Physics of Turbulence
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Lecture - 24
Energy Transfers Spectral Energy Flux; Shell-to-Shell ET

I will use of the energy transfer, mode to mode energy transfers to compute useful quantities, one of them will be energy flux and which is very useful for understanding turbulence. So, that is what is the topic now for this one, energy flux and shell-to-shell energy transfers, ok. So, I will focus mostly on energy flux.

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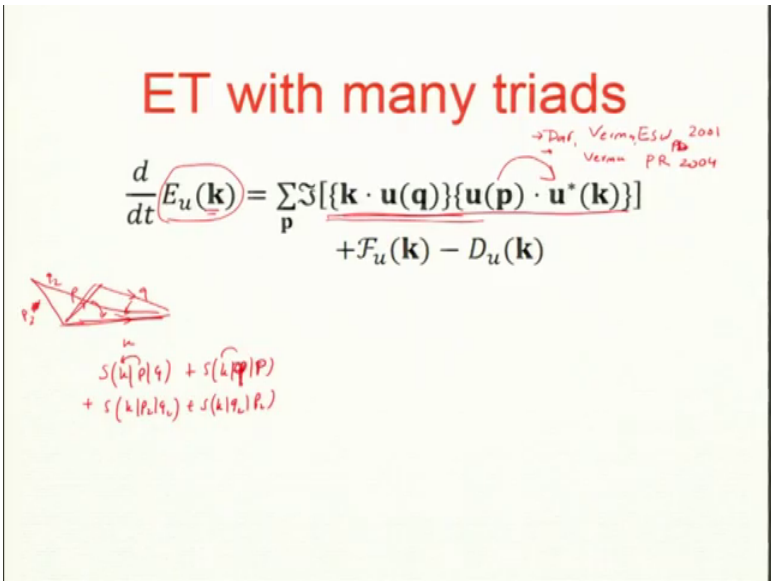

So, let us just try to motivate how to come to energy flux. And some of it, well there was a question yesterday what happens when there more than one triad. Now, in general will have many triads; if I look at fluid flow the example is in the atmosphere or in normal systems there are many triads, but we can easily extrapolate our result to work with more triads. So, this is what I wrote.

This is the full energy equation coming into wave number k. So, this is model energy, right; so, the model energy which is getting contribution from a single triad that getting contribution energy going from. In fact, we know energy going from $\boldsymbol{p}$ to $\boldsymbol{k}$ with $\boldsymbol{q}$ acting as a helper, but there are many $\boldsymbol{p}$'s.

Now, look we derived yesterday energy transfer. So, let us use this $\boldsymbol{k}$ notation. So, if I do this $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}$. So, here is one triad we get energy coming from $\boldsymbol{p}$ and $\boldsymbol{q}$. So, there is a formula derived.

Now, if you have more triad, let us say I have this triad another triad. So, this let us call it $\boldsymbol{p2}$, $\boldsymbol{q2}$. So, $\boldsymbol{k}=\boldsymbol{p2}+\boldsymbol{q2}$. Now, from this energy transfer is additive.

We can see that energy is additive, is that clear. Had it been some other function, then I could not do this operation. Here is you simply add the contribution because this equation tells us that energy is additive, this quantity is additive.

So, what will I do? Energy coming into $\boldsymbol{k}$ will be from here I do for one triad and do for another triad. So, I derived formula for a single triad was $S^{uu}(\boldsymbol{k}|\boldsymbol{p}|\boldsymbol{q})$. Then, I had $S^{uu}(\boldsymbol{k}|\boldsymbol{q}|\boldsymbol{p})$. I can simply add them, right. More triads no problem I can just simply add keep adding them. Everybody is convinced with this.

So, deriving for single triad was for getting the idea, but once you have done for one we can do it for any, I mean you solve one integral of the same given sort then you can do for many, right, so that is the idea. So, we can do it for many triads, and we will use it. So, my energy transfer has total energy is this, but it also has few more terms which we need to do today. There were some questions yesterday on what happens when there is a viscosity.

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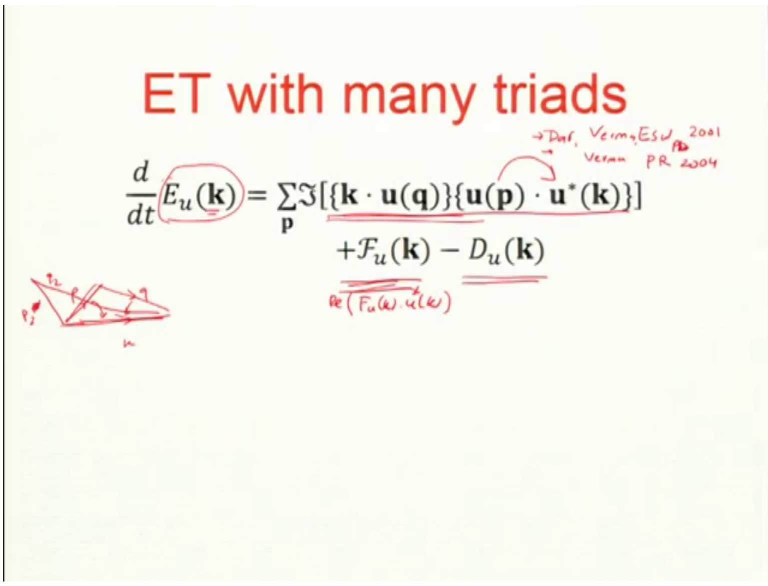

So, viscosity will deplete the energy, we will take away energy from kinetic to heat, you know dissipation basically takes it to heat. And this is external force, external force can feed energy like for example, we saw how convection can feed energy you know temperature through buoyancy feed energy. So, this F_u is energy coming into kinetic energy from external force, ok. So, if you recall this was $\Re[\boldsymbol{F_u}(\boldsymbol{k}).\boldsymbol{u}^*(\boldsymbol{k})]$, ok. So, now we can you were basically you can add more terms.

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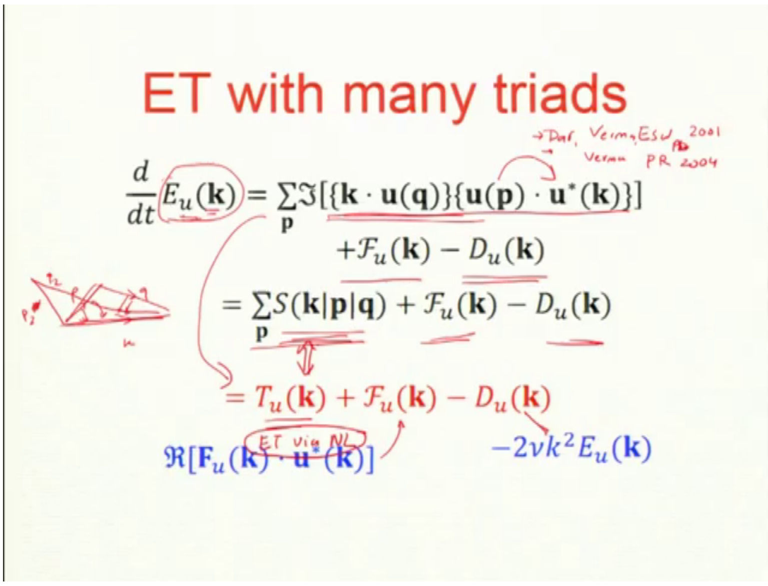

So, as I just said my energy coming into $\boldsymbol{p}$, energy coming into $\boldsymbol{k}$ model energy is gaining something is just sum of all the transfers with all $\boldsymbol{p}$s. So, here I had written it we I just erased it, from everywhere from $\boldsymbol{p}$ to $\boldsymbol{k}$, $\boldsymbol{q}$ to $\boldsymbol{k}$, $\boldsymbol{p2}$ to $\boldsymbol{k2}$, $\boldsymbol{q2}$ to $\boldsymbol{k}$ for this two triads. So, we just have to sum over all triad, ok. So, this is total energy.

