Physics of Turbulence
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Lecture – 27
Kolmogorov's Theory
Energy Spectrum & Flux

So far we have made the formalism. How to write down the Navier-Stokes equation, compute energy transfers compute the flux right the formalism of flux. So, you can also compute the flux given the Navier-Stokes equation, I can compute energy transfers but you need a profile right. Given velocity field you can compute flux.

Now, we are going to do some theory. Given a profile of course I can compute flux, but like to know what happens in nature like within the atmosphere there is a flow is there a some property of the flux. And, turns out it is not very complicated you can understand what should be the nature of flux or what should be the nature of spectrum or energy of the Fourier modes, we can compute them we can give a theory of it.

The first theory was the Kolmogorov's which I will discuss right now, first theory for flux and spectrum, energy spectrum and flux for 3D hydrodynamics, we have to find it for 2D hydrodynamics or MHD (Magnetohydrodynamics) or convection, they show different behaviour.

All of them you start from Kolmogorov's theory or the formalism of Kolmogorov's theory you see how they are different. So, it becomes cornerstone like the starting point for other fields as well including plasma turbulence and so on. Wave turbulence like water waves you know ocean there is a turbulence in that. So, lot of it starts from here then you make modifications, but idea of flux is very important know when I will show you how it is.

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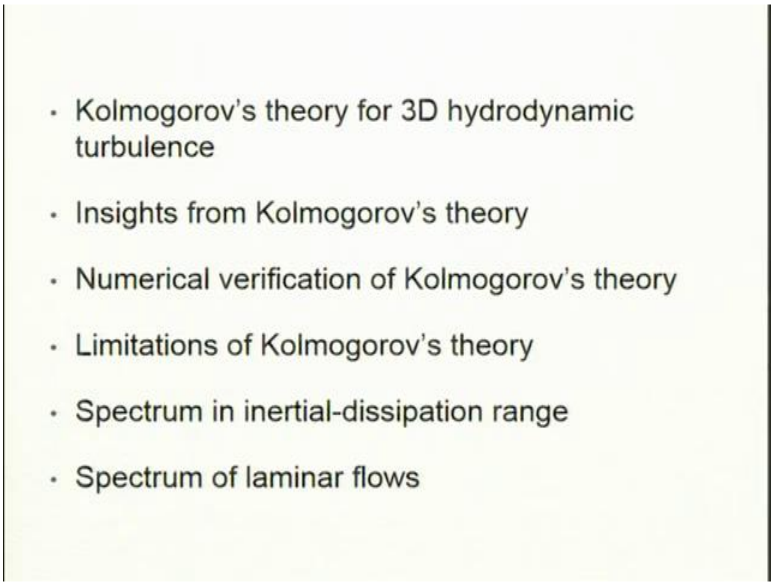

The outline for this set of lectures will be I will derive the Kolmogorov's theory I will just explain what it is. In fact, I will derive it ok, but in our own way it is slightly different ways with different than what Kolmogorov did, but the result is the same. So, insights from Kolmogorov's theory what can you say about fluid, so we can say more things then numerical verification.

I will not get into experiments, but in computer simulations we can find the flux and the spectrum and we find that is very similar to what Kolmogorov's products. And, then limitation of Kolmogorov's theory and we can also generalize it to dissipation range and to also laminar theory ok. So, this is the outline, so we will do it probably in two lectures.

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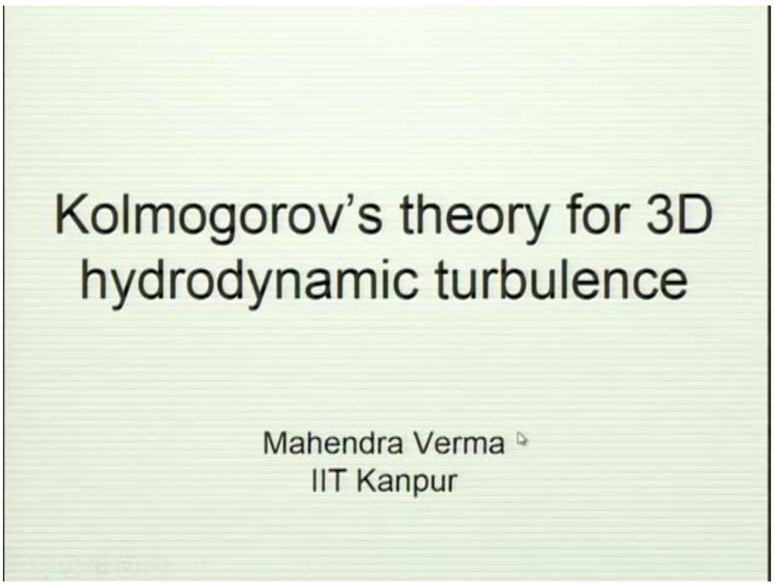

So, let us start it, Kolmogorov’s theories for 3D hydrodynamic turbulence.

