortex tube. So, this vortex and vortex tubes.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \mathbf{F}_\omega + \nu \nabla^2 \boldsymbol{\omega}$$

Now, the equation for vorticity is $\nabla \times (\mathbf{u} \times \boldsymbol{\omega})$ ok. So, this is the form and this is coming from $\nabla \times \mathbf{F}_u$ and this a viscous term. So, instead of $\nu\nabla^2\mathbf{u}$, $\nabla^2\boldsymbol{\omega}$ ok. So, this has been derived if you look at your notes or look at the notes I gave you; so you find them. So, u dot grad u has become, an pressure has basically disappeared because of curl of pressure is 0, but u dot grad u has written is of this form.

So, in Fourier space; so flux means you work with Fourier space ok, that is best though there is some derivation of flux in real space, but Fourier is most convenient.

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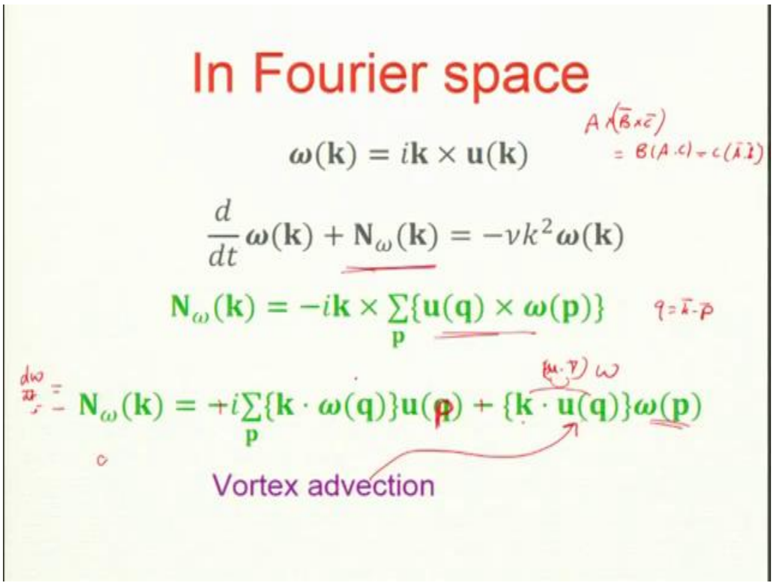

So, we will work is Fourier space $\boldsymbol{\omega}$ becomes i**k**×**u**(**k**) and so I take the old equation which I showed you in last slide convert into Fourier space. The non-linear term becomes k the u cross omega becomes convolution of the curl. So, it is not a dot product; it is not a cross product. So, **u**(**q**) × **ω(p)** and **q** is equal to **k** minus **p**. So, a sum over all p's and this is a viscous term.

Now, this **A**×(**B**×**C**) can be written in terms of some vector identity. So, this **A**×(**B**×**C**) is a very good thing to remember **A** × (**B** × **C)**; what is that? **B** (**A**.**C**), the first one minus A; A dot 1 and 3 and **C** (**A.B**). So that is what I have done here. So, i**k**· **ω(q)** **u**(**p**) and i**k**.**u**(**q**) **ω**(**p**). This we derived it before, but it just a recap, but you should pay attention; so this is important we need this.

If you look at equation, this is basically d**ω**/dt. So, if I put a minus sign of this it becomes plus and this plus. So, d**ω**/dt – **N**(**ω**) forget the viscous term. So, the first term is **k.ω(q)**. So, that your this p this is the type; so qp qp qp; so I have again made one mistake here, so this is a mistake. This is vortex advection this one. So, is carrying; so **ω** is carried by **u**(**q**). So, this comes as$(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$; this comes from this term $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ right because this will come from grad and this comes from u is that clear to everyone?

A real space there are two terms this $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ and this is the minus sign also this is I made a. So, this is sign error as well know one of them come with a minus sign. So, this comes with a minus sign and this comes with a plus sign in the right hand side ok. Let us just talk about $\mathbf{N}_{\omega}$. So, I think there is there is error of sign; so which is plus sign?

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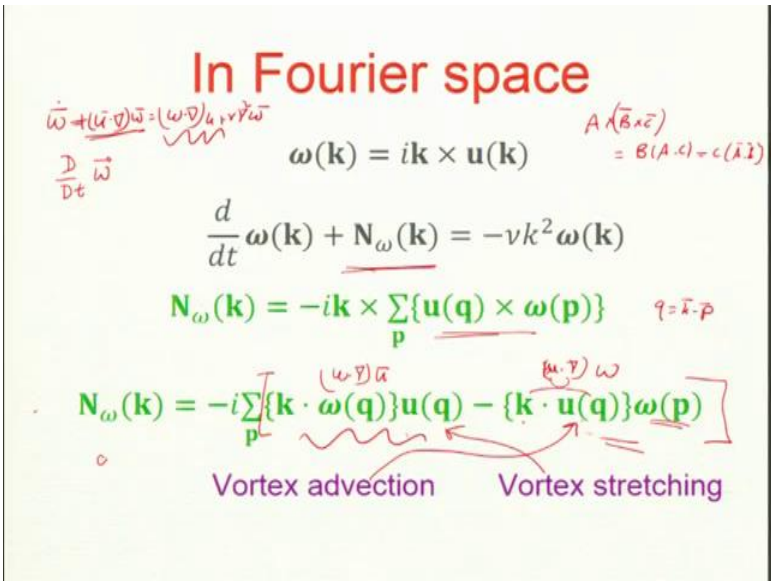

Student: (Refer Time: 05:20).

This plus sign this supposed to be this and there is a minus sign the minus becomes plus ok. So, this term is coming from $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ and this term coming from $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ .In real space, let me just write this real space $\dot{\boldsymbol{\omega}} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$ .This is equation of $\boldsymbol{\omega}$. This term is giving you this non-linear term in Fourier space and this one is giving you this non-linear term; so this term is advection.

What is the advection? So, this is basically total time derivative of omega on left everybody is done this fluid mechanics, So this is material derivative. So, omega if you sit with then when you go along with it. This is pushing this the mean flow is pushing this vorticity is that clear to everyone? So, this term is advecting term and this term is vortex advection and this term is $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ is something Stretching**.**

Student: Stretching.

So, this is vortex stretching. So, I did not look at it, but the other terms are. So, k dot omega u is vortex stretching; ok, Now let me ask a question which one is going to increase omega overall? If I look at omega squared of the whole system you can make a guess.

Student: Sir.

Advection will not increase advection is just porting things; is just taking one same material from one place to other place; it does not give new material. But stretching can increase omega or decrease omega is expected to increase omega. So, vortex stretching is coming from the second term. So, there will be two kinds of energy transfers or enstrophy transfer one is because of advection other one is from stretching.

Now, we can make this quantitative.

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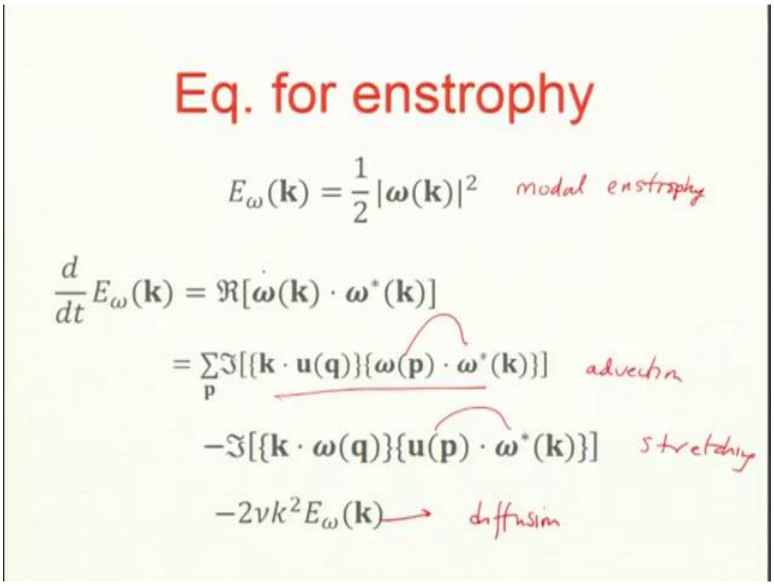

Now enstrophy I define as $|\boldsymbol{\omega}(\mathbf{k})|^2$. $\boldsymbol{\omega}(\mathbf{k})$ is the complex quantity. This is modal enstrophy, modal means for a given mode what is the enstrophy.

I will take the time derivative of this. Since $\boldsymbol{\omega}$ is function of time. The $|\boldsymbol{\omega}(\mathbf{k})|^2$ will also function of time. It will be just straight forward. So, this half will go away and this is very similar to what you do for velocity field. I will get now $\dot{\boldsymbol{\omega}}$ as two terms. So, it is now also will have two terms. So, this coming from advection and second one is stretching.

Now, this stretching and diffusion or dissipation let me call. Now, in advection term $\boldsymbol{\omega}$ is multiplied by $\boldsymbol{\omega}$ and here $\mathbf{u}$ is multiplied by $\boldsymbol{\omega}$ with dot product multiplication. Here is dot product ok.

We can do the same analysis like mode to mode transfer. This is energy now let us do, what happens for a single triad now this is for all triad now. So, let us try to dig further and try to get some picture about how energy is flowing by the way. So, this is this one work

I do not see many papers now, but people look at it, book by (Refer Time: 09:43) and (Refer Time: 09:44) and (Refer Time: 09:45) this is a lot of discussion on mechanism of vortex stretching. So, flow should go like this and it should stretch this is all in real space.

Now, there is interesting discussion when you stretch it frequency will increase just because of mechanics know if you stretch it, then (Refer Time: 10:04) increases. Now, I would want personally to do it in terms of Fourier space. So, understand in terms of transfers. So, I will exactly tell you what we should look for in terms of mode to mode transfer and it can be done.

So, we look for mode to mode enstrophy transfer. So, same formula we for kinetic energy we can generalize this to enstrophy.

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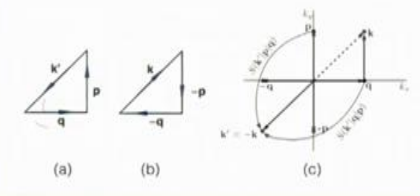
$$\frac{d}{dt}E_{\omega}(\mathbf{k}) = \sum_{\mathbf{p}} \Im[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\boldsymbol{\omega}(\mathbf{p})\cdot\boldsymbol{\omega}^*(\mathbf{k})\}] - \Im[\{\mathbf{k}\cdot\boldsymbol{\omega}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\boldsymbol{\omega}^*(\mathbf{k})\}]$$

k’+p+q=0



Instead of working for all possible triads in the flow, we focus on a single triad. Single triad and its pair for complex conjugate reality condition requires that also – **k**, **-p**,**-q**. So, we have a **k′ p q** and this is k minus k prime which is k **-p -q**.

So, I have only one triad. So, for a single triad instead of sum I will get few terms you know; the sum will reduces to only few terms.