Now, is it easy know, I just want one line and in fact, one formula I just say this is the total energy coming via non-linear term. And I will show you how the Kraichnan formula is deficient, it cannot capture some quantities because they do not have this nice formula, ok. We will come to it towards the end.

Now, this is coming from force, external force and this is coming from dissipation. Dissipation is D is always positive, so $-D$ will always be negative. And if you can come with both the signs and this quantity can also come in both the signs. So, if it is negative then; that means, modal energy is losing, the modal energy $\boldsymbol{k}$ is losing to other modes. If it is positive this one, then is gaining from other modes. And this is totally non-linear.

Now, there is a notation. This, this quantity is called also $T_u(\boldsymbol{k})$. So, energy transfer via non-linear term, ok. So, this is the notation in textbooks it is called T_u. They do not well, the, in the textbooks they write this is T_u, ok. Our contribution is that we write this as sum of $\boldsymbol{p}$, ok. Now, I will use this relation and as I said D_u is this one was coming from viscosity, so this I just wrote some time back, ok. So, now what I have done here is with many triads with force included as well as viscosity included.

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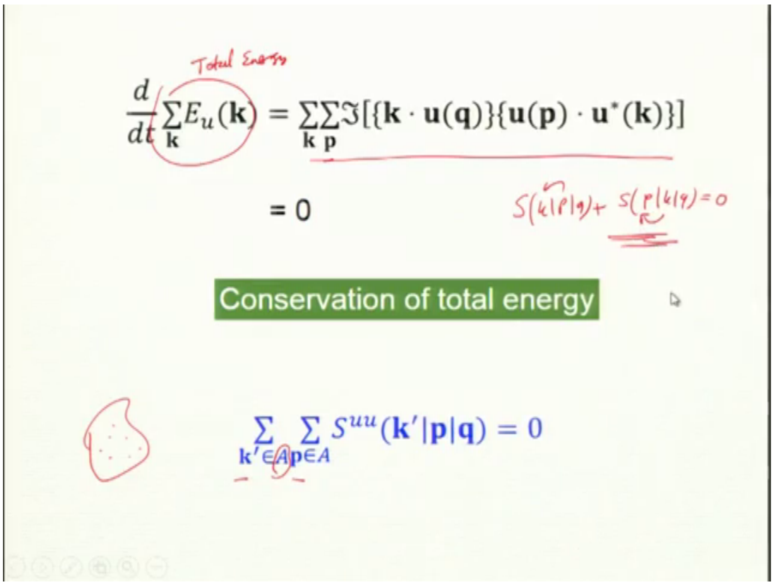

Now, so if I just sum this for all wave numbers, now that was for given k remember the old equation. Now, we do it for all $\boldsymbol{k}$'s. So, this gives you total energy, total energy. So, right hand side there is this earlier there was sum over $\boldsymbol{p}$, now I have sum over $\boldsymbol{k}$. Now, there will be I do have things, ok I am turning up right now the viscosity term in the external force term. So, what is this quantity? Now, I gave the answer, but what is, what is this are we convinced it is 0 or not convinced?

So, this is energy exchange among themselves it is like a party where they are exchanging money among themselves. So, a gets from b and b gets negative of that from a; so, overall $\boldsymbol{k}$ to $\boldsymbol{p}$, and if just sum over when the sum for $\boldsymbol{p}$ comes is negative. It is just quite trivial; you do not need to go. I mean basically in any room if there is any transaction overall there is no gain for the whole room right, when we just exchanged. So, a loses to b, I mean, so a will be positive, but when you sum over b, how much b has gained is negative.

So, it just total is 0. So, this comes, this 0 is coming basically because one property which we did in the last class $S^{uu}(\boldsymbol{k}|\boldsymbol{p}|\boldsymbol{q}) + s^{uu}(\boldsymbol{p}|\boldsymbol{k}|\boldsymbol{q}) = 0$

$\boldsymbol{p}$ giving to $\boldsymbol{k}$ is equivalent opposite to $\boldsymbol{k}$ giving to $\boldsymbol{p}$, ok. So, you will get this sum. When I do the double sum, this will come always in pairs and they give you 0. So, this is reinforcing our earlier statement that non-linear term conserves total energy in Navier-Strokes equation, right. We did this for when we solved for a real space, we also solved in Fourier space I could show that it is 0, the real space I did so for periodic boundary condition or vanishing boundary condition this is true.

Now, I am doing Fourier that means, I am in periodic boundary condition in general, ok. Now, we can modify our vanishing boundary condition also to the Fourier. So, let me make this statement know because I will come back to it maybe later that if this vanishing boundary condition then I can use sin function the sin vanishes of the both the points. So, a sum is not exponential $e^{i\boldsymbol{k}.\boldsymbol{x}}$ which involves both sin and $cosine$, but I am taking some terms of that, ok. So, it is possible, but we are assuming right now there is this periodic just assume periodic boundary condition, ok.

So, this is same result as before that for periodic boundary condition my in total energy is conserved for Navier-Stokes equation via non-linear term.

So, this is same statement. So, we should be able to see that result even with this new formalism. So, this conservation of total energy, and, but ok so this is true. But now with mode to mode we can discover some more finer results. What is the finer result? I can do this for any region of Fourier space. So, region of Fourier space has some wave numbers. So, it is also true as I said there may be many transaction between among all of us or we can have sub transaction during some set of 3 or 4 people and cut off the other transactions, or you do not count other transactions.

So, I do not mean to say cut off is it is like non-interacting. We want the interaction to be there, but we only count the ones which is among set some set of people. Do not count, like I may be getting money from other people, but I do not count that money, I count only among whoever I am interacting with, is that clear. So, we call this region a. So, the set is a, set of mode is a.

And if I sum over $\boldsymbol{k'}$ and $\boldsymbol{p}$ again there will be pairs and that pair when you sum it up you will get 0. So, this is true even for a set of modes if you keep only subspace of those modes, ok. This is useful quantity for this thing. By the way, please keep in mind this is only for hydrodynamics. If I include magnetic field then it will not work, this, this is a proof, is that clear. And this will follow from this relation, a to b is same as b to a with a negative sign, ok

Now, I define spectral flux. So, I want to say that in energy flux in usual physics is what is energy crossing unit area for unit time, that is what you would say know. I mean the particles hitting the area let me say some energy is crossing this area. So, any quantity in fact in is defined what is flux, in usual physics is that quantity crossing the flux of the quantity crossing per any time per unit area. But here and that is normally well depends, it could be so that is a definition for particle flux, energy flux and so on. But this flux is very different. This flux is energy going from one scale to other scale. So, let us draw this. This is Fourier space, ok.