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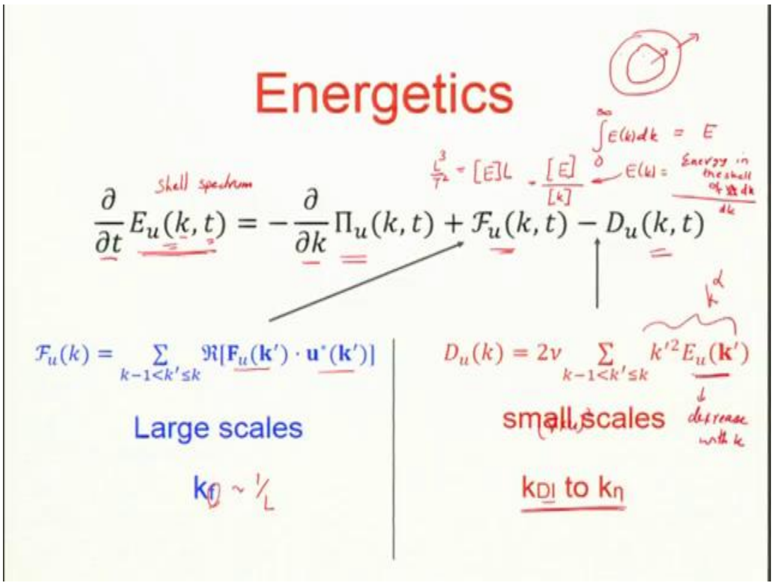

So, this is the equation, see the above figure. This is shell spectrum not a mode spectrum. How do I convert mode to a shell spectrum to from the mode spectrum? I just sum over all the modes within the shell right, there are many-many modes you in the shell it has radius k and for in computer simulation we make the radius as 1.

But in general you can choose dk as a variable dk difference between the radius of the two circles. So, here $E(k)$ is basically defined in the following way because we need for Kolmogorov’s theory, $\int_0^\infty E(k)dk$ is a total energy if I sum over 0 to infinity.

So, $E(k)$ is energy in the shell of radius dk So, that will be shells shell spectrum and that is what Kolmogorov’s five third theory. So, we should know exactly what is $dE(k)$. So, what is the dimension of $E(k)$ by the way in Kolmogorov’s theory or in physics language see the divided by dk. So, it is going to be energy dimension. So, this is a dimension right energy dimension divided by dimension of k. So, that will be k is dimension of one by length, so this is a dimension of energy times length or energy is what is velocity squared. So, I am not keeping mass in it mass is one, so it is L square by T squared.

So, this is dimension of L^3/T^2. So, that is what is meant by energy shell spectrum you must understand this is not energy within energy of the shell. But energy of the shell divided by it is thickness. For computer simulation for like in our code we would make dk

as 1 more quite often. If it is not then you divided by the width. Now, so this is energy of a shell well it is not really a shell, but I mean I have divided by dk throughout ok. So, this like in different in calculation we divide by δx so this is a density.

Energy in the shell, as I told in the earlier classes, you can change by the flux, if flux goes out then energy decreases in the shell. I just want you to keep this in mind these are shell energy changing. So, the two fluxes flux coming from the inner shell inner sphere and flux goes from the outer sphere. It should be the difference of the flux. Flux is always for a sphere and I did tell you in the past that this flux is not like a flux energy going out per unit time per unit area. It is a scalar quantity it is not a vector quantity. It is a difference in the flux for the radius dk divided by dk this we derived in the earlier classes. That is why these are the partial derivative. Now this function of both k and t can change in time. So, that is why we have this ∂t. Energy can also shell a change in a shell by some external force, external force could be buoyancy or some magnetic force you know there could be lots of forces that can change the energy of a shell or it would change by dissipation.