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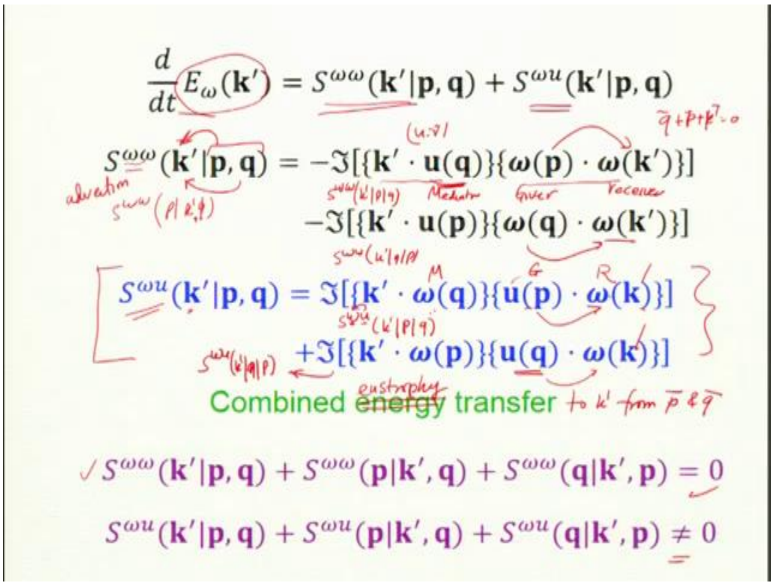

And so this is transport advection and this is stretching ok. So, the sum will involve two terms for stretching and two terms for advection. Straight forward know I mean these are just long formulas, but if you understand physically that is what they mean. So, this is ω ω multiplication k dot u. This is coming from $(\mathbf{u}\cdot\nabla)$. So, this is advection term.

So, $\{\mathbf{k}'\cdot\mathbf{u}(\mathrm{q})\}\{\boldsymbol{\omega}(\mathbf{p})\cdot\boldsymbol{\omega}(\mathbf{k}')\}$ and $\{\mathbf{k}'\cdot\mathbf{u}(\mathrm{p})\}\{\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(\mathbf{k}')\}$ and $\mathbf{q} = \mathbf{k}\text{-}\mathbf{p}$ So, $\mathbf{q}+\mathbf{p}+\mathbf{k}'= 0$. Let us keep this notation vectors. Now what about ωu? It just follows from the sum. So, sum as the u is replaced ω and product is u with ω.

So, $\mathbf{u}(\mathbf{p})$,$\boldsymbol{\omega}(\mathbf{k}')$ I missed prime here $\mathbf{k}'$ $\mathbf{u}(\mathbf{q})$,$\boldsymbol{\omega}(\mathbf{k}')$ this is for a single triad. But kinetic energy has only one term the first term and these are all us everything was u. Now with vorticity of course, there will be more complication. So, now we have vorticity to vorticity and velocity to vorticity. So, these are the terms and this should be telling you is a stretching term. So, we should analyze this for stretching when you stretch it how does the vorticity change. So, mode to mode and from the DNS; we should pick the data you know daily see whether it is stretching.

Let me just do few more steps and close my derivation; this is enstrophy increase rate of change of enstrophy at a wave number $\mathbf{k}'$ by two terms so and two wave numbers. So, there is increase in $\mathbf{k}'$ from both $\mathbf{p}$ and $\mathbf{q}$ so, it should be combined enstrophy transfer. So, this is a combined oh actually sorry this should be combined enstrophy transfer ok.

So, combined enstrophy transfer to $\mathbf{k}'$ from $\mathbf{p}$ and $\mathbf{q}$ and now this satisfies some properties. Now as I said you can do it the first one you have to write for $S^{\omega\omega}$ $(\mathbf{p}|\ \mathbf{k}',\mathbf{q})$ and same thing for q by q; sum them up incompressible condition has to be used and if you use that then you get this is 0.

But intuitively it is obvious because this is not stretching is just advection. So, you just porting from one place to other place; you just sweep your floor, but you just transferring the dust from one place to other place you differently seen the dust unless you take it out. So, this is just advection and does not change the enstrophy the first term.

But this one can change. In fact, you can just do the sum you will; you cannot show that it is 0. So, if I have again three terms from $\mathbf{p}$ and $\mathbf{q}$ just add them up this is not equal to this. So, sum is not equal to 0 ok. So, that is the result for combined enstrophy transfer, but now can we think of mode to mode transfers?

Now, same question this is this two terms together give the rate for $\mathbf{p}$ to $\mathbf{k}'$ and $\mathbf{q}$ to $\mathbf{k}'$ is total, but can I say individually how much goes from $\mathbf{p}$ to $\mathbf{k}'$ and how much goes from $\mathbf{q}$ to $\mathbf{k}'$. It turns out this is the proof which I will not prove it right now is that in the book. This term is very similar to what we do kinetic this is $\mathbf{p}$ to $\mathbf{k}'$ whatever is product of ω to ω is a giver and receiver. So, this is a giver and this is a receiver.

So, $\mathbf{k}'$ is a receiver right So, I identify whatever is dot product $\boldsymbol{\omega}(\mathbf{k}')$ with a giver for this term this is a receiver, this is a giver is that clear. And here what is this $\mathbf{u}(\mathbf{q})$? Is the mediator because it is multiplying with $\mathbf{k}'$ which is $\mathbf{u} \cdot \nabla$ this term k with u. So, this is a mediator it is a advector something advecting; is only driving the thing this is its taking things one place to other place, it does not participate in the transfer.

So, this is ω to ω transfer right; so giver is ω receiver is ω . So, ω to ω transfer. So, this is basically $\omega\omega$ $\mathbf{k}'$ $\mathbf{p}$ $\mathbf{q}$ and this one is $\omega\omega$ $\mathbf{k}'$ $\mathbf{q}$ $\mathbf{p}$ because k $\mathbf{q}$ is giving and $\mathbf{k}'$ is receiving.

Now, same idea here $\mathbf{u}(\mathbf{p})$ is giving and $\boldsymbol{\omega}(\mathbf{k}')$ is receiving. So, these are giver and these are receiver these are mediator. So, this is $S^{\omega u}$ u to ω right. So, the notation the second argument here in the super superscript, u is a giver ω is a receiver, $\mathbf{k}'$ is a receiver, $\mathbf{p}$ is a giver and $\mathbf{q}$ is a mediator.

And what about this one is $S^{\omega u}(\mathbf{k}'|\mathbf{q}|\mathbf{p})$ because $\mathbf{q}$ is a giver; $\mathbf{u}(\mathbf{q})$ is giver and $\boldsymbol{\omega}(\mathbf{q})$ is the receiver. Here this is like giver is somebody else is not ω is getting from u field is stretching. So, it is not among themselves but it is some other party. So, this is what it is ok.

So, the proof I will not do it here is quite long, but it is very similar to what you do for kinetic energy. So, complete as I read in the book, but is just intuitively this; this is what it is.

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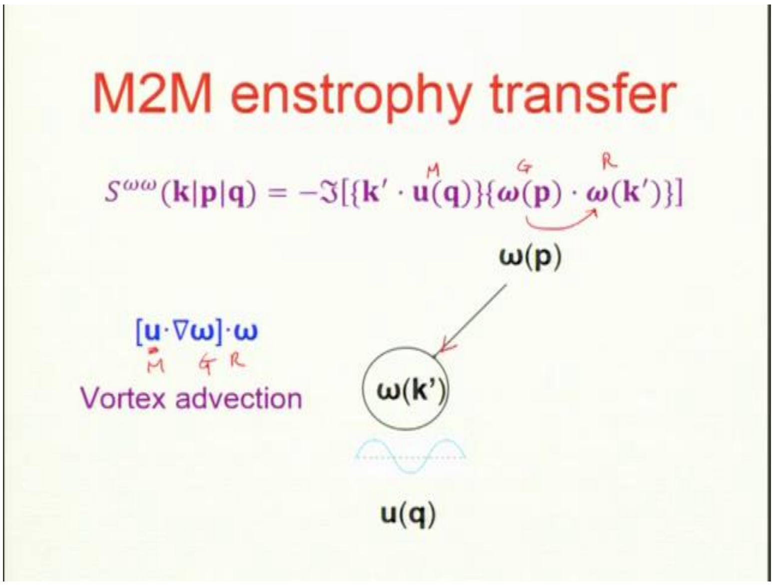

Let us see this result. So, without proof I am just stating that this is $\boldsymbol{\omega}(\mathbf{p})$ to $\boldsymbol{\omega}(\mathbf{k}')$ with $\mathbf{u}(\mathbf{q})$ is a mediator, giver, receiver, mediator and the picture is it comes from u is advecting mediator; this is this is giver and this receiver and the picture which is; so it is coming from here $\boldsymbol{\omega}(\mathbf{p})$ to $\boldsymbol{\omega}(\mathbf{k}')$ with $\mathbf{u}(\mathbf{q})$ is a mediator. So, this is mediating wave is mediating.

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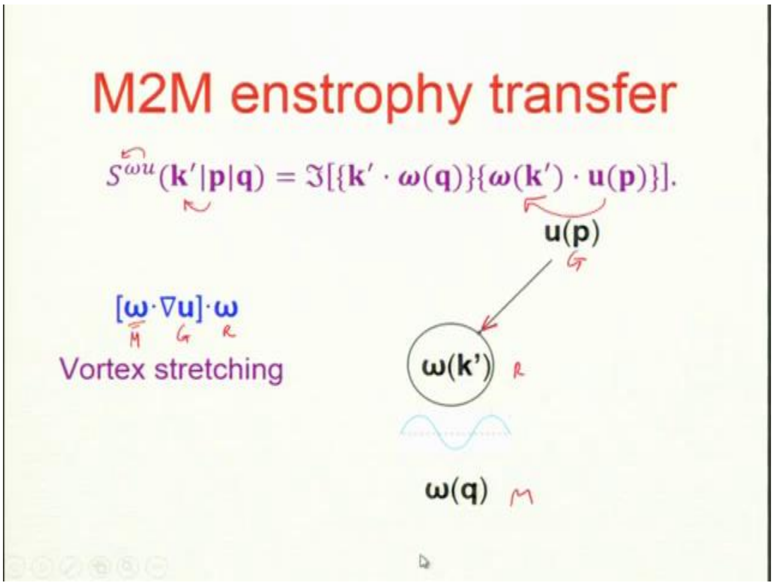

Now therefore, other one ωu; here $\mathbf{u}(\mathbf{p})$ is giver and $\boldsymbol{\omega}(\mathbf{k})$ is receiver. So, this is here and its comes from ω these are mediator giver, receiver and this is a vortex stretching and up is the arrow is not very up is a giver this is a receiver and this is a mediator. So, this is mode to mode u to ω; We just discussed mode to mode transfer. So, this vorticity to vorticity and this is a velocity to vorticity.

Now, we can define flux with that you know. So, non-linear term that was in a triad, but you have many triads. So, that will define energy going from one wave numbers sphere wave number sphere inside to outside. So, it is just like Kolmogorov flux very similar.

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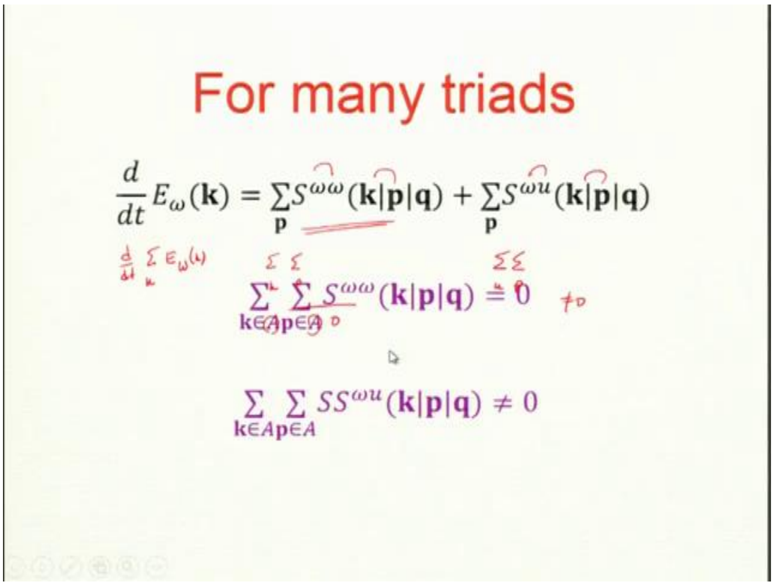

So, this was ω to ω advection and this is u to ω. Now, I do it for many triads; so I derive it for single triad mode to mode, but just do it for many triads. So, this will give you. So, $S^{\omega\omega}$ is basically sum over all **p**'s. This is coming to **k** from **p**.

