Physics of Turbulence
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Lecture - 22
Energy Transfers Mode-to-mode Energy Transfers continued

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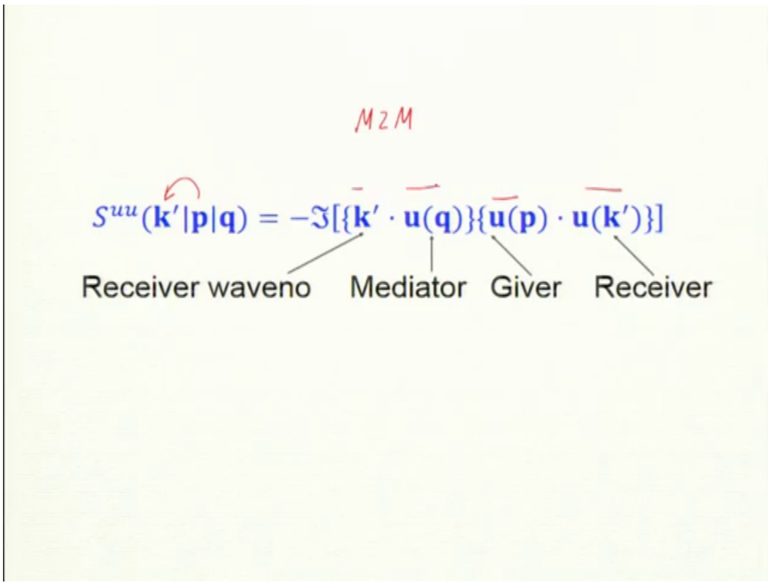

So, this is what I showed, ok. You seem to be happy which is nice. So, more Energy going from Mode to mode, this is mode to mode m to m. So, $\boldsymbol{p}$ to $\boldsymbol{k}'$ it is dot product of the receiver and giver mode and dot product of wave number receiver wave number with mediator.

Now, So, the question is the solution which I wrote, is this solution unique?

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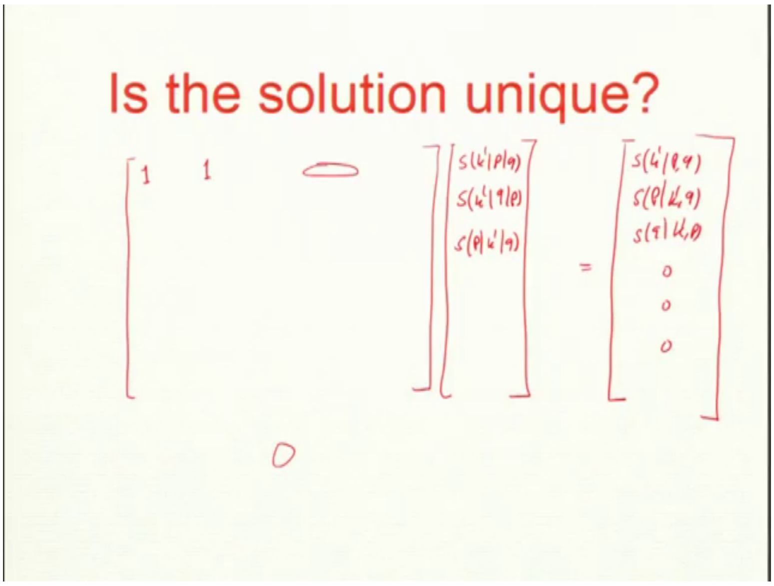

So, the 6 linear equations. So, we can write them as a matrix equation, so we have this matrix. So, unknowns are in this column and known are in the right-side column. So, unknowns will be $S(\boldsymbol{k}'|\boldsymbol{p}|\boldsymbol{q})$, $S(\boldsymbol{k}'|\boldsymbol{q}|\boldsymbol{p})$ and so on, see the slice above.

Now, the first two entries will be 1 1, right, this plus this gives you that, and remaining are 0s. So, you fill up this matrix. Their entries are 1 and minus 1 only, ok. That turns out you take the determinant of this matrix is 0. It is a tragedy you know that we did get a solution what that looks like, it is not a unique solution. So, we will do some more work.

So, in that paper we wrote, so we get some arguments that even though it is not a unique solution, we can make a meaningful answer giving the solution, ok. But I will not follow that path; that path did evoke lot of controversy and that was one reason why our formalism did not pick up so much. So, now, I have constructed a new argument, why this is a good formula, this unique formula, but need some more conditions. From the 6 equations, I cannot prove that this is a unique solution, but I need to put some more constraints and they came come from symmetry arguments.