Dissipation always will decrease the energy. So, this $D_u(k)$ is negative, minus $D_u(k)$ is decreasing. Kolmogorov's assumes that we as we force the large scale this is an assumption which is made in Kolmogorov's theory. So, what does it mean? I take a bucket of water and stir using a big stick like this a large scale you do not force a small scale ok.

It is possible that I can have small scale I mean I have charged particles I shine electromagnetic field that will force it small field. But that is not what Kolmogorov's theory is, I force a large scale and large scale means size of the box of the order of size of the box, it could be one fifth half like but it is box size. So, it is important to keep in mind large scale and we divide we make a notation is k_f is a forcing scale with the order of 1 by length of and L is a box size.

Now, what about dissipation? So, dissipation is defined like this $2\nu k^2 E(k)$. We know in fluids that it is it will be some kind of energy at large scale but it keep decreasing, this will decrease with k. So, k^2 is a positive, is increasing function, it turns out all of this one this quantity is k^α, α is positive for intermediate and large scale.

So that means this function is increasing with k. $D_u(k)$ increases with k. So, dissipation is more and more if you go to higher and higher k and that basically comes from the gradient, you know ω^2 this is coming from the ∇, ∇^2 is giving k. So, it is small scale the gradients become stronger and stronger. So, dissipation is active more and more active at small scale in hydrodynamic turbulence. Situation may be different in others other physical system.

But in hydro 3D if I take a bucket of water stir it very fast then you find that it is only that this is a situation. So, this is active more at small scales and so I am going to make a picture. We have forcing as well as dissipation. I wait for some time, so that the system becomes quasi equilibrium that means this energy of the shell fluctuates, but fluctuations around the mean.

If you wait for some time you keep stirring it then bucket water will keep you say average energy will be constant it will fluctuate in time, this is called steady state. But it is not steady state in the sense that it is constant, but it is steady state in the sense that its average is constant.

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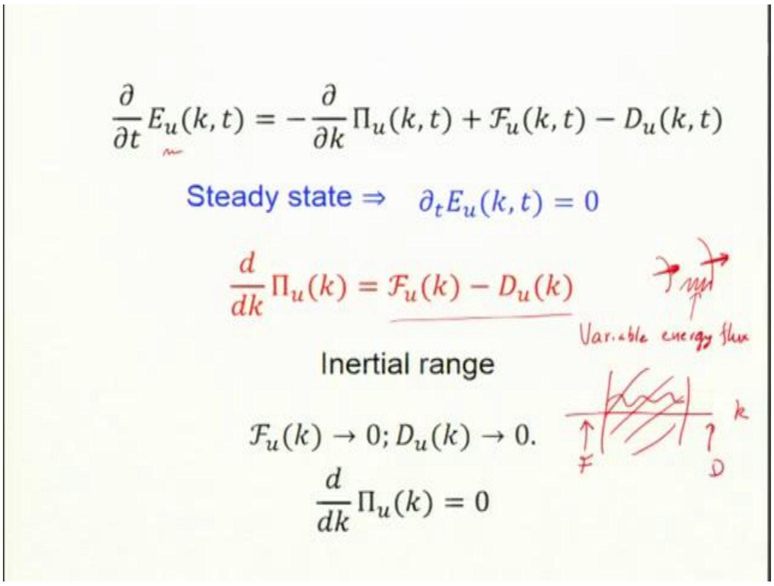

So, make this assumption steady state where I can drop the $\partial/\partial t$ term, this is fluctuating on this one but is changing very slowly. I make instead of partial derivative make a ordinary derivative or total derivative like this.

If outgoing flux is more then ingoing flux under steady state. How this is possible? I mean my energy this energy does not change here know. So, if somebody is like giving more money to people you know donation or whatever. So, it is guaranteed that that person has more income and less spending otherwise they cannot sustain it right.

So, what does it mean if this is positive then my F_u must be bigger than D_u from this equation is straight forward. So, F_u is earning and D_u is dissipation, spending. And, Π_u is like giving to others or taking from others ok. So, these like taking in the left arrow and this is going giving to others you know. So, this is what is so if you give more, then that means you must have more income under steady state ok.

So, this equation for flux, so in fact we I call this variable energy flux is a very nice idea, it is a very simple idea but it is very useful idea we can we can really say a lot with this. Now I make an assumption of the inertia range. What is the inertia range? So, I define two ranges one is forcing scale, where forcing function will be determining the fluid behaviour like in bucket I just stir this.