Now as I said if I sum over all k's I sum over all k's d by d t of this. So, this is sum over k sum over p sum over k sum over p this is 0 sum over p is 0 because advection is basically conserving enstrophy. And this is this sum is not equal to 0 this is enstrophy coming from velocity field by stretching.

So in fact, this is only for all, but is true for any region, but giver and receiver in the same region you know you sum over. So, $S^{\omega\omega}$ is 0 this you can show for like from the conservation law which I told in the last set of slide and this is not equal to 0. Now from this; so again you can do more formal thing which I did in the verticity, but velocity field, but I will skip all that you can define flux.

And definition is once you have the transfer from one region to other one mode to other mode then you can define flux that is a formula is any no need to motivate further.

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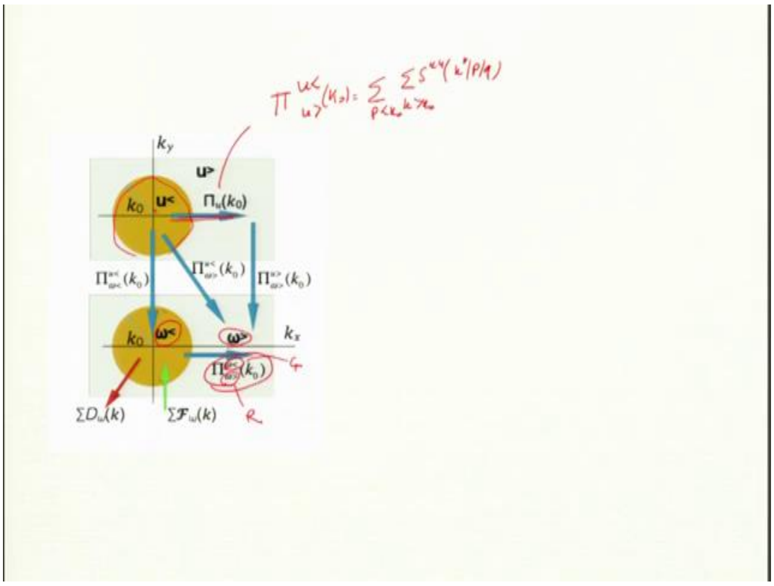

So, there is the picture looks very complex, but if you look at the picture slightly more carefully then you will find the idea. So, these are velocity sphere the brown is velocity sphere the mode inside is u< mode outside is u> and we also define vorticity sphere. So,

ω< is the modes within this sphere ω> is the modes outside this sphere, is that clear? Though ω and u are related, but we define two different spheres.

Now, u sphere u< sphere can give now this is Kolmogorov flux right modes inside giving more energy to modes outside and this was written as $\Pi^{u<}_{u>}$. In fact, it can be written as like this we did in the last lecture radius of this sphere is k_o. It is sum over p which is giver. So, it is less than k_o and the receiver is outside. So, k greater than k_o, $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ where, $\mathbf{k}'$ is -**k**. So, this we define now we can also define flux for enstrophy. So, let us look at the simple one first this one. So, what is this? $\omega^<$ and $\omega^>$, who is receiver and who is the giver?

Student: (Refer Time: 23:23).

who is the giver? $\omega^>$ is the receiver and $\omega^<$ is the giver. So, giver modes must be within this sphere and receiver must be outside this sphere.

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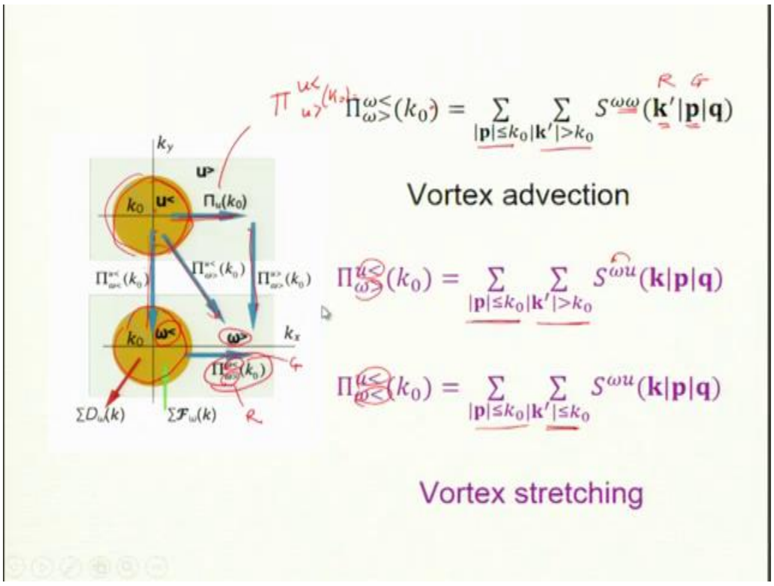

So, I just define it here; so let us erase this one. So, $\omega^<$ to $\omega^>$ is defined as receivers. So, ω to ω transfer which are defined mode to mode giver is **p**, receiver is $\mathbf{k}'$ so this receiver giver. Giver must be within this sphere and receiver must be outside this sphere, you just sum it up. So, this is ω to ω. In fact, this is within ω there is like Kolmogorov type very similar, but ω to ω and this is advection vortex advection.

Now, ω get from u by how many channels? You got one two three channels. This is important point u cannot give to u itself within this sphere, but now ω within can get from u within. So, this blue, this line is u< to $\omega^<$, it can be nonzero and so we have u to ω, but both receiver and giver are in this sphere, but giver is in this sphere and receiver is in this sphere oh; sorry this other one let me focus on this one first this one both are within. So, this is u< to $\omega^<$.

Now, we have u< to $\omega^>$. So, here to here this is one; so this p is giver less than k_o, but receiver is more. So, this vortex stretching plus there is one more which I did not write it here this one; giver and receiver both are outside. Now, this what needs to be computed in simulations and this work is incomplete. In fact, nobody has done this computation; as far as I know I mean no. In fact, mode to mode ω transfer nobody has given the formalism. So, we are the people who construct all this formalism.

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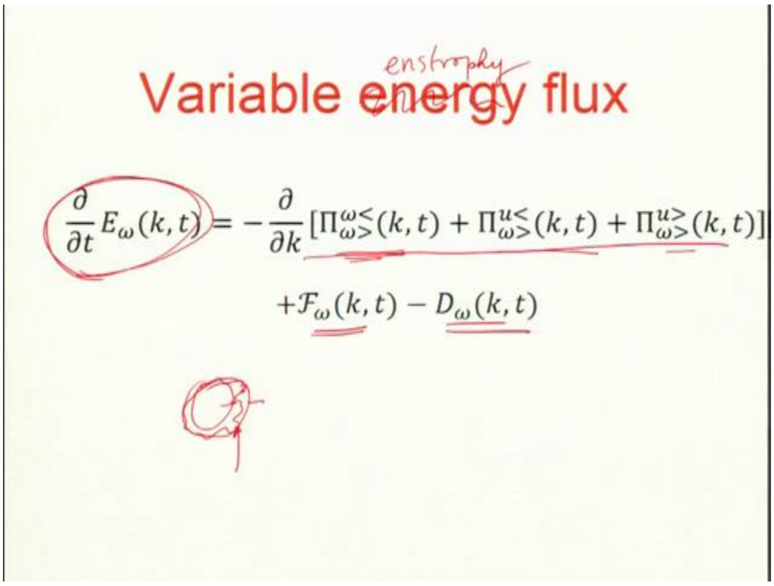

So, we can also define enstrophy flux. So, This is very similar for variable energy flux. But if I look at enstrophy of a shell, it can change by fluxes, but they are not free fluxes. There is one flux within itself, we had what kinetic energy flux here minus flux there, but now these are the flux coming from outside. So, this derivation I am not doing it here, but I derive in my notes.

So, variable enstrophy flux it involves three fluxes. So, there this one is coming from external force and this is dissipation and this one. So, if this is 0 under steady state then we should get and even initial range; then this must be constant. But I am not sure whether this is study this also should be studied whether energy enstrophy keeps increasing or it saturates I would like to see in simulation. Now, this is I am just flashing, it is a something to test in simulations.

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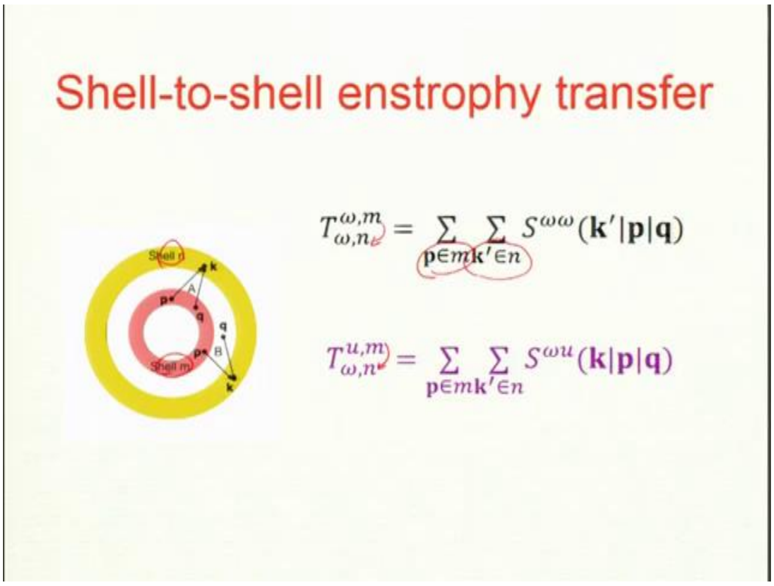

We can also define shell to shell transfers. Now the two of them ω ω and u to ω, Remember the shell this is the giver shell and receiver shell. So, **p** belongs to m giver shell and $\mathbf{k}'$ belongs to receiver shell. This is shell to shell; so is m to n ω to ω; we also define u to ω from shell m to shell n. So, there two channels of transfer; ω to ω and u to ω and they give you shell to shell enstrophy transfer.

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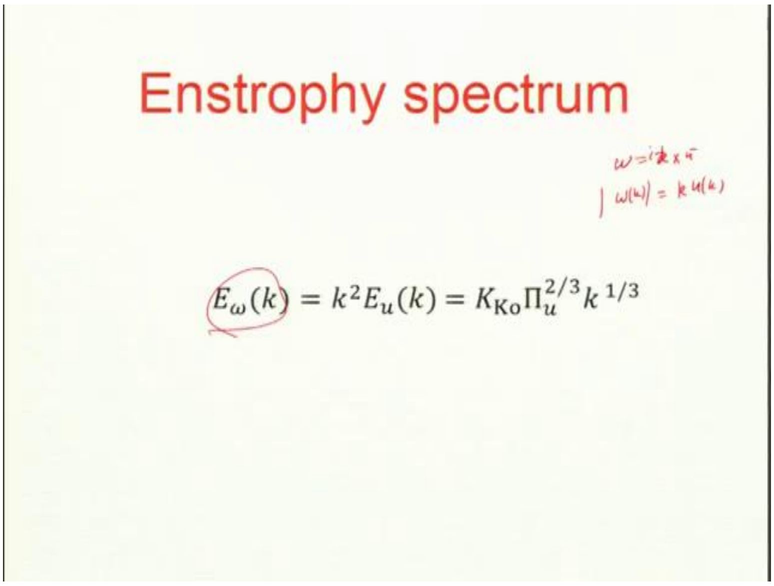

What is enstrophy spectrum? that is by dimension not dimension because $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. So,i$\mathbf{k}\times\mathbf{u}$; $\boldsymbol{\omega}(\mathbf{k})$ is; mod of this is ku(k). So, spectrum of E_ω= $k^2 E_u$.So, its spectrum is straight forward just Kolmogorov theory. By the way velocity field follows Kolmogorov theory; this is a funny situation my velocity field is Kolmogorov, but enstrophy is giving somewhat new well I mean it has not been studied that is why I mean enstrophy has not been studied so carefully in Fourier space. So, spectrum is this is hundred percent sure because it just follows of Kolmogorov theory, but the fluxes are what we need to compute.