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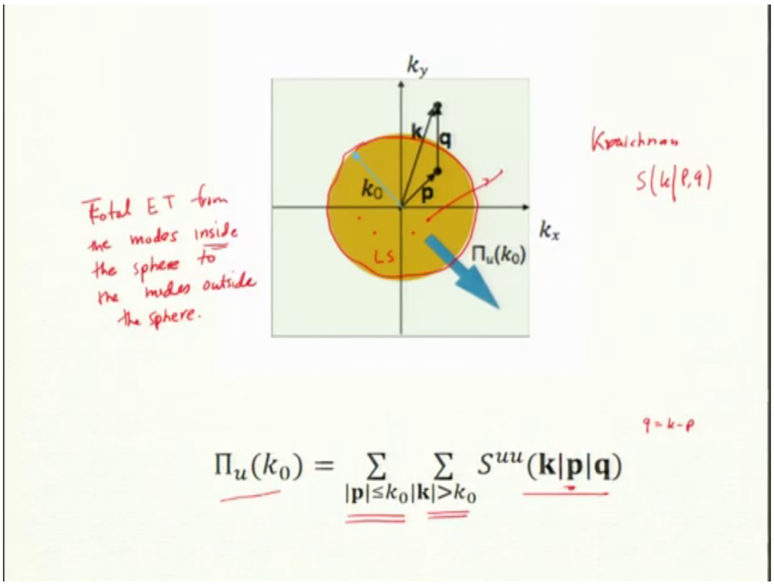

One thing this is not a real space and is defined as energy crossing this wave numbers sphere. Energy coming out of the wave numbers sphere via non-linear interaction; so, the modes here inside this, in this yellow sphere will interact with modes outside, right non-linearly.

So, there will be some energy going out. You sum up all the energy count all the energy going out of by this non-linear interaction, some, all, scalar quantity this is not a vector quantity, ok. It is because summing of the energy. So, you sum of all the energy going from the modes inside this sphere to outside the sphere, ok.

So, let me write this. So, energy total energy total energy transfer from the modes inside this sphere all the modes, to the modes outside this sphere is that ok. For, for well for consistency in our numerical simulation, we also keep inside means including the surface modes. So, inside I include the modes on the surfaces. Surface guys that well count me you know. So, either it counts for outside or count for inside you count from inside. There will be problem with the energy balance, I mean there will be some missing out if you do not count them, ok.

So, how will I write the expression now? So, this was in fact, his quantity postulated well Kolmogorov did not postulate this, but this understanding that this is large scale know, inside is like wave number is inversely proportional to length. So, these means larger scale and outside will be smaller scale compared to the ones inside the box inside this sphere. So, if the energy going from large scale to smaller scales is like many cascades, I said from let us say from one bigger companies to smaller companies. So, this is what is this flux means.

Now, it was computed by Kraichnan and he gave a formula Kraichnan using combined energy transfer, and the formula is quite complicated. Well, it involves quite a lot of derivation, but with mode to mode is straightforward. So, how will I write this formula with mode to mode? I have a $S^{uu}(\boldsymbol{k}|\boldsymbol{p}|\boldsymbol{q})$, right, so the giver should be where?

Outside this sphere, so that is it. So, we write this flux. So, this radius of this sphere is k_0. So, we have this stuff. So, $\boldsymbol{p}$ is the giver, must be within this sphere including the surface and the receiver is outside this sphere $\boldsymbol{k}$ greater than k_o.

Carrier comes from the constraint $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{p}$ the carrier could be inside this sphere or outside this sphere that is immaterial. And that is where Kraichnan had a problem, because in combine energy transfer it is $S^{uu}(\boldsymbol{k}|\boldsymbol{p}, \boldsymbol{q})$. So, both $\boldsymbol{p}$ and $\boldsymbol{q}$ give. So, if $\boldsymbol{q}$ is anywhere then this guy does not know how to count them, whether it should be it is in the giver zone or in the receiver zone and that is precise the difficulty. So, $\boldsymbol{q}$ is indeterminant and that is where we have to account, I mean he has to account properly and that is why the derivation is long, ok.

The formula is which I will show in the later part of this, well I will not prove it I just state the result, ok. Its proof is quite easy with a modern perspective, but it is quite long with in the Kraichnan's paper, is that clear. So, this is how we compute flux.

Now, we also can get this flux from some other formulas. In fact, we need some more well, this is a formula for flux which we use it for computing in computer simulations in various calculation. But let us see it from some other perspective.

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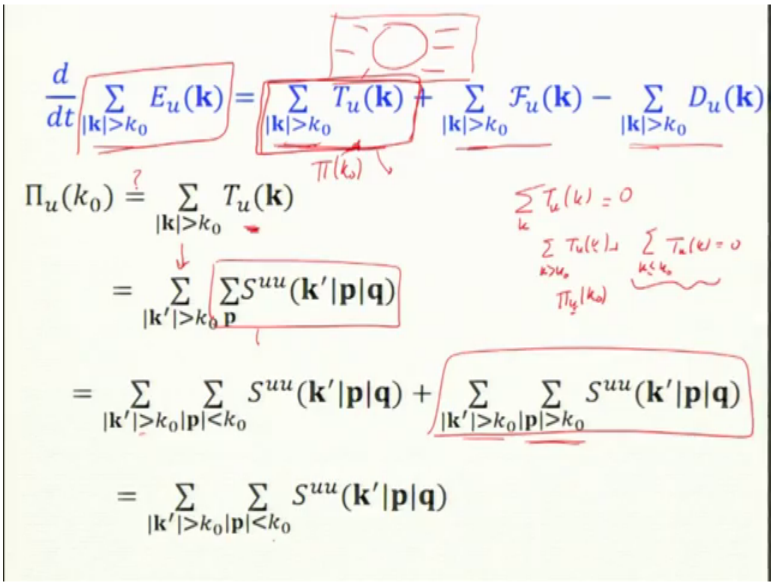

So, this is the formula. Well, I wrote a formula without the sum. This sum was not there. So, remember $\frac{d}{dt}E_u(\boldsymbol{k}) = T_u(\boldsymbol{k}) + F_u(\boldsymbol{k} - D_u(\boldsymbol{k}))$, now I am just introducing the sum for modes outside this sphere. So, what does it mean? This corresponds to sum of the energy of the modes outside this sphere. Are you happy with this?

So, this $\frac{d}{dt}$ of the energy in the Fourier space outside this sphere, outside the sphere of radius k_o, ok. So, I had this box which includes all the modes, but you exclude the modes inside, ok. Now, this will be non-linear term. So, this is the sum, the this is also sum over force contribution in this dissipation.

Now, let us look at this term, sum of $T_u(\boldsymbol{k})$. Now, I will show you that that is equal to flux. In fact, these also equal to $\Pi_u(k_o)$, ok. How will I prove it? Now, let us use our idea of mode to mode. So, $\Pi_u(k_o)$, this is my question mark. Is it true or not? So, I have one definition already I gave you; $\boldsymbol{p}$ within this sphere and $\boldsymbol{k}$ outside this sphere.