So, at large scale there is a forcing, so that is how I stir it that will determine the energy at that scale. The small scale where dissipation is more active dissipation is everywhere, but dissipation is strong at small scales. But, these range of scales in between that is called inertia range. And is in the assumption that inertia range it does not know much about forcing neither about dissipation. So, inertia range is a universal function, the spectrum in the inertia range is a universal function that is independent of forcing as well as independent of dissipation.

So, you can see it kind of intuitively in what I am going to say. So, inertia there is a scale in between, so we have forcing this wavenumber. So, small scale will be forcing it small case for it small scale large scale and this is a dissipation this F and this is a dissipation. What happens in between, there is no forcing right in this also dissipation is weak. So, in the inertia range I can say that both are negligible.

So, my flux will be constant and this is assumption of Kolmogorov’s theory, well it is a derivation of a Kolmogorov’s theory. In fact, so if I am energetic I can say the flux will be constant. Of course, flux is fluctuating but on the average it will be constant.

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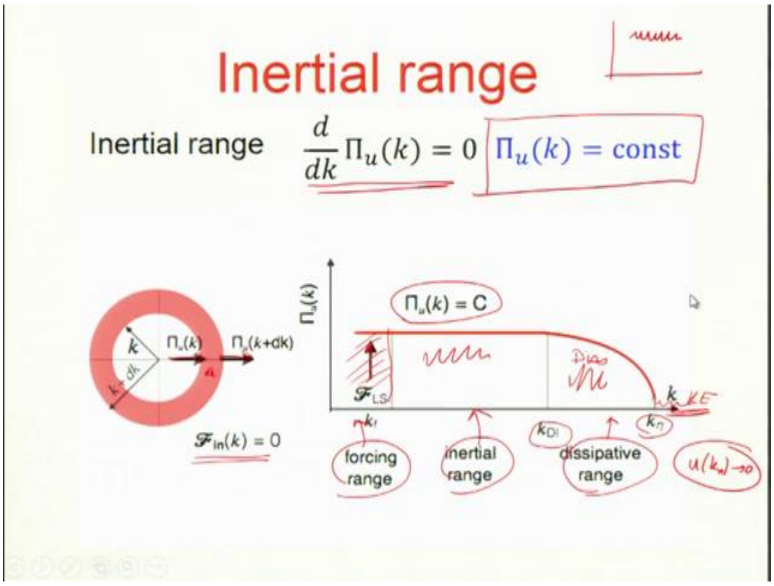

This is a my conclusion in the inertial range. And, if I am not really making any assumption really I start from energetic and it derive flux must be constant. But, as I said this flux will be fluctuating in time, so if I just take this flux so it will fluctuate the time. But, if I just ignore the fluctuation then my mean value average two are slightly longer time is constant.

This is a shell, is my pink region in the shell is the 3D is a cut, so its radius is dk. Now I make this in this if it is k is in the inertial range, then my injection by external force is 0. This is assumption now by Kolmogorov’s theory that I force only the large scale not the intermediate scale. So, these what I force only the large scale. So, you see these are k is a one by length.

So, this k_{Di} is where dissipation starts to become very dominant, in k_η I am going to define where basically kinetic energy is very small after k equal k_η. So, this region has hardly any kinetic energy. So, I was telling the last class, so there is energy spectrum if you draw then energy is almost 0 for k greater than k_η. So, this region has negligible energy. Now negligible in computer simulation will be 10^{-8} in non dimensional unit.

So, this is a region where flux is constant and these called inertia range. So these inertia range is here dissipation in this called dissipation range or dissipative range and this is a forcing range. So, we are know that flux must be constant. So, it is true in a bucket of water I supply energy at large scale then it is cascading, cascade means this is going down in scale it is not going on in real life I mean real space. If you are look at this thing there is no energy equal is going from one scale to other scale. In some sense is very similar to money flowing from large scale to rich people to small middle class then too small, so it is going down like that.

So, very similar analogy and it very small scale this is a transfer to the heat and after that scale there is no kinetic energy. So, if I if you look at zoomed view of the velocity field you see this big vortex, then smaller vortex smaller vortex. But after some time you just becomes a fuzzy blob. There is no definition of velocity field at that scale ok, you just diffused structure it very tiny scale and you cannot define velocity. If there is no motion large scale motion, well I want to this more no motion at there is no visible motion.