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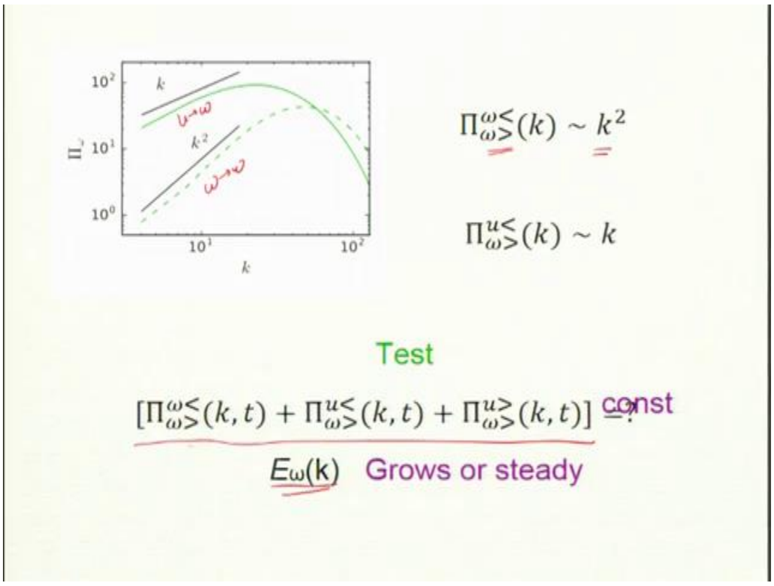

So, spectrum is fine. So, in numerical simulation; this was done by Subhadeep who is in our lab student. So, he computed this fluxes has been there in Tarang, so somebody has to run it. These are fluxes from $\omega <$ to $\omega >$ this that ω channel is k^2 and u< to ω channel only k, one channel has been studied and this is ω to ω and this is u to ω and they are not constant.

Now, my question is, it does what about the time derivative of this is it when this ω is growing or become steady in the under steady state I believe should become steady. But one should check and what happens to this in the initial range and also as a picture of stretching in Fourier space. How does it look? So, which mode we can make a triad, but we can construct triads you know examples which I have constructed in the class.

You try to do it not well many of them do not give stretching in fact, they give you decrease in ω. So, this is important point let me make this point clear if I construct any arbitrary example for sin cos or cos sin which you have been playing around compute; the stretching or just see whether ω will increase or not mode to mode transfers it turns out ω may decrease in fact, I find ω decreases.

It tells us one important point, it is a organized structure of the steady state that will give you stretching any arbitrary field composition will not give you stretching. So, stretching increase of enstrophy which is say that stretches is for flows which are already reach steady state or is evolved states not all any combination of velocity you does not give stretching; is it clear to all of you?

We need to look at the simulation data construct triads and then look at stretching. In fact, we just we need to analyze the data of the developed flow. So, the fluxes are so if I take any arbitrary flow field a compute Kolmogorov flux what will I get? Not from the simulation, just generate any velocity field.

You will get if basically get 0 flux random, but is evolve field this is it its phases are in such a way that it is organised itself that gives you constant flux. So, organization comes by evolution this is equation is leading towards that organised flow. So, this stretching is what we should look at from the developed data. So, these are some things one needs to work out. So, I think we stop.

Thank you.

Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 34
Two-dimensional Turbulence

So today we will discuss Turbulence in 2D flows ok.

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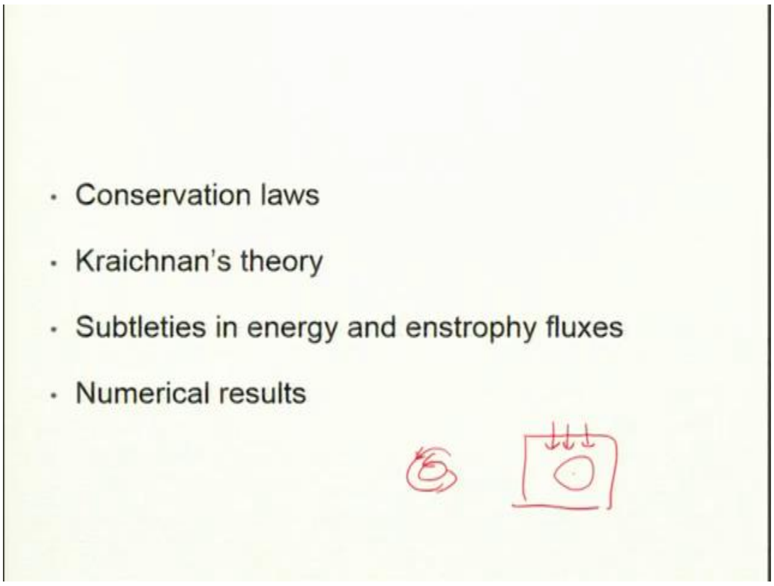

So, my outline is so conservation laws. So, there are which we discussed before, but Kraichnan theory and then there are some subtleties in energy and fluxes, numerical results. But 2D flow is it real, I mean do you really see it in nature, it turns out real 2D flow difficult to get; but there are like Soap film know these people have done experiments. In fact, this one lab in IIT itself, Sanjay Kumar's lab in Europe they do experiments in Soap films. So, you can flow from the top and it is coming down and you can make turbulence

So, Soap film is 2D is velocity only in x y plane, but in it is in real flows like hurricane is almost 2D. So, you also seen this velocity field is like that, the main velocity field and the fluctuations within it. So, there are some u_z components, but they are much small compared to u_x , u_y also that atmospheric flows in general. So, there is horizontal velocity is around you know, it would be 10 kilometre per hour, the wind blowing right now outside. It would be bit less, but during the monsoons it is quite high; but vertical velocity is much smaller. So, that is also considered to be quasi 2D, and lot of ideas we discussed for 2D flows are applicable to those flows.

So, the 2 D flow is very clearly important role in many many systems, like in astrophysics strongly rotating stars or galaxies that typically 2D. And, I am going to show you that 2 D flows show strong structure formation, very strong formation and the reason will become clear after this presentation ok.

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So, what are the conservation laws which we did it before? So, can you name for the conservation laws for 2 D?

Student: Energy conservation.

And.

Student: Enstrophy.

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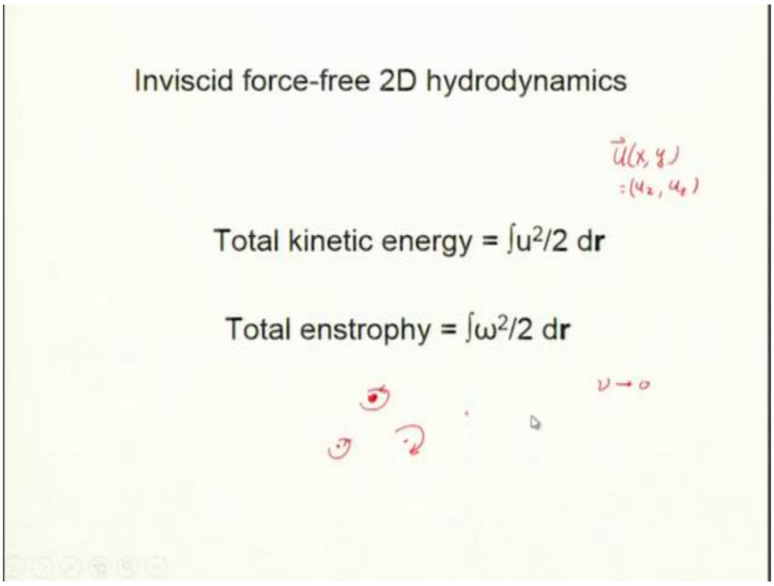

Enstrophy. So, one thing is energy total kinetic energy is conserved and enstrophy is conserved. From yesterdays lecture also you might have seen, the 2 D flows the $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$ term essentially advects vorticity. It does not increase or decrease, this was stretching of vorticity in 2 D. So, this vortex are you should also keep in mind that vortex are 2 D lines, infinite lines there is no change along z. So, please remember that **u** is only function of x and y, and **u** is also only u_x and u_y component there is no u_z component ok, is only function of x and y.

So, if there is a vortex column it will be same along z. An analogy which I am not going to describe here, I think I did probably make a remark that current carrying wire, the infinity current carrying wire has magnetic field; the equations are exactly the same for velocity field and for 2D velocity field and this current carrying wires. So, this is approximately good analogy. So, you can have many current carrying wires and they will also attract repel; attract repel is not very correct analogy that is different which I will not discuss right now. But other than that is basically these are wires which are infinite wires and they are interacting. So, that is good analogy, but there is no change along z ok.

So, the field lines are they are strong vortices; for if viscosity is tending to 0, then they are tending to 0 very tiny, but they must be small, but non-zero. Then you get very small this vortex and velocity field is there could be like that also ok. So, they are come in cyclonic, anti-cyclonic and they are point vortices, viscosity going to 0. Viscosity is not equal to 0, but finite; then this they get a flux, there is a core has some size this called vortex core, anyway that is not what I will discuss right now.

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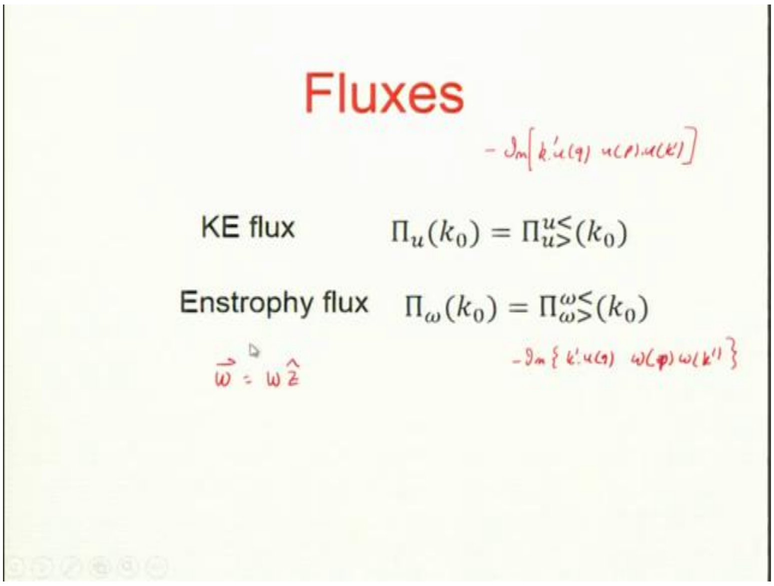

So, given two conserve quantities you can have fluxes. In fact, all are quadratic quantities u^2, θ^2, helicity. Product of two variables you can define flux that comes from that, you have three products product of three variables and that defines flux. So, we have energy flux and enstrophy flux ok.