Now, we will see whether that formula matches with this one. So, I showed that $T_u(k) = \sum_{|\boldsymbol{k}'|>k_o} \sum_{\boldsymbol{p}} S^{uu}\,(\boldsymbol{k}'|\boldsymbol{p}|\boldsymbol{q})$. This was done two slides back. Now, this sum is coming here well I put now the $\boldsymbol{k}'$ is same as $-\boldsymbol{k}$, ok. For energy we can interchangeably change it, so $\boldsymbol{k}'$ or $\boldsymbol{k}$. Now, these can the now $\boldsymbol{p}$, so $\boldsymbol{k}'$ is $\boldsymbol{k}, \boldsymbol{k}'$ is greater than k_o, but $\boldsymbol{p}$ can be anywhere, and $\boldsymbol{q}$, $\boldsymbol{q}$ will come from $\boldsymbol{k}-\boldsymbol{p}$. So, I write this sum, two sums, $|\boldsymbol{p}|$ less than k_o and $|\boldsymbol{p}|$ greater than ko. So, then I summed it this one is $\boldsymbol{k}'$ is getting but, $\boldsymbol{k}'$ is getting from all $\boldsymbol{p}$'s. So, are you happy with this? This I go from step 2 to step 3.

Now, what is; there are two terms in this step 3. So, these the second term is 0 because $|\boldsymbol{k}'| > \boldsymbol{k_o}$ and$|\boldsymbol{p}| > \boldsymbol{k_o}$. I had said no, this is true also for any specific region. So, this is that quantity for all these spheres, all the modes outside this sphere. If the giver and receivers have the same region you sum it up, you will get 0. So, this is 0. So, the left-hand side is this.

Now, is it not it exactly the same formula which I wrote in the last slide. So, $\boldsymbol{p}$ is a giver which is inside this sphere and $\boldsymbol{k}'$ is a receiver which is outside this sphere, ok. So, this indeed is our flux. In fact, this one of the steps of Kraichnan's derivation, but of course, he does not go through the derivation, but he argues in different ways, ok.

Now, if I sum over $T_u(k)$ over all $\boldsymbol{k}$ then what will I get? 0 know, this we showed this is 0. That means, so you from here I do this is $\sum_{|\boldsymbol{k}|>k_o} T_u(\boldsymbol{k}) + \sum_{|\boldsymbol{k}|\le k_o} T_u(\boldsymbol{k}) = 0$. This is $\Pi_u(k_o)$. So, this must equal to what?

$-\Pi_u(k_o)$. So, this is by the way subscript means, kinetic, because we will also work with other fields. So, we have notation the subscript is u means is kinetic.

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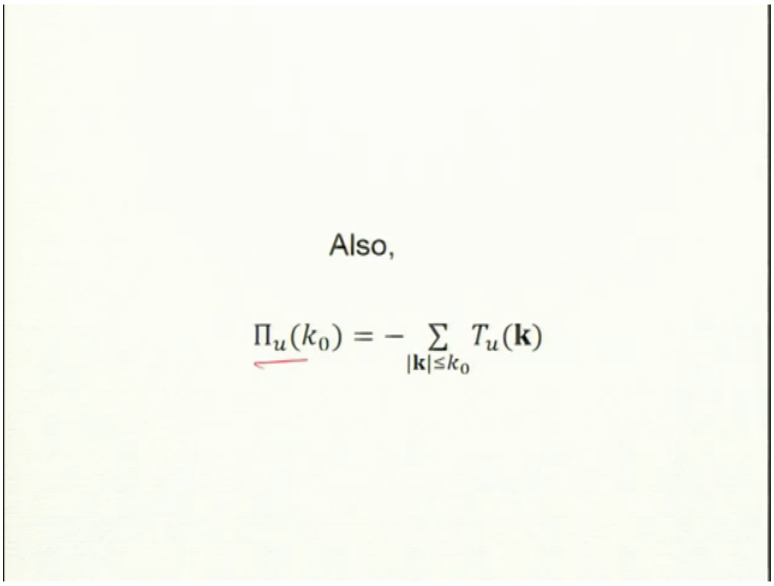
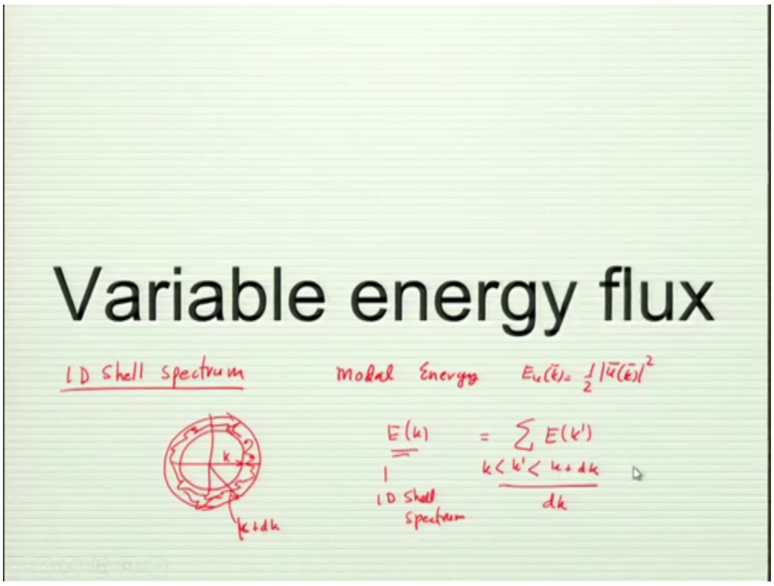


So, this is what I write in the next slide. We also write $\Pi_u(k_o) = -\sum_{|\boldsymbol{k}|\le \boldsymbol{k_o}} T_u(\boldsymbol{k})$. So, we can use both. So, this is very useful formula for my next derivation, ok. Now, this straightforward know, this straight away comes from energy equations.



Now, we define something called variable energy flux. Now, this concept this wait I mean forget about variable right now, we are going to derive how the flux varies in from one wave number to other wave number. The flux is function of k, so it is not necessary that is same everywhere know. So, let us see how it varies with wave number.

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$$\frac{d}{dt}\sum_{k<k'\le k+dk} E_u(k') = \sum_{k<k'\le k+dk} T_u(\mathbf{k}')$$

$$+\sum_{k<k'\le k+dk} \mathcal{F}_u(\mathbf{k}') - \sum_{k<k'\le k+dk} D(\mathbf{k}')$$

$$\frac{\partial}{\partial t}E_u(k,t) = -\frac{\partial}{\partial k}\Pi_u(k,t) + \mathcal{F}_u(k,t) - D_u(k,t)$$

$$2\nu \sum_{|\mathbf{k}'|=k} k'^2 E_u(\mathbf{k}')$$

So, now, so let us define it. 1D shell spectrum, did I define 1D shell spectrum?

So, let us go back in the earlier slide, ok. I am going to define 1D shell spectrum, ok. But anyway, let us even if I define I will do it again. So, we have something called modal energy. What is modal energy? It is the energy of a mode. So, $E_u(\boldsymbol{k}) = |\boldsymbol{u}(\boldsymbol{k})|^2/2$. So, the energy of a given wave number $\boldsymbol{k}$. If the system is, ok, see if I make wave numbers sphere or let us make a wave number shell, this is the wave number shell we saw the shell now this is $\boldsymbol{k}$ and this is $\boldsymbol{k} + \boldsymbol{dk}$. So, my shell is this, is that clear to everyone.

So, they are modes within the shell the shell thickness is dk. Now, we define, so if system is isotropic then modal energy of any mode inside the shell must be equal, right. If I take a mode here on the average, and mode here, since this isotropic we expect that energy to be equal. So, is a good well we know we have too many modes, is a well less sum them up and get a number for a $\boldsymbol{k}$, wave number for a shell, shell of radius k with thickness dk, we can get energy content. So, is defined like this. So, $E_u(k)dk = \sum_{k<k'<k+dk} E(k')$. This is the definition.