Now visible need not be my eye, but visible is like if I make a coarse grained picture in continuum picture I take many 50 particles and see whether there is a mean flow. If there is no mean flow at that scale then you say 0 velocity. So, this is the k eta, now I will not discuss this but these are interesting physics. This is not my mean free path length, the particles please do not mistake this one this is not kinetics a mean free path length it is where my velocity field loses meaning that is what it is k eta ok.

It is that my velocity field $u(k_\eta)$ is tending to 0, that is my definition of k_η. Now given flux can I get a formula for the spectrum. Now, what do I do? One idea is to use dimensional analysis. So, let us do dimensional analysis.

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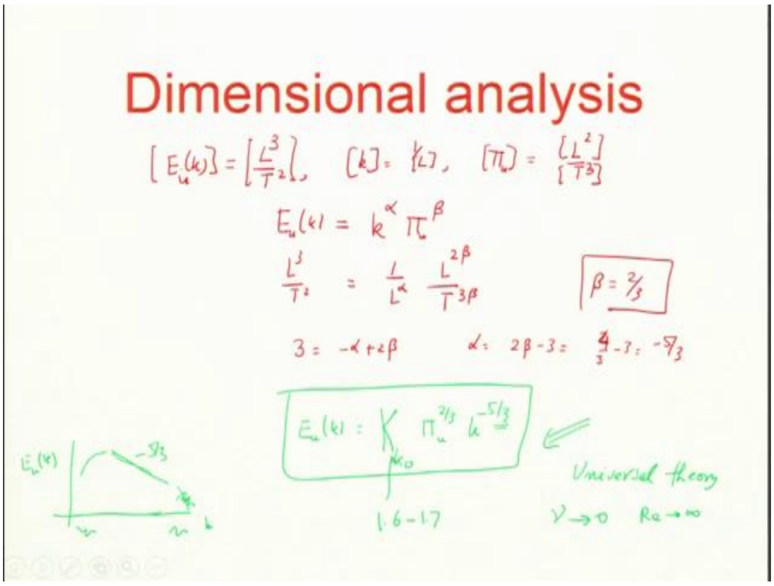

So, it just very straightforward from here, so what is the dimension of $E_u(k)$. So, dimension is this, I derived this I told you in the last slide.

Now, what could what could u depend on? So, let see u now we make an assumption that in this intermediate range flux is constant, I say I believe that it will not depend on how I force the system if irrespective of forcing. It also it does not depend on how it is dissipated how energy is dissipated. So, it is independent of viscosity or how kinetic energy is being dissipated, so it does not depend viscosity as well.

So, it can depend on this flux which is cascading down. So, it has the information of how much energy is being fed at large scale which is cascading. Now because locally I know this is coming down and when it goes through me. So, any scale shell knows that how much which is cascading now ok. So, it has information about how much flux is going through. So, k^{β} and local way number is this local scale know. So, it depends on k, so k^{α}.

So, these two quantities I assume is an assumption that depends on flux and local wave number, it does not depend on viscosity it does not depend forcing this why it is called universal. Now, if I can find it. Now what is dimension of so let us say I will do dimension analysis ok.

I can write down the formula, for energy spectrum $E_u(k) = K_{Ko}\Pi_u^{\frac{2}{3}}(k)k^{-5/3}$, K_{Ko} is a dimensional constant, Ko means Kolmogorov's.

This is a prediction for the energy spectrum formula from this energy argument, derived from dimension analysis but it is great I mean this. So, it turns out this formula works for all flows as long as hydrodynamic, if I put magnetic field there is some indication that it. So, that is different part MHD or convection. So, somewhere also it works, but in hydrodynamics in 3D it works people have done experiments on ocean atmosphere in labs in various internal stuff, so this is seen everywhere.

And that is quite is called it is independent of your forcing mechanism how is so. So, whether I turn like this turn like that or to left to right it does not matter how we do it this will always be there or whether it is water or air or liquid helium or whatever fluid, viscosity is not in this. So, it is independent of which fluid you apply. So, it is like

Newton's law, Newton's law does not care whether you are applying Newton's law to stone or to star or so Newton's law works at classical scale.

So, these are universal theory that means it works for all fluids, all fluid means hydrodynamic with water or mercury and it works for all sorts of forcing in the intermediate range not in forcing range. So, when in the picture which I had shown before, so there is a forcing band this is the dissipation band in k. So, in this band there will be $-5/3$ in the intermediate, this region could be depend on forcing this region could also depend on dissipation.