So, energy flux is I already, the formula is exactly the same which is $S^{uu}(\mathbf{k'}|\mathbf{p}|\mathbf{q})$ will $-\Im[\{\mathbf{k'} \cdot \mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p}) \cdot \mathbf{u}(\mathbf{k'})\}]$, I am not putting vectors $\mathbf{k'}$ minus. So, this is for u to u transfer. And if I just sum over all the modes inside this sphere to modes outside this sphere you get kinetic energy flux. You also enstrophy flux, but this is only from $\omega^<$ to $\omega^>$, since there is no stretching, there is no u to ω transfer there is only ω to ω transfer; and that is coming from $S^{\omega\omega}(\mathbf{k'}|\mathbf{p}|\mathbf{q})$ is $-\Im[\{\mathbf{k'} \cdot \mathbf{u}(\mathbf{q})\}\{\boldsymbol{\omega}(\mathbf{p}) \cdot \boldsymbol{\omega}(\mathbf{k'})\}]$, $\boldsymbol{\omega}$ is a vector ω is a scalar.

Now this we did before $\boldsymbol{\omega} = \omega\hat{z}$, along z ok. So, I do not need a scalar product sorry, $\omega(\mathbf{p})\ \omega(\mathbf{k'})$. So, $S(\mathbf{k}|\mathbf{p}|\mathbf{q})$ by sum over modes inside to modes outside, I get these fluxes, good. So, we want to see whether in steady flows, well turns out in 2 D there is no steady flow. So, which will also become clear after this talk, that if you have somewhat give us some time for the flow to organize then it has certain properties; the flux is one show some properties and that is what I would like to discuss today.

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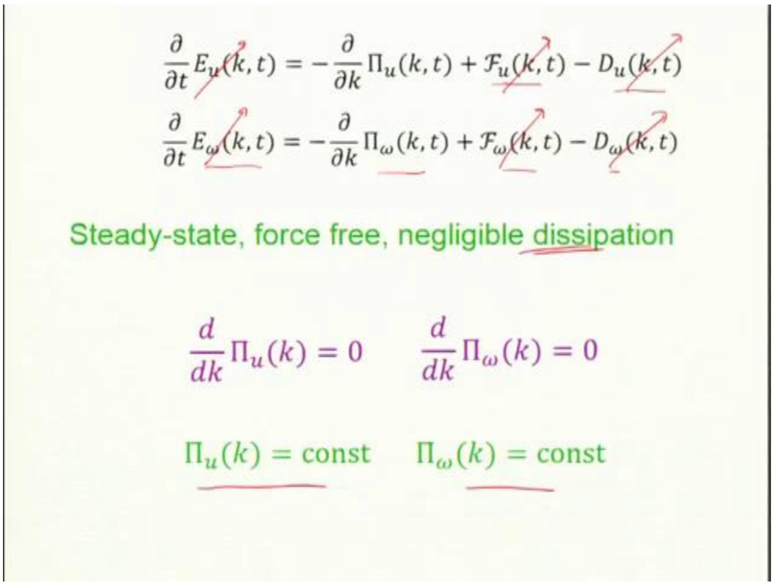

So, these are standard equation for the energy, you know this we did before. So, energy of a shell can change regards the fluxes, external force, and dissipation. Enstrophy can also change by enstrophy flux, so this is important know. Now let us assume that is quasi steady at least, is not changing much. So, we will drop these, these terms then we can get some steady flow. And study is important, you know do not keep time dependency; that is like to it is already complex problem and you make it more complicated. So, study is force free, negligible dissipation.

So, force free is I also turn it off and dissipation is weak; that means, I turn this off. So, we get basically $d\Pi_u/ dk = 0$ for kinetic energy flux. So, flux Π_u , kinetic energy flux must be constant; it follows from this equation when I am not doing an assumption. Of course, assumption in this one, and steady state is an assumption, actually turns out steady state is an assumption. And, from the second equation I will get enstrophy flux to be constant, but kind of steady flow we will show this. So, now the question is, these two flux conserved quantities and let us see by the both of your positive and negative, where does it seen sign if it all.

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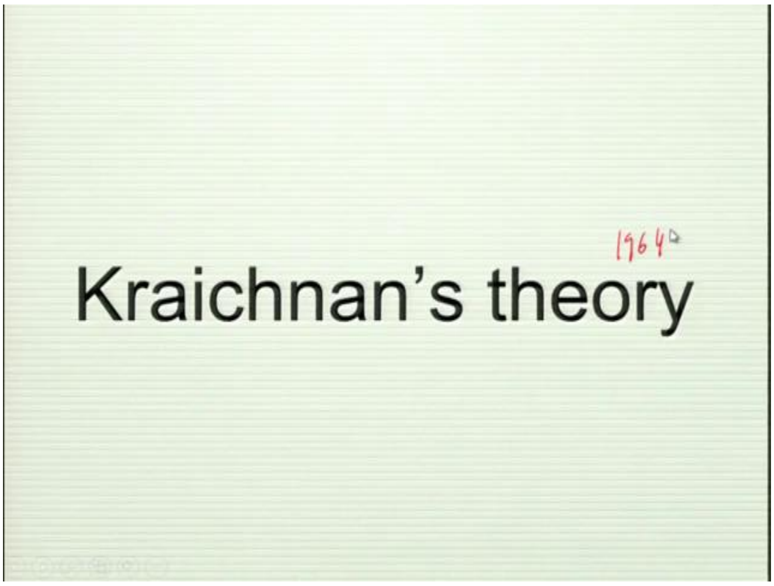

So, this is what was discovered by Kraichnan. In fact, way back in 1964 or 62, I mean very I mean this like 60 year old theory, 50 year old 55.

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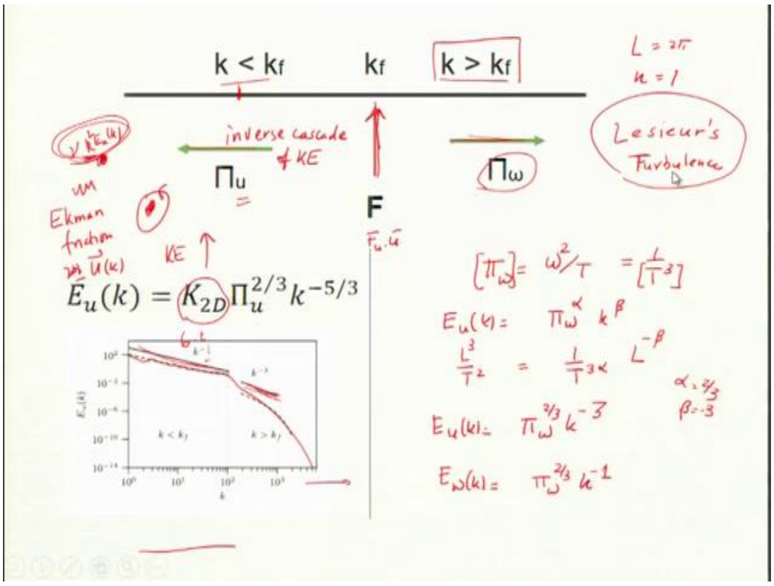

So, what is Kraichnan's theory say? So, these are wave number line. So, I forced at some intermediate scale, not at large scale; in 2D we force intermediate scale. So, like ocean, is huge. So, ocean you, ocean could be 1000 of kilometres, but your forcing could be your large scale is 1000 kilometre; but your forcing could be 10 kilometres ok, one hundred thousand size. So, my wave number is not normalize to one, well my box size is 2π then my lowest wave number is 1, right; 2π/L is 1. But I will say I will force at 50, 100, 10; 50 is more like it; so, you to get more space for both left and right.

So, we will have wavenumber less than k_f and wavenumber greater than k_f. So, according Kraichnan theory which is slightly detailed field theory and so on, which I will not prove it; but there are ways to show that it is reasonable, what is being told is reasonable by just the triad interactions ok. But I will not discuss it today. So, I will not say why, how you motivated, but is doable there are intuitive theory; if you want to look at you can look at by Lesieur's book, Lesieur's fluid mechanics, turbulence book. You will discuss this stuff. So, I have just state the result.

So, you want to conserve both kinetic energy and enstrophy, you can discuss it is off; that will force you to have kinetic energy flux going backward. So, this thing you know, in 3D kinetic energy was going forward; that means, large scale to small scale, but here kinetic energy is going from small scale to large scale. So, what will that do, is something goes from small scale large scale; that means, large scale will become stronger.

So, joint hurricanes are created by this. In fact, mechanism is this. So, energy is going from small scale to large scale, see if I force somewhere normally want energy to go to smaller scale you know, cascade to heat in 3D; or in 2D it goes other way round, it just becomes larger a large scale. And if that is how hurricanes are born, 2D structures are form and huge structures is if I will discuss maybe bit later. A rotating turbulence we get huge cyclones in our simulations, this big vortex going zooming fast ok.

So, these primarily because of the inverse cascade, so this call inverse cascade of energy, of kinetic energy; forward is from large scale to small scale or small wave number to large wave number; but this other way round. Now what happens to $k > k_f$this region. So, this region according to Kraichnan, enstrophy goes forward. So, kinetic energy is not going forward, kinetic energy move backward; but this region Π_ω is dominant, I will discuss bit later today. That there is a Π_ω here too Π_u, but that is weak; here Π_ω is strong here Π_u is strong, this we can see in many different ways.

But I will not prove it today, but you require some kind of you know field theory for rigorous proof, Kraichnan paper is field theoretic which I will not discuss in this lecture. Now, so this is a picture. So, enstrophy goes forward, kinetic energy goes backwards. Now these were the picture is. So, if I supply energy here. So, given force we supply energy, energy supplies $\mathbf{F_u} \cdot \mathbf{u}$ right that is energy supply, force times velocity that is power. So, it will go backward. So, this dissipation is weak here right, because dissipation is $k^2 E_u(k)$, well if a strong E_u; then there is dissipation $\nu k^2 \mathbf{u}(\mathbf{k})$, it is not the dissipation is 0.

Student: (Refer Time: 13:25).

But because k is small, it becomes weaker. So, the dissipation is weak. So, that is why getting steady state is difficult, because energy is piling up at large scale. In fact, it keeps piling up and our computer simulations break, the energy just from keep going; and when energy velocity will very large, then computer is not able to time step and it just blows up. So, kinetic energy tends to go up in getting steady state is somewhat difficult for 2 D; but it is possible, in our code we do something, well our code we do not know anything, we just let it run it reaches quasi steady state, it becomes kind of, but it keeps increasing energy it does not blow up.

But people normally tend to put some viscosity friction here, additional friction. So, put some additional friction at large scale and that also has a name. So, this called Ekman friction, friction at large scale; that is not neatly squared this is not of this type, but it could be constant, is just proportion to velocity field $\mathbf{u}(\mathbf{k})$, $-\mathbf{u}(\mathbf{k})$, not $k^2\mathbf{u}(\mathbf{k})$. Now, what is this spectrum? So, this is a flux, now let us imagine that flux is these two fluxes are constant; a kinetic energy flux is constant and enstrophy flux is constant. So, what do I expect for this spectrum, in the left we can, you can guess.

Student: (Refer Time: 15:05).

Left must be Kolmogorov, because Kolmogorov was derived truly from dimensional argument five third. So, I assume that any wave number here in between, the spectrum there will depends on the flux and wave number; if you do the dimensional matching is same as 3D derivation. So, in the left space spectrum is just Kolmogorov. What do you expect? So, this is, but the Kolmogorov constant can be different. So, the integrals involved you know, dimensional matching does not say anything about the constant. So, in 3D it is around 1.6, but in 2 D this is around 6, well 6 point something.

What about right, right side I have to do the dimensional let us apply the same dimensional argument; but the dimension of Π_ω is different in dimension of Π_u. So, what is dimension of Π_ω? It is ω^2 by time and what is dimension of ω, is 1 by time.