See, with if I divide well if I make dk will go to 0, this quantity with the sum will also be very small number. So, I can divide this by dk I can get $E_u(k)$, ok. So, this is called 1D spectrum, 1D shell spectrum. In computer simulation, we make $dk = 1$.

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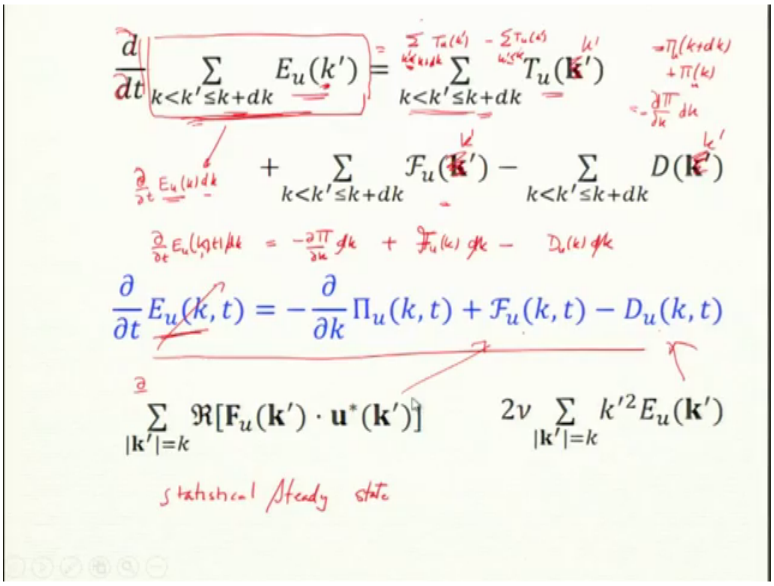

Now, that is what I have done. So, instead of summing from 0 to k, 0 to k_o or k_o to ∞, we will now do the sum of. Now, this is 1D spectrum, so there is no, this is not a vector anymore, this is a scalar. It is a magnitude of $\boldsymbol{k}$. So, k' goes from k to $k + dk$. So, right now I am including the well this equal is here, $E_u(k)$ equals here.

Now, so I am just doing the energy equation. So, I made a mistake this should be k, k' and k', no more vector. So, I am summing over shells. So, shell has a number, ok, fine.

Now, this $T_u(k)$ so, let us look at the next step now. $T_u(k)$ is connected to flux, now this is for the two spheres, sphere of radius $k + dk$ and sphere of k is this quantity is the difference between the two spheres. I can sum over $T_u(k)$ over bigger sphere and subtract from the smallest sphere I get that quantity in the box; so, the right, so the first quantity is well.

Now, take the $dk \to 0$. So, what let us write this one. What is this quantity? So, it is $-\Pi_u(k + dk) + \Pi_u(k)$.

We can make also no, this is I generally should make a k' for a shell, ok, is that clear; in difference between the two spheres. So, this is that. Now, if $dk \to 0$, then what is this quantity?

This is $-\frac{\partial \Pi_u(k)}{\partial k} dk$ a partial because it can also depend on time. Yes, what about this quantity?

We can write this as $\frac{\partial E_u(k,t)}{\partial t} dk$, id $dx \to 0$ then again, any quantities $f(x)dx$. Similarly, these quantities are $F_u(k)dk$, $D_u(k)dk$. So, dk gets cancel, but dk is removed dk going to 0 limit its cancels. So, I get this equation. So, my flux is changing with k also with time. And energy is also changing with time and space, sorry time in time in Fourier k. So, this is equation. See the slides for details.

Now, let us work on under steady state. So, I am going to make the assumption of steady state. So, how can fluid turbulence become steady? If I supply energy and wait for some time, then there is a flow of energy to small scale. So, energy supply matches the dissipation, then I get a steady state, ok. Like my income and my spending are roughly equal then I have a steady life you know my bank balance does not change very much and think.

It is not a saturation, it is like system is still fluctuating, but it has a on the statistical average, for energy it is not a saturation, it is evolving. But it does not change on the whole. So, energy of a given sphere or given shell will not change, it will be average quantity. It will fluctuate around a mean, but if the mean will not change local mean will not change at k. So, steady state is when my force and dissipation are balancing each other. So, in my energy of a given wave number will not change in time.

So, I said; so, steady state normally you know people say steady state with $\frac{d}{dt} = 0$, thing should depend only on k not on t. But in turbulence of course, things change in time, but we are talking about average. So, I sum over time unit, then is one time unit, then another time unit of same length I should get the same number, ok. So, this is statistical steady state, in fact, that is the word, which is used statistical steady state, ok.

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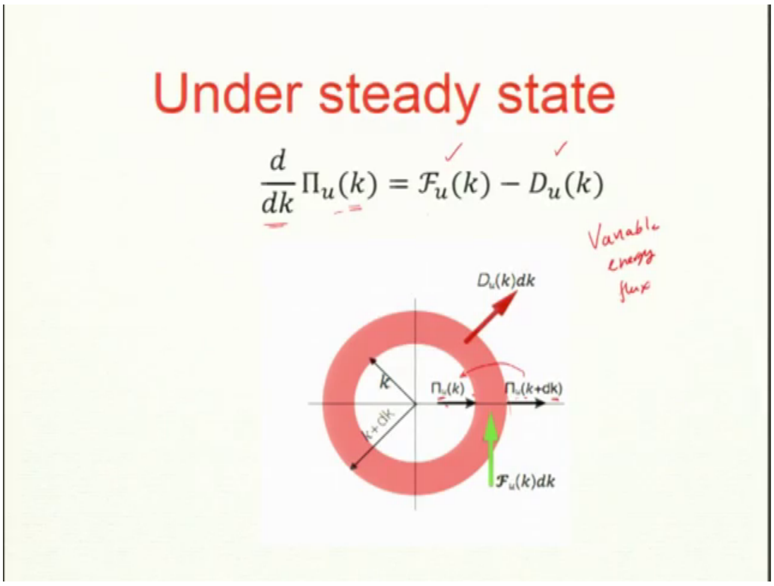
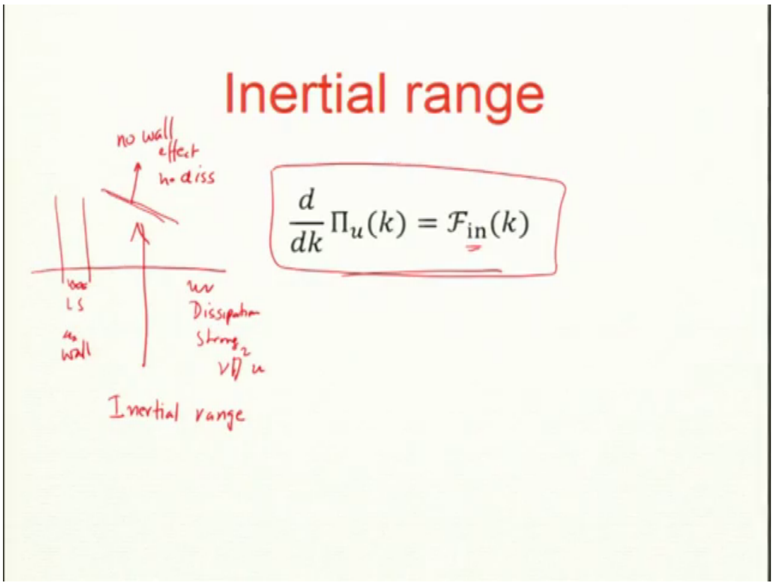


So, let us look at what happens in statistical steady state. So, my $\frac{d}{dt}$is gone. So, let us let us go back to this equation once more. So, I can take this to left hand side. So, $\frac{d}{dk}\Pi_u(k) = F_u(k) - D_u(k)$, for any k, this is for any k.