It is Kolmogorov's constant, has also been calculated by experiment simulation and also by theory, there is a theory, field theory it is around 1.6 to 1.7. So, if these are not like fine structure constant or in turbulence, we cannot get a number with some three decimal places it does not work. There are fluctuations there are effects of the forcing so this is not very precise.

I must say that in real life, of course there is a dependence on. So, I am going to show you this theory has it is own limitations is not exact theory. There are some so if I make experiment on grid and simulation, then this Kolmogorov's depends on what grid resolution I am using, but that is experiment limitation. What Kolmogorov's says that new going to 0 limit, that means the Reynolds number going to infinity limit this is true.

But, no experiment can claim that it is I am infinite Reynolds number. Asymptotically you have to wait for steady state, so these are mathematics theory. So, in experiment there always will be some difference, but in many experiment even though they claim you know great accuracy every experiment has experimental error bars is so on. So, this is a theory of Kolmogorov's which is this called five third theory you know the spectrum is $-5/3$.

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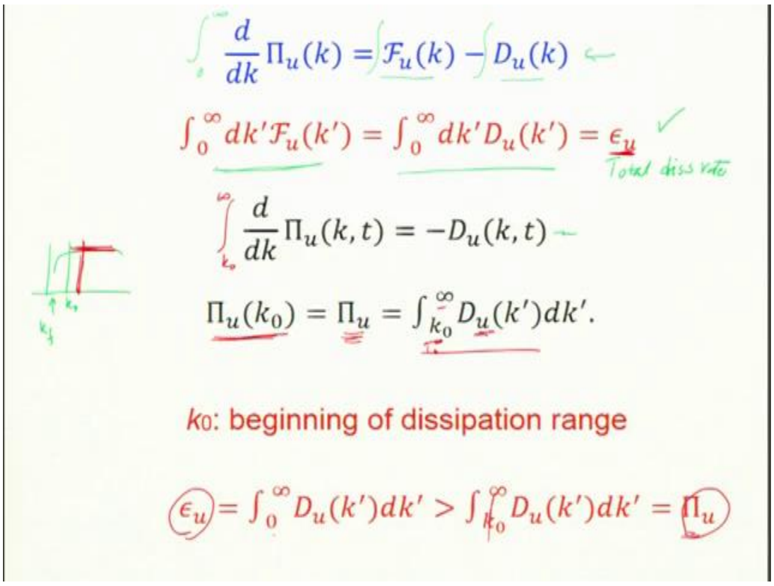

Now, so let us do some more analysis so about the flux. Now under steady state so this is the under steady state now this formula I just showed you in the last slide. If I do the integral over full space full wave number space, so what happens to this 0 to ∞. So, there will be flux at infinity minus flux at 0. So, what is flux at infinity?

Student: 0.

Because there is no more outside to give energy to right I mean this is basically nobody no taker of your modes. So, flux at infinity is 0. So, left hand side will be 0.

The integral is an energy supply rate, so my energy supply rate by external force. So, then you supply rate and what is this integral 0 to infinity is a dissipation. So, under steady state is obvious that whatever I supply energy must be balanced by total dissipation, but this one is 0 to infinity so this exact result. And now we call ϵ_u is the dissipation rate this total dissipation rate.

Now, I can do some more, so the region where there is no force. So, I start from the region where so this was forcing here. So, I start from k_0 where force is not there anymore. So, this is I turn off the forcing, forces well now I do not turn it off, I go to the wave number region where forcing is 0 for $k > k_0$. Here, k_f is a forcing wave number we should left of k_0.

So, let us go now I will integrate this, but I integrate from k_0 to ∞. So, that will give you $\Pi(k = \infty) - \Pi(k = k_0)$. So, I am going to relate the flux with the dissipation rate, this is relating input energy with the output energy this one, but I want to relate it to flux.

So, this is constant flux, where forcing is basically not affecting anymore. I can say that ϵ_u must be bigger than Π_u because, Π_u is k_0 to infinity integral and ϵ_u is 0 to ∞ integral.

So, in your simulation we should test whether you must keep in mind that ϵ_u must be bigger, your flux will be always smaller than ϵ_u, flux will fluctuate a bit. But, flux does not really a fluctuate a lot in simulation which is average to are many-many modes. So, this is one condition which I encourage students to test when you do the simulations. So, I think this is about the first set of slides.

Thank you.