Student: 1 by time.

Right ω is $\nabla \times \mathbf{u}$, so, 1 by time. So, the dimension of 1 by T^3. So, now, I will I am looking for $E_u(k)$, I will say that is $\Pi_\omega^\alpha k^\beta$. So, $E_u(k)$ is dimension of L^3/T^2 right; that we did it before, this has $\left(\frac{1}{T^3}\right)^\alpha$, k has dimension of L^{-1}. So, that is very straightforward, α is 2/3 and β is -3. So, my spectrum in the right hand side is $\Pi_\omega^{2/3} k^{-3}$. So, the spectrum in the right hand side, k^{-3}, so kinetic energy.

What about E_ω? E_ω is I multiply k^2,$\boldsymbol{\omega} = \nabla \times \mathbf{u}$. So, if I multiply k^2 is going to be k^{-1}. So, enstrophy spectrum is k^{-1}, and kinetic energy spectrum is k^{-3} ok. So, let us see what you get in. So, the sketch is this, this from simulation which she is doing some of the simulation. So, the -5/3 in the left is kind of nice line minus five third, this five third is here ok. Right side is not quite -3 right, because minus 3 is this line and we are getting steeper and that is because of dissipation.

So, there is a dissipation of enstrophy and the dissipation is steepening; dissipation will always take energy out there, somebody constant flux is giving you some spectrum. If you increase dissipation, if you add dissipation then energy will be steeper; because somebody is just eating up the energy you know. So, there is not -3, if you want minus 3 which we are trying to do that we need to increase the range in that direction. We need a bigger grid or you force somewhere here, you shift the forcing range; then you get enough dissipation range sorry enough inertia range and that is where we expect minus 3 ok.

So, this is Kraichnan theory. So, I did not prove it the derivation of fluxes, but you can look at this book ok.

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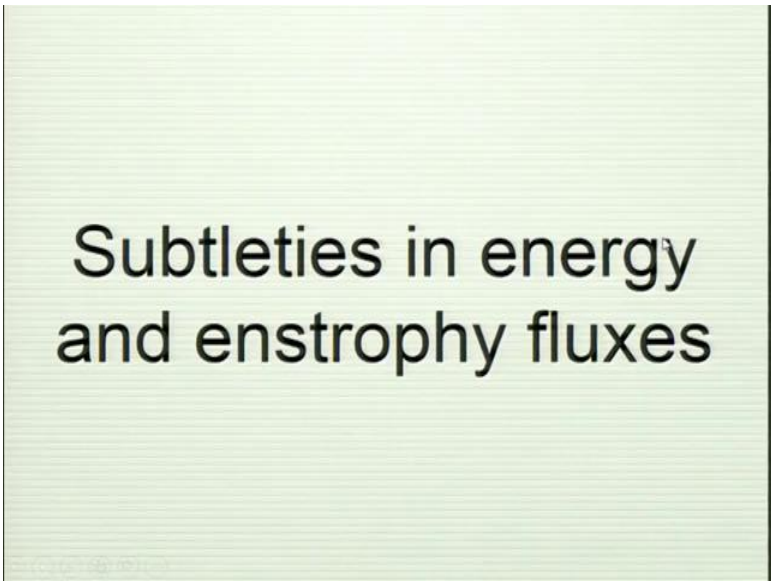

Now, there is certain problems.

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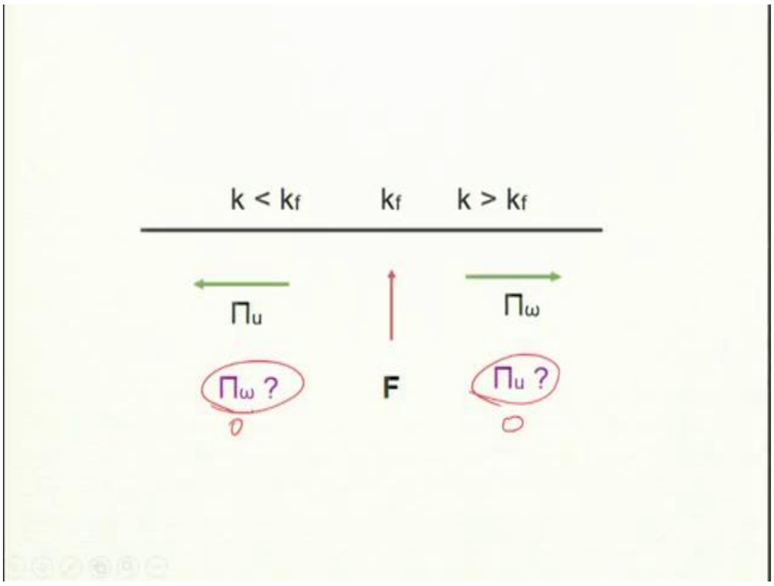

Now, of course, Kraichnan theory tells you about Π_u and Π_ω; but question is I would like to know what is Π_u in the right hand side? So, what is Π_u, and what is Π_ω in the left? So, there are quite a few papers with several, Π_u is 0 here, and Π_ω is 0 here or very small and that is not quite correct ok. In fact, both of them are not constants simultaneously; one way to see this is the following ok. So, this is what we would like to answer, what are this Π_ω in the left and Π_u in the right.

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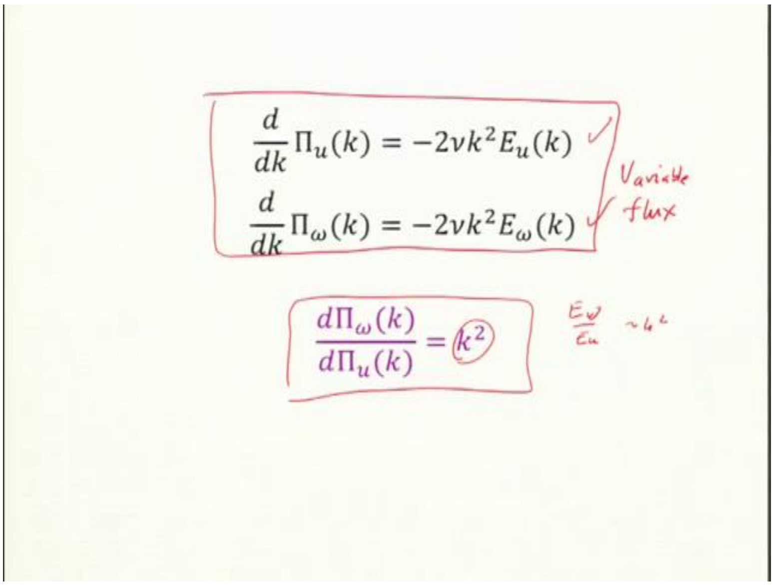

So, let us look at these two equation which I did it before, assuming steady state that is a difficulty; but we will assume steady state. Now if I take the ratio of these two, if I take the ratio what will I get. So, by the way this will cancel νk^2, but I get E_ω/E_u, and what is the ratio of E_ω/E_u; k^2. So, this is the ratio. So, the ratio is k^2. So, both of them can not be constant at the same time. So, one of them should be decreasing, when one guy is constant. So, something like that should happen. So, this is telling us that we should start with these formulas and try to derive the other one.

In fact, for me I am very fond of these equations, under steady state use the flux, variable flux equation we call this variable flux know. So, the name is variable flux and here dissipation is changing the flux; whereas the force is there in a small band. So, force is 0 for the initial range, both left and in the right. So, let us try to solve them. So, let us work out the flux and spectrum already showed you. So, let us look at the fluxes, both the fluxes E_u and E_ω to left of k_f.

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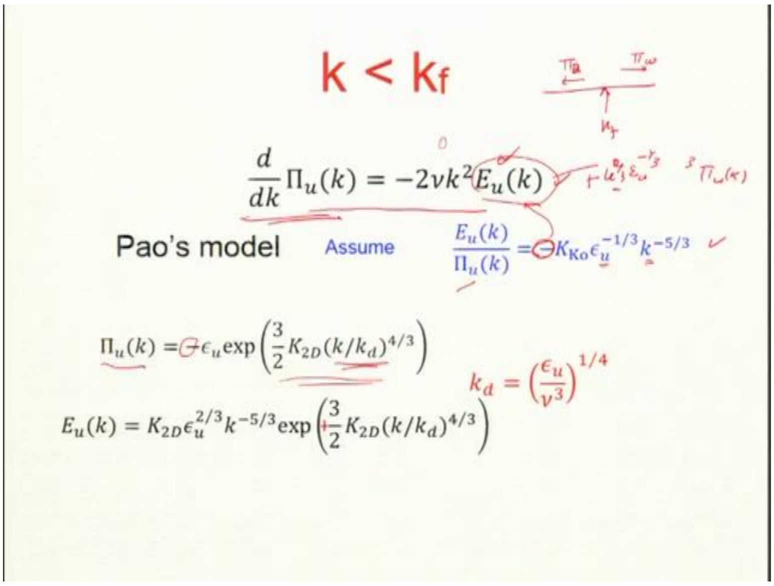

So, we are forcing it k_f, to left of k_f and to the right of k_f ok. Here Π_u is constant; here Π_ω is constant. So, if Π_u is. So, $k < k_f$, Π_u is constant. So, this is. So, this we satisfy this relation right; but as I said I am interested in both dissipation, and well I am interested in solving this more general equation. So, if you make it 0, then I get Π_u constant; but if I have the dissipated term, then I will get more general formula which will be not constant, but it will be some function of k.

So, the two unknowns Π_u and E_u. So, then we will apply Pao's from Pao's model and according to. So, I will just modify Pao's model in 3 D. So, what is the Pao's model Π_u/E_u, or E_u/Π_u is only function of k and dissipation rate or flux is independent of ν or forcing ok. So, this is. So, E_u/Π_u is only function of ϵ_u and k and of course, proportionality constant is also there. So, this basically follows from the spectrum and assumption ok, this is an assumption we could depend on ν know when, but it does not depend on ν this assumption was made.

If I, now I have two equations this equation, this equation I can solve for both. Now please remember the minus sign here, because a spectrum is positive, but flux is negative. So, put a minus sign, because minus minus will become plus, this minus and minus when I substitute E_u here, so I will get function of Π_u right. So, I replace this by $\epsilon_u^{-\frac{1}{3}} k^{-\frac{5}{3}}$, this is k^2 already and Π_u. So, I get 1 D, first order ODE in Π_u ok, and this k^2 and $k^{-\frac{5}{3}}$ gives you $k^{\frac{1}{3}}$; but is a plus sign, remember this plus sign I am not writing ν and so on.

So, plus sign makes Π_u increase with k, not decrease with k. So, is easily solvable is 1 D. So, I will escape all the algebra. So, Π_u is exponential, because is one third integrated this will get k four third and this minus sign here. So, because I know the flux is negative ok. So, this is the minus sign. And so, k_d is much bigger than k ok. So, this exponential term is not very large, is it is one actually, order one; because k_d is large. So, k/k_d is approximately 0, but it shows some effect, fit is better with this.

Now, what about spectrum, once I know Π_u I can substitute and I get E_u, so straightforward. So, it is again minus five third, but there is a correction exponential correction, but it comes the plus sign; in 3D it is comes as a negative sign. And what is k_d? k_d is usual, because is the derivation is exactly same as 3D, except the minus sign has become plus sign. So, this is a plus sign inside and this because of this minus sign ok. So, this is a formula. So, I will show you in computer simulation this seems to be a better fit.

What about Π_ω? So, I solve for Π_ω using similar equation is this. So, we also have pi equation for Π_ω. So, what is the equation for Π_ω, this equation.