So, this is local quantity and please remember it is not the total. So, if it in any k this will work, under steady state. So, this is a sphere, our thin sphere which is radius dk starts from radius k to k plus dk. Now, we are supplying energy $F_u(k)$ is dissipates $D_u(k)$ and then they need not be equal $F_u(k)$ and $D_u(k)$ need not be equal.

So, this $\Pi_u(k)$ here and here they can be different, but they will not change in time, this quantity is independent of time, this quantity is independent of time, so, but I know for sure that the difference between these two divided by dk must be equal to $F_u(k) - D_u(k)$. So, this is income, and this is dissipation for the shell. So, if F_u is bigger than what will happen? Which will bigger than D_u then which should be bigger? If $F_u(k)$ is bigger, then $D_u(k)$ let us say this.

And if $F_u(k)$ is smaller than $D_u(k)$ then other way around, ok. In fact, this is very useful conclusion. This is very simple idea but is very useful. And this we call variable energy flux. So, yeah, our energy flux is changing with k.

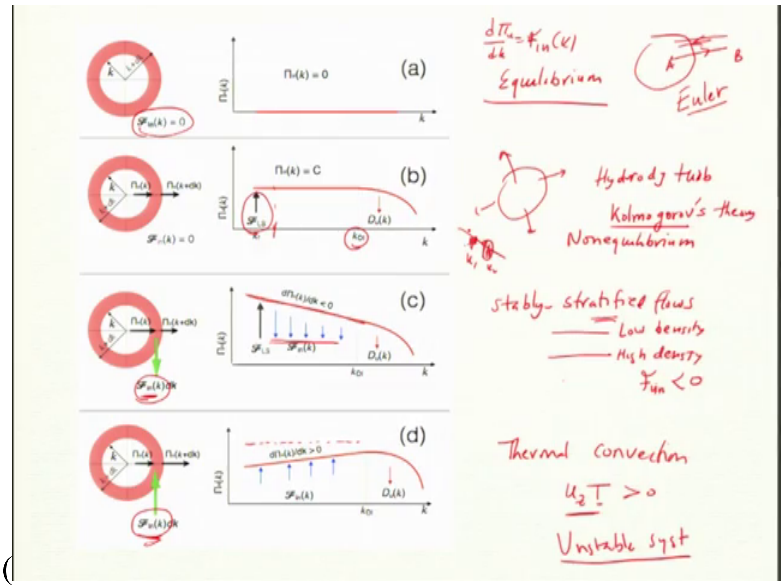


Now, let us focus on inertial range. So, I am also now going down further and further narrowing myself. So, what inertial range is, what is inertial range? So, we divide in our wave number space in 3 regions this is typically done in hydrodynamic turbulence, but that is to follow I will do, follow up I will do.

So, there is one region where there is some external forcing. Well I of course, I has a forcing sitting here, but let us imagine well there is some large scale, you do not call external force, there is large scale, where the box effect will come into play know the walls and so on. This is the region where dissipation is strong, dissipation strong. In between is the inertial range where dissipation is ignorable, and the wall effect is also ignorable. So, no wall effect, no dissipation and this is called inertial range.

So, inertia means, well this is a slightly technical, I mean in the sense actually I will not say technical it is community notation, the fluid dynamics notation is $\boldsymbol{u}.\nabla\boldsymbol{u}$ is the inertia term, acceleration term you know $\boldsymbol{u}.\nabla\boldsymbol{u}$. So, if the region where inertia is important; so, $\boldsymbol{u}.\nabla\boldsymbol{u}$. is like it is not viscous term viscous term is $\nu\nabla^2\boldsymbol{u}$ So, here $\nu\nabla^2\boldsymbol{u}$ is important. Here some wall effect is important, in left side wall, in between is where only the non-linear term is playing a big role, non-linear and pressure, both I would say, fine. So, and these are notations.

So, we are focusing on the middle region and in the middle region I will get this because dissipation is weak. So, I drop the D_u and instead of F_u I write F_{in}. So, there could be some force here in large scale, ok. So, I will call inertial, in for inertia. So, force in the inertial range. In fact, it is just that I just drop the D_u term and I come I look at the local force, ok. So, this is our important equation to look at the property of flux in the inertial range, ok.

(

Now, my next slide is slightly dense slide. So, there are 4 cases know, I will go one by one. In fact, that will answer your question. So, let us consider case where my inertial range force is 0. So, if inertial range force is 0 then $\frac{d}{dk}\Pi_u(k) = 0$. So, flux will be constant. Now, flux being constant has two scenarios flux is 0 or flux is a constant value it also be negative, but in 3D energy flows from large scale to small scale, so it is positive, in 3D, in 2D it can be negative, but we will do it later. Let us assume we are working in 3D and it is positive; so, two scenarios, 0 and positive. Now, when do you get 0? Can you think of situation where we get 0 flux?

$F_{in} = 0$ i, ok. Now, this is a scenario when there are many modes at different scales, but a giving to b same as any region and there is equilibrium, ok. This is called equilibrium. But things are existing at all levels. So, there are companies, big companies, small companies, rich people well there is not small, but this is not possible. Equilibrium means somebody has to give, non-equilibrium is well rich will always spend more. I mean these analogies I like to make. So, there will be more this money flowing from rich to poorer, ok. So, this is non-equilibrium; that means, one is giving more than other.

So, what is equilibrium? It is any transaction must be equal a to b should be same as b to a, not equal and opposite, a gives 5 rupees to b, b gives 5 rupees to a, ok so, not my, it is not that b receives. So, equilibrium means there is no net coming to anybody. If there is no net energy coming to anyone then what happens to the flux. Is will there be any transfer from any level to any level?

Student: No.

No, right. So, this flux was transferred from one scale to other scale in fact, large scale to small scale from the sphere. If it is not this balance of transfer from any to any then there will be 0 flux. So, if I draw this sphere, if a to b and b to a are equal then what happens? So, it is 0 flux, right. So, there are modes outside this fine, there are modes inside, but they do not exchange on the average, so that is like thermodynamics, you know thermodynamics is supposed to be equilibrium. So, equilibrium means there is no net transfer.

By the way, equilibrium well, thermodynamics normal assume single scale. The molecules in this room are not of different sizes or not having different energies. They are roughly; so, this is richer the scenario is different scenario where there are different scales. So, bigger molecules, small molecules, small molecules size is different know k. So, it is not exactly analogy with thermodynamics, is different. But there are situations where it does is claimed to how happen the Euler equation, when there is no viscosity then Euler equation has something similar physics is this, where each mode is not any mode is not giving any energy to another mode, ok. But we will do this bit later. So, I will not worry about Euler, but this is equilibrium scenario.

Second scenario, when there is a net flux. So, there is energy going out of this sphere. So, energy goes from large scale to small scales in 3D, so that is a constant flux. In fact, I do some stirring at large scale, I shake up this fluid and that finally, goes into heat. So, this is our scenario of hydrodynamic turbulence. And this will lead to Kolmogorov picture which we will do it after the in the next class.