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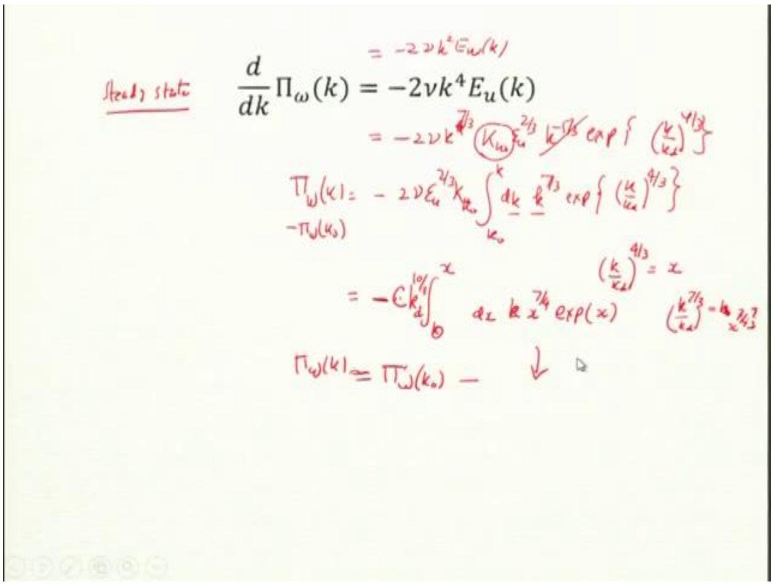

So, let us try to find out what is Π_ω. So, equation is $d\Pi_\omega/dk$ is $-2\nu k^4 E_\omega(k)$, but $E_\omega(k)$ is $k^2 E_u(k)$. So, you get this equation. Now see I know u k, so I substitute, so $-2\nu k^4$. So, Kolmogorov constant, $\epsilon_u^{2/3}$ two third know; actually this is not Kolmogorov constant, this where Kolmogorov constant, but for 2 D. So, is around 6 not 1.5, $k^{-5/3}$ exponential apart from constant is $\left(\frac{k}{k_d}\right)^{4/3}$ ok.

Now, this five third will correct this four to seven third, now I integrate this. So, it is best to do in a computer $d\Pi/dk$ is -2ν, all the constants come out K u k Kolmogorov integral d k k seven third exponential k by k d four third, well this is if you want mathematical formula. So, what is the formula, which function is this; right side function where do you want to look in a table. Earlier this kind of integral where there was no computers, so people used to look in a table. So, which table we should look at.

Student: (Refer Time: 27:03).

Well you will go from k_0 to k. So, I must write pi omega half k k 0 ok; but k_0 can be assumed to be small ok. So, this is if you see it is a gamma function, you can relate into gamma function.

Student: So exponential.

Right. So, exponential is there, but gamma is a positive argument; but gamma can be positive and negative. So, you have to make a change of variable. So, I just want to make you give you a idea. So, four third we should write as x. So, the right hand side is the constant C k know. So, we basically worry about x. So, replace k four third by x. So, it become $\exp(x)$ and $k^{7/3}$ will be. So, I have to say what is k seven third, is going to be k_0. So, k by k by k naught is x.

So, these together will come out outside will come out, all remove get all the dimension out, these another trick for do integral. All the dimensions outside, so I should remain take the k, k has dimension of k_d. So, k the natural unit for k is k_d. So, they will come out and make it k towards 10/3 and everything now is k/k_d. So, $\left(\frac{k}{k_d}\right)^{7/3}$ will be actually tell. So, the one k, yeah this is what is going to come; this is going to give us x to the power three quarter into 3 by 7 by 3 7 by 4, I get $x^{\frac{7}{4}}dx$ and this goes up to x so, 0 to x.

Now, this we can see in a table for gamma function, but computer can do it better and I would like to. So, we do not know, I mean this is what we want to; if assuming steady state which is not extremely clear whether it is steady state, earlier state we should get these behaviour, this is a minus sign here. So, $\Pi_\omega(k)$, now I am not, well we are not sure what is Π_ω at small value.

Normally expect with 0, what is a 2 D; the big vortex sitting there, it may give some vortex flux. So, these are something which we are trying to investigate ok. So, this one, so what is this form? So, I will show you numerically what you get, but this model is what we like to fit, if it fits; but this is being discussed or this is being investigated that is what we are doing right now clear.

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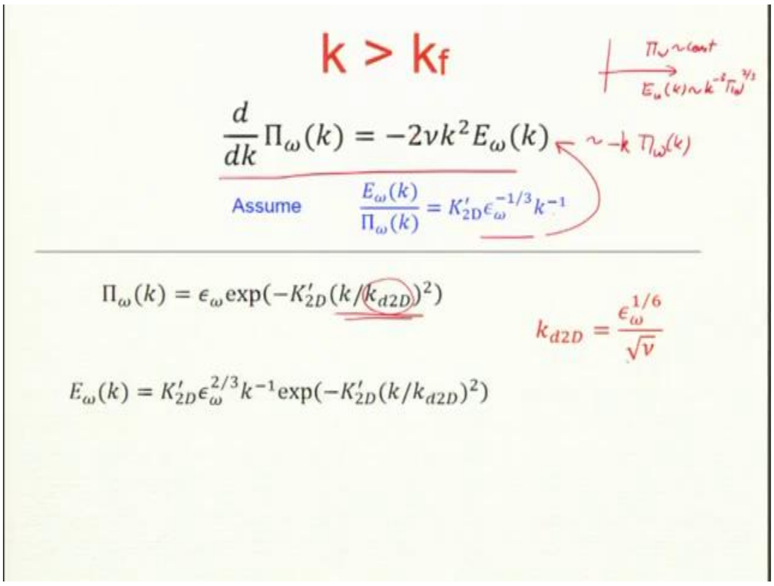

So, this for $k < k_f$; for $k > k_f$ too we are sure about one thing that Π_ω is constant approximately. If there is enough range for enstrophy flux; but dissipation will also start playing a role so, the equation to solve is this equation right. Now let us assume Pao's model again works, you know also. So, is in theory this what you try, you try to see whether you are model you can extend that model to something else.

So, in mathematics by the way there is another. So, if you, if you talk to on computer science or computer science specially; you want to solve a problem then you see a problem which already solved, and then use that solution to solve this problem ok. So, there are tricks solve a general problem then use it takes special case, or solve a special problem generalize it. So, one idea if you know how to solve one problem, then you try to use a solution elsewhere; or a code works then you use a code recycle it for something else.

And it uses the same idea which seems to work, it is quite nice for this model that we use Pao again. So, E_ω/Π_ω is independent of ν, is only function of ϵ_ω and k now. Since E_ω is k^{-1} right E_u is $\Pi_u(k)k^{-3}$. We proved that spectrum in the right side. So, this $k > k_f$, in this region Π_ω is constant and $E_u(k)$ is $\Pi_u(k)^{2/3}k^{-3}$k minus 3 5 omega two third. So, E E_ω will be k^{-1}, because we multiply k^2. So, this is what we will get, for from dimensional analysis from Pao's model.

Now, I substitutive it here, so I will get only 1 D, ordinary differential equation first order. So, which is straightforward and now see the power; power is k^{-1} will here, come here and k into k k in to k minus 1 is k. So, I will get basically this. So, if I integrate this, I will get k^2. So, now, exponential k squared with a minus sign. So, this one is minus. So, the solution is Π_ω is exp $\left(-K'_{2d}\left(\frac{k}{k_{d2D}}\right)^2\right)$. So, this is a constants ok, which I will show you right now, and is and this Π_ω is positive flux; that means, goes from large scale to small scale.

And E_ω will be once I know Π_ω then I know E_ω, and once I know E_ω I know u which is k minus 3 and what is k_{d2D}, k_{d2D} is this it comes by non-dimensional is this, well basically it comes from this equation ok. Now, what about Π_u? So, we can compute Π_u by same trick which we did before. So, I know E_ω I know E_u.

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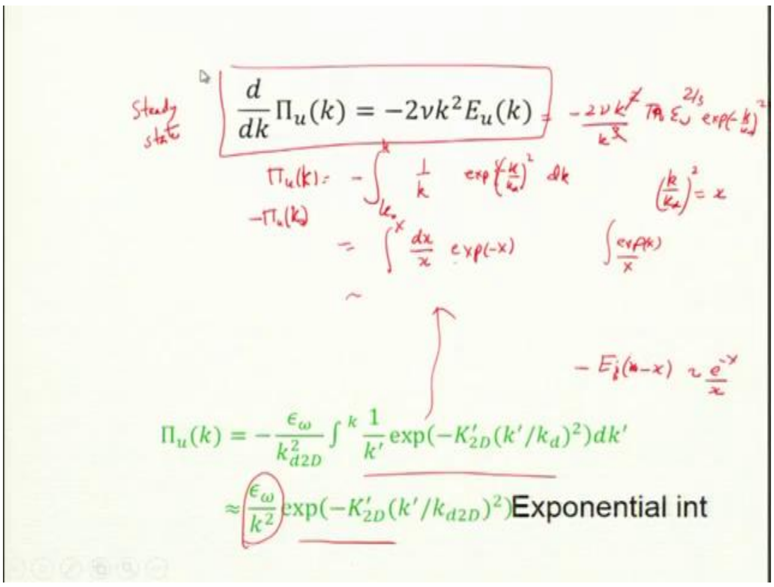

So, I will use this equation. So, this is general, only thing we assume is steady state; there no forcing, we are beyond forcing. So, integrate this. So, how will I integrate, this is $-2\nu k^2$, now E_u is k^{-3}. So, $\epsilon_\omega^{2/3}$ and they are constants which I am ignoring. So, this becomes $1/k$. So, integrate this. So, Π_u is k is integral well sorry, I forgot exp $(-\left(\frac{k}{k_d}\right)^2)$. So, $\frac{1}{k}$ exp $(-\left(\frac{k}{k_d}\right)^2)\mathrm{d}k$, $-\Pi_u(k_0)$. So, $-\Pi_u(k_0)$ is this and this k_0 to k.

This function we can again relate it to some known function ok. So, one thing is to make a change a variable $\left(\frac{k}{k_d}\right)^2$ is x, similar idea. So, this becomes. So, $\mathrm{d}k/k$ is dimension less. So, you can easily check that these $\mathrm{d}x/x$ exp $(-x)$. So, now this also this is name for this function know, this is not well this is not called gamma function is called exponential. So, this integral is this ok, I mean I already made some more simplification, k is k is squared is x ok.

Now, this does fit well because there is identity. Now I let me just tell you that, this call integral exponential integral, it has a name called exponential integral and $\mathrm{E}_i(x)$, well this exponential integral, this is an identity well it is a asymptotic, it is not a and it is called asymptotic for large experts of behaviour. So, $\mathrm{E}_i(-x)$; so, in the tables it is given for $\exp(x)/x$, not for $-x$. So, I made a change of variable. So, x to $-x$ so, this is minus of this is $\exp(-x)/x$ ok. So, if this function goes is $1/x$, but x is already k^2. So, $1/k^2$ ok. So, this is what we are asymptotically claiming therefore, large x it will be this will be this formula ok

So, Π_u is not Kraichnan theory is silent, Π_u it does not tell you Kraichnan pi u it only says Π_ω is constant. So, it using Pao's formula and this equation to compute Π_u and now let us. So, is that clear. So, this derivation is just algebra; but we now have all, energy spectrum of course, in that in turn gives you enstrophy spectrum as well as Π_u and Π_ω in both the regimes. Now we will try to see whether it fits with the simulation data.