Kolmogorov theory says energy flux is constant, but I derived it from energy equation, I did not assume it. If there is no net force in between in Kolmogorov theory, Kolmogorov assumes that we supply energy only at large scale. So, there is external force, but that is not in the inertial range, in these inertial range, this between k_f to k_{DI} means, DI means dissipation. So, these are wave numbers. In this region there is no force, is that ok.

So, I derived it my energy, I really did not assume anything. Energy equation put force to 0 flux must be constant whether this is independent derivation and so, this is Kolmogorov derivation is different which we will do later in the course. So, this is coming straight from energetics in Fourier space, fine. So, flux force being 0 gives you two scenarios.

A third is $F_u(k)$ this one is negative. So, then flux according to this definition our formula it should decrease. So, $\frac{d}{dk}\Pi_u(k)$ must be negative. So, flux will decrease and I can see that the $F_u(k)$ is decreasing with k, so it is negative, you see $F_u(k)$ is negative, so my flux will decrease. And this does happen in nature.

Now, one scenario is which we will I do hope to find time to discuss this for stably stratified flows. Like earth atmosphere the stable stratification high density below and low density above and I can stir this. In fact, in earth atmosphere there is some turbulence, but is a stable system, so whatever energy you try to feed in these are the gravity will in fact, it turns out which I do show it in those research papers that because of stability at the system the external force which is buoyancy will take the energy out. So, we can show that this is negative, ok for stably stratified system.

Now, the proof is not so straightforward, but we will do it hopefully later. But we can show for this system F_u F_{in} is less than 0. Actually, so this is since its stable. Now, let us look at the other scenario, when my force is positive. So, this green arrow going into the sphere shell means it is feeding energy. So, my force is positive, force feed, energy feed by external force is positive. So, what will happen to the flux? Flux should increase which is what is the scenario. Now, can it happen or where does it happen?

Inertial range only, but physical system. In real life we had to find a scenario when this happens. This happens in thermal convection. In thermal convection we know that buoyancy, so it is a fluid is at rest. So, what is kinetic energy flux? If it is rest.

0, because velocity is 0. If I turn on the heater and in the so start convection. So, what happens? Every sphere has got every shell has got energy. So, I fed the energy through buoyancy. So, buoyancy is always this is positive. You so, buoyancy well in fact, we know that hot fluid rises, right. I mean this is, so u_z and times temperature. So, this is the energy feed know. So, buoyancy is proportional to temperature and multiply by velocity.

So, F times u is in fact, is proportional to this, force is proportional to temperature and I multiply by u, so which is u_z. So, this is positive. Hot, if T is greater than 0 greater than mean then u_z is positive. So, energy feed by buoyancy is positive anything. But this is straightforward proof, and the cold flumes come down, so T is less than mean and u_z is negative. So, they are come in the same sign.

So, here my flux should increase with k; that means, lower wave number modes, higher wave number modes small scale fluctuation should become more violent, more and, ok. This is what we try to do this simulation and this analysis. We find that it for relevant and convection does not really increase it is flat, ok. For reasons which are more complex which I cannot discuss right now. But in thermal convection is possible that the flux can increase, ok. So, this is the example.

And other things like bubbly turbulence when you put in bubble or unstable systems. All unstable systems come in this category that is my statement. There may be some exceptions, but that is my stable systems come under this category, if there is the force it all scales unstable systems come under this category. So, variable energy flux can help us formulate and understand the energy flux via these formalisms, ok.

So, it is very useful, though it is like based on energy and it straight away comes from energetics arguments you can understand flux. And once I understand flux, I can also say something about energy in a in a sphere, we call 1D energy spectrum, ok. So, let us go further. So, is that clear to everyone, I mean this is important we can discuss some more. So, let me just summarize. So, $\frac{d}{dk}\Pi_u(k) = F_{in}$. Now, if F_{in} is 0, then I have two scenarios, my flux is 0 which is equilibrium, flux is constant it is non-equilibrium. I also would like to say that this is non-equilibrium.

So, thermodynamics ideas do not work here, these are net transfer of energy from one scale to other scale, so that is now equilibrium means a to b is same as b to a, there will not be change in energy or there will not be any transfer net transfer in equilibrium, but here there is a net transfer though energy does not change, mind you. So, whatever I get; so, this big is supposed to be whatever I get I must pass on it a smaller scale.

But the question is why does energy is more for given scale then other. Here you will find typically the energy is larger its $k1$ compared to energy at $k2$, when $k2$ is larger. Why it is so? Because they are more modes here; so, is in fact, this analogy is reasonably accurate that their poorer more people with less money receive from fewer people with more money. Looks like there is some kind of inequality is built-in in this non-equilibrium. This is unequal know when there are few poor rich people. So, energy is more here because there is fewer rich people.

The number of modes in a wave number $k1$ is $4\Pi k_1^2$. So, lower k less number of modes. And so, we will discuss some of it later when we do turbulence, but this net transfer and when the when F_u is negative and then my flux should decrease $F_u$is positive flux should increase, ok.

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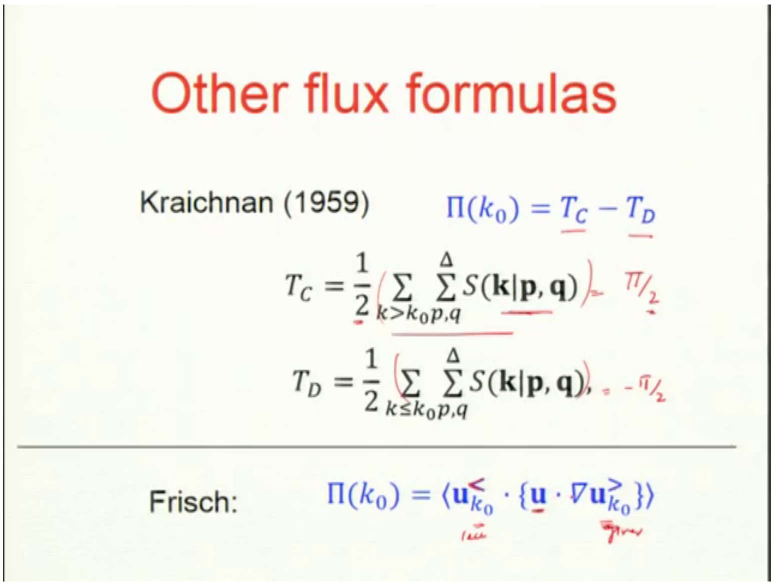

Now, let us go further. So, other flux is Kraichnan formula. Follow the above slide.

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But we can get more detailed picture by something called shell-to-shell energy transfer. So, relate quantity shell-to-shell energy transfer which gives you more information. So, flux was energy going out from inside the sphere to outside the sphere.