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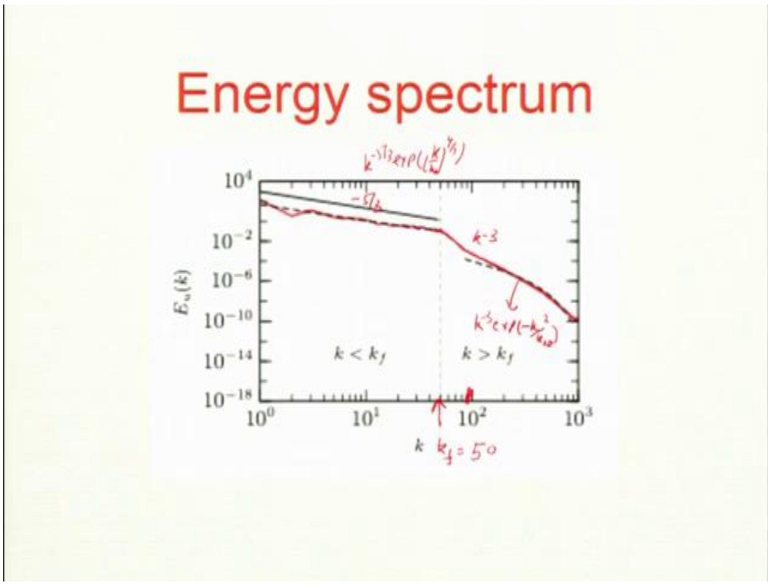

So, this is a spectrum. So, we are forcing it k equal to 100, k_f is 100, around here k_f is. So, we are falling somewhere here, this why this line is somewhere here well 10, 20, 30, 40 actually this is not 100 is this is a new run this is a 2001, 2, 3, 4 40 is force at 40 change of year is 40 not 100, 100 was some other run.

Student: It is 50.

50, 20, 30, 40, 50, sorry yeah 50; now to the left this line is the dash line is the fit ok. So, it is $k^{-5/3}\exp\left(\left(\frac{k}{k_d}\right)^{\frac{4}{3}}\right)$. You can see that viscosity small, so k_d is large and it is not changing much this is five third line; right side since you are forcing at 50 and maximum wave number is around 1000, there is not enough range for k^{-3} spectrum. Right side expected k^{-3} spectrum you know, but I do not get a power law, it is dropping exponentially.

And this one is $k^{-3}\exp\left(-\left(\frac{k}{k_{d2D}}\right)^{2}\right)$. So, exponential is dominating, there is minus 3 parts, but there is a domination of exponential part. So, it steepens further ok; so, $k > k_f$ is dominated by the exponential. We need to make a run where the force somewhere here, let us say k equal to 10; then you will get a large range for minus 3 regime and we should expect minus 3 ok, is that clear to everyone. So, minus 3, to get minus 3 you need simulation which has enough scope of getting minus 3.

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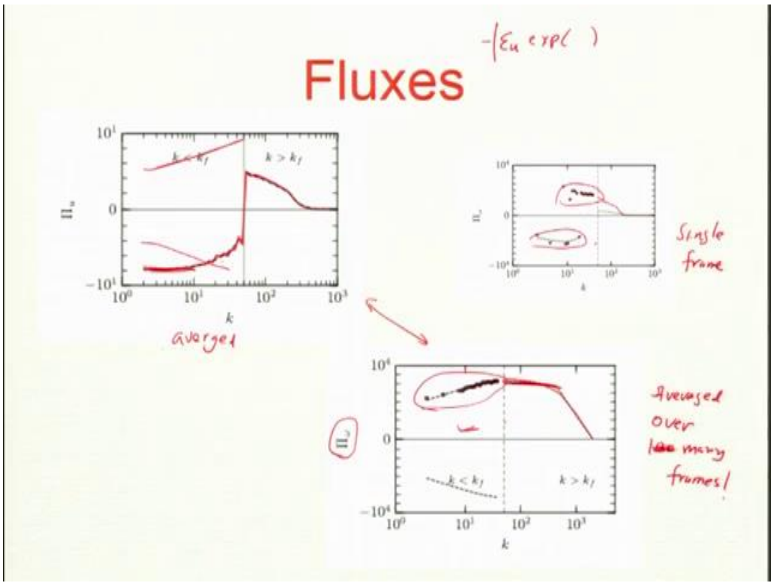

So, let us look at the spectrum now, a flux, so that spectrum. So, flux Π_u is not constant all over, but it is fitting nicely Π_u this one. So, this is what is a Π_u spectrum and we are getting constant here, Π_u I expected to go increase know negative why is it pi constant minus. So, it is minus of that ok. So, is it is constant. So, is ϵ_u minus exponential of that. So, it should have, in a case this part is constant; but it should be, if I take a mod it should increase with k we need to check, so this part.

So, mod means this if I take a mode if I come here it should have increased. So, it should have gone through that ok. So, this part is needs to be seen, this right side is that Π_ω is constant here; right side Π_ω, Π_ω is expected to be constant in the right. And Π_u is decreasing in k, $1/k$ constant by k squared moderate by exponential. In fact, even though we do not get k^{-3} spectrum, but we are reasonably getting Π_ω constant, and this part is that part ok.

Now, Π_ω to the right is looks ok, but to the left is very strange; left what has both positive and negative parts. So, either it is some steady state assumption is not correct. So, this is summed over, now this plot and this plot are different; this for a single frame and this averaged who had many frames where around 100 frames, where I am not sure well many frame.

Student: And left one is also average.

This is also averaged.

Student: Average.

So, these two are average, and this is single frame you wanted to see whether it is fluctuating vortex indeed fluctuating and we are getting positive according to theory it should be negative this ϵ_u ; unless there is a big flux at $k = 0$. So, this part we do not understand ok.

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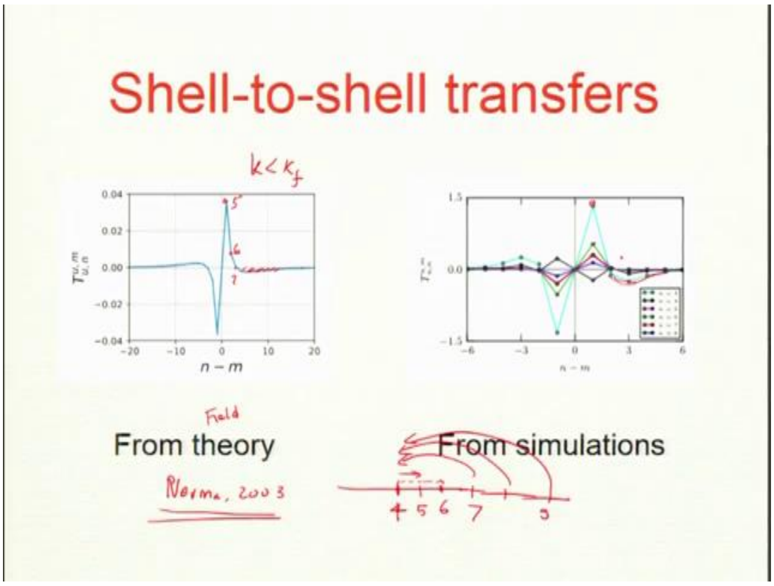

So, we need to see what is happening here. Shell to shell is also surprising, but we know from theory. So, this was what I did in Pramana 2003 I think; so, energy in 2D, so this is for $k < k_f$. So, though energy is going backward all full energy, but there is a local forward cascade. So, what is local forward cascade? So, if I look at shell 4 it gives positive energy to shell 5 it 4 gives, but 6 what is 4 to 6, now it is distant; actually 4 to 6 also positive, but small, but 4 to 7 is inverse.

So, 7 is giving to 4, 8 it is giving to 4, 9 is giving to 4. So, all these people are giving energy backward. In fact, there lot of energy mode which are shells which are giving energy backward from. So, this is next shell, this next to next. So, this will be 5, this is 6 if m is 4 ok, and this is 7, 7, 8 all the way up to something like 18, 15 or so, here for this particular this from analytical theory, this called field theory. We compute this stuff, which I will not tell you how to compute them, but this gives you negative.

So, though I give energy, so if I look at right side I give to labour; but I get lot of it from the small shops. So, I had to give to my neighbour, but I get lot of from the right side. So, on the whole money is flowing from right to left; so, for each shell is just flowing backward, if I sum up all of it. So, this submission you can see in this theory, if I assign the book as well I believe I did that.

Now, this is what is plotted from simulation. So, simulation is n to $n+1$ is positive, but in simulation right away these are negative ok; except the black which is negative with the for the next itself, rest are all this is positive then negative, red is positive than negative, this for different m, $m-n$ sorry different m. So, this is not visible here, but for I. So, this is $n-m$ know. So, I choose different m, then I get different different plots. So, for m equal to 5 I get 6 7 like that, m equals 6 I get 7 8 9 10 like that. So, in the right this called data collapse.

So, this theory is not, where the simulation is not very resolved and probably the lot of fluctuations ok. This 2 D seems like the steady state is not, assumption of steady state is not very very robust. So, that is why here there are not collapsing to single plot, single plot collapse we should have all of them should have fit on the same plot; then would have been stronger theory. So, this theory is not very solid.

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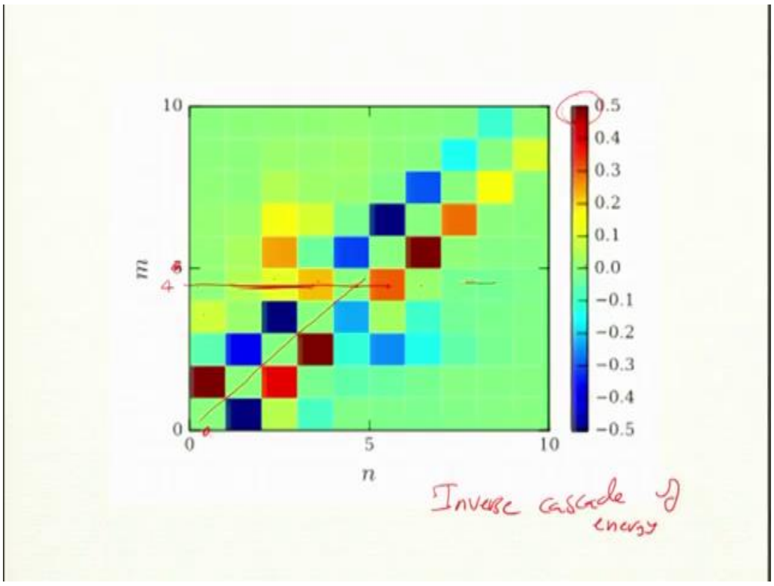

So, we can also look at density plot. So, what is the density plot give you. So, if you look at shell, let us say this one. So, this shell diagonal, so which one is this shell 0 1 2 3 4, so the shell 4. So, this index is not ok, shell 4. So, shell 4 to 4 is 0, I do not give money to myself. So, 4 to 5 is positive, 4 to 6 is almost 0, green is 0, red is positive; but then it search getting negative this blue. So, it is negative.

So, like what we saw. So, I gives to next neighbour, but I get from more distance neighbours and the negative side is also telling you. So, these guys getting from negative, not the giving to negative. So, 4 is giving to 3, 4 is giving to 2. So, I give to my left; that means, the things are going flowing backward that is inverse cascade. So, energy is flowing to n equal to 1, all energy is going basically towards n equal to 1, so this consistent with inverse cascade of energy, kinetic energy ok, so u^2.

Now, we want to do the same thing for enstrophy. So, enstrophy should forward, but not clear whether it be there be non-local component, local component. So, this energy transfers give you lot of insights; how energy is flowing, and who is getting more energy ok. So, shell to shell give us that information ok. So, stop.

Thank you.