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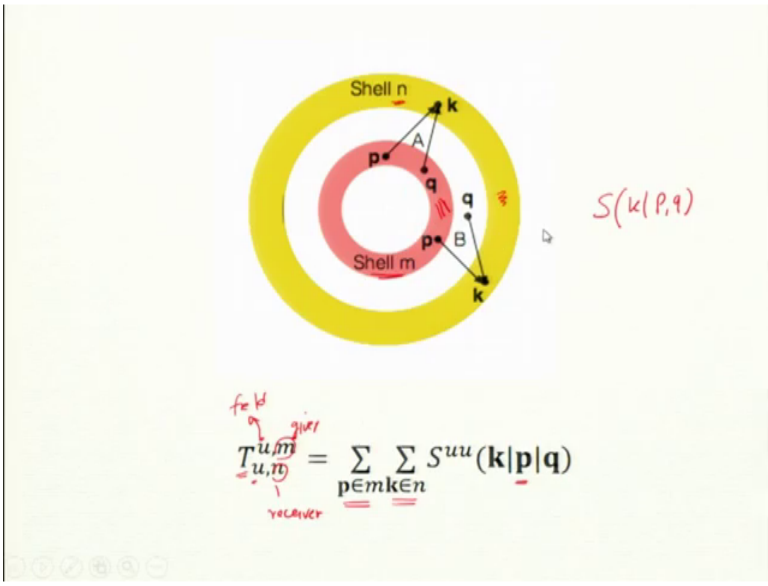
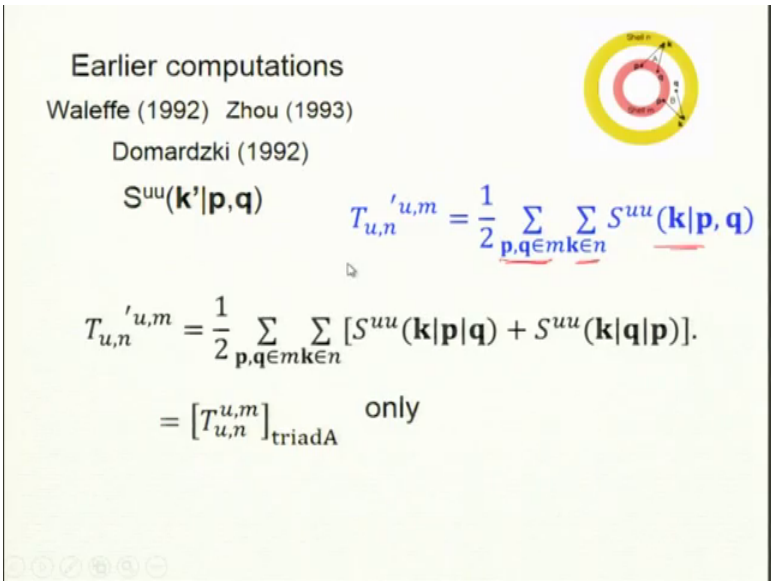


Now, we can get more detailed picture by this. Energy going from shell m, this shell m, there are two shells I am drawing here. This is again Fourier space in yellow shell is shell n, ok. So, we just label them. So, these are supposed to be lot of modes in it in shell n and shell m, thickness do not worry about it right now. I am not telling you that the same thickness, but they are shells.

So, we can think of energy going from shell m to shell n, right or shell n to shell m. There are two different regions. It is equilibrium of course, there will be 0, but non equilibrium there will be transfer. So, how do I compute? Energy going from shell m to shell n so, here the notation is this shell-to-shell transfer $T_{u,n}^{u,m}$. The giver is above, so m is a giver and n is a receiver and the first symbol is field, right now I am thinking only about velocity field, but when you do MHD then we have magnetic field know, so that is a notation we keep u is the well.

So, this is the field. So, right now is u to u, but we want to keep the same consistent notation that is why we write that here. So, how do you compute this by shell-to-shell transfer, by mode to mode transfer? Straightforward know, the giver is pink shell modes in the pink shell, receiver is modes in the yellow shell. So, $\boldsymbol{p}$ lying in $\boldsymbol{p}$ is part of shell m belongs to and receiver belongs to shell n and just sum it up, right, so which is the one line. So, this is how we compute.

Now, if I use Kraichnan formula then there is a problem. Can you see the problem in Kraichnan formula? Kraichnan is $\boldsymbol{k}$ receiving from $\boldsymbol{p}$ and $\boldsymbol{q}$. Now, it is a triad only you know. So, we need all 3. So, it is possible that $\boldsymbol{p}, \boldsymbol{q}$ lies in pink shell, a $\boldsymbol{p}$ lies in pink shell and $\boldsymbol{q}$ out $\boldsymbol{q}$ lies, anywhere $\boldsymbol{q}$ could also be in yellow. So, $\boldsymbol{p}, \boldsymbol{q}$ does not lie in the pink shell, ok. So, there is ambiguity, that is that is so. In Kraichnan formula it is not possible to compute shell-to-shell.



So, I will give you some examples which people did try to compute with Kraichnan formula. So, these are names Waleffe, Zhou, Domardzki and this is a formula of this sort; see the above slide. So, will it give the correct answer? So, what will it give you? So, this is a picture here, you can see the picture smaller picture. $\boldsymbol{p}, \boldsymbol{q}$ lying within m, and $\boldsymbol{p}$ is lying here and $\boldsymbol{q}$ lying there.

So, it is only that triads of type a. So, I can write this combined energy transfer as two terms, right this one, this is by definition combined transfer. Now, there is a definition know where. So, this half is basically I am doing the sum twice. So, this is shell the modes triads of type a, triads of type b are not included at all in this. So, triads of type a, type b is not included. So, this is not correct.

So, these were the mode to mode formula wins. It can compute the shell-to-shell transfer exactly. And this is a problem with quite a few papers, ok. And there is a variation which I discuss in the book, but I will not discuss that. So, triad b is missing in this, ok.

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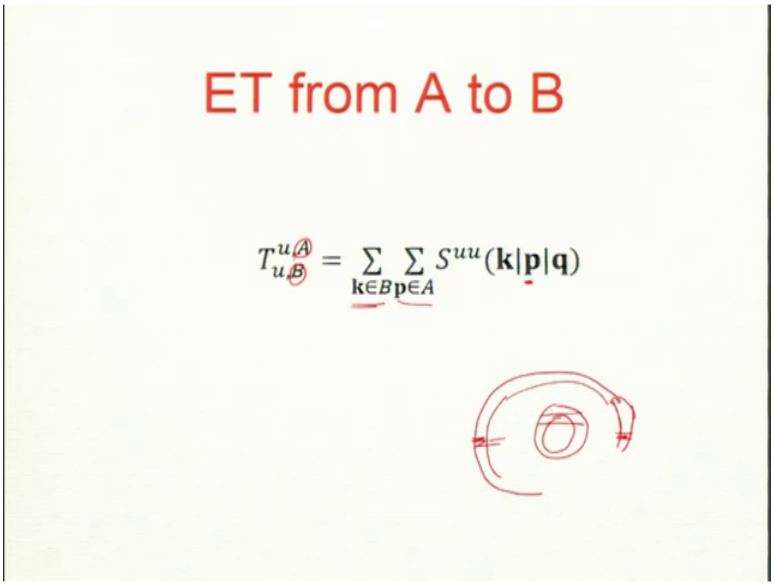

We also talk about energy transfer from any region to any region; so, region A to region B. So, for anisotropic turbulence we define rings, so not shells we can go further down. So, like in a in an earth we can think that dynamics near the pole region is different near the equator, right because is an isotropic, is more heating at the equator then poles. So, we can define this that p belongs to A giver and receiver belongs to B. So, this will be region A to region B, it should be ring or should be any region you like. And this can give you energy transfer from a region to another region.

So, these are very useful formalism to compute various types of energy transfers. So, the 3 types flux, shell-to-shell and in fact, generalized any to any. So, one example I am giving you is ring. So, ring is; so, shell was this know, one shell to other shell, but we can have ring like this and a ring here. Well, I am sorry I should join. So, the ring here equatorial ring and polar ring, and we will do that later. So, ring to ring, I did not really draw a picture, but it is general, ok. So, stop here.