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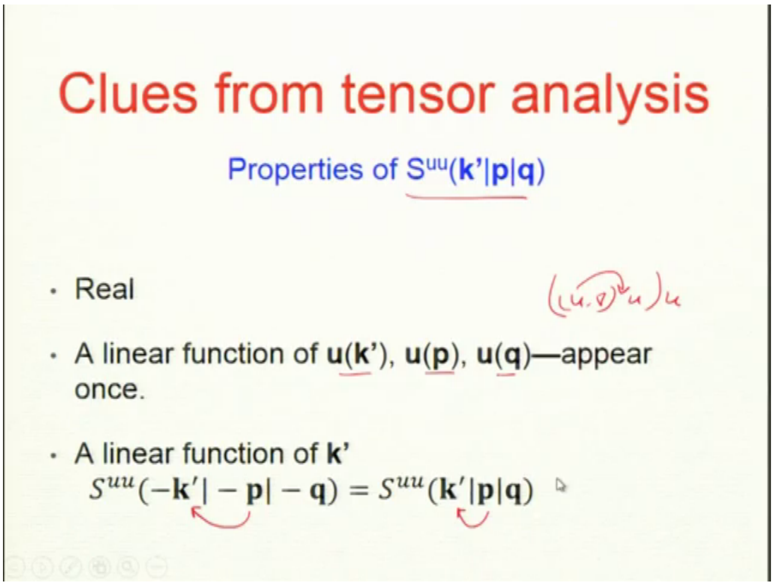

So, for a physicist I think you should be happy that this argument put in more constraints and that will give you, that we will convince you that this is a good formula, ok. So, I must go back a bit.

Now, let us do some more algebra, and tell so that this indeed is a unique solution is a good solution, ok. So, let us try to use tensor analysis. And I will try to state some properties of $S^{uu}(\boldsymbol{k}'|\boldsymbol{p}|\boldsymbol{q})$. Now, of course, I will use some of the properties of the equation. So, this is like you know physics arguments or argument by physics, mathematician will not be happy, but I am not going to make them happy, I just want physical argument. So, what should be the property; what should be properties of this object?

So, energy going from $\boldsymbol{p}$ to $\boldsymbol{k}'$ with $\boldsymbol{q}$ is a mediator. First thing, it should be real energy transfer must be real. Second, it should be proportional to $\boldsymbol{k}'$, it should be linear function of $\boldsymbol{u}(\boldsymbol{k}')$, $\boldsymbol{u}(\boldsymbol{p})$, and $\boldsymbol{u}(\boldsymbol{q})$. Why? Well it comes from kraichnan the first formula, kraichnan formula, all 3 of them appear and appear only once it comes from the structure of the Navier-Strokes equation. There must, so we had $\boldsymbol{u}.\nabla\boldsymbol{u}$ and $\boldsymbol{u}$ and each of the u's are coming here is 3 numbers, 3 wave Fourier modes. So, all of them are appear only once. So, it is a linear function of these and all of them must appear once, ok these 3 must appear.

So, I do not have $\boldsymbol{u}^2$, $\boldsymbol{u}(\boldsymbol{k})^2$. So, all of them appear once and is not quadratic or cubic by the way it is this structure of Navier-Strokes had it been held Navier-Strokes been like this, then it will have 4 terms, ok. But our Navier-Strokes has this, and it has 3 of them in a pretty much. Because of grad it is only linear function of $\boldsymbol{k'}$, ok. So, because of incompressibility grad comes here and we get linear function of $\boldsymbol{k'}$ that comes from the energy equation.

And one more condition that, so we have a negative pair know the complex conjugate pair. So, energy going from $-\boldsymbol{p}$ to $-\boldsymbol{k'}$ is same as energy going from $\boldsymbol{p}$ to $\boldsymbol{k'}$. This by again symmetric argument, this $\boldsymbol{k}$ this call parity argument you know parity.

Now, with these 4 assumptions I can show that this indeed is a solution. Now, I am putting additional conditions, ok. So, let us put them now. These arguments are again used for a more complicated hydrodynamics like MHD we can apply similar argument, but I will not do them for MHD when I do it, but this please see this carefully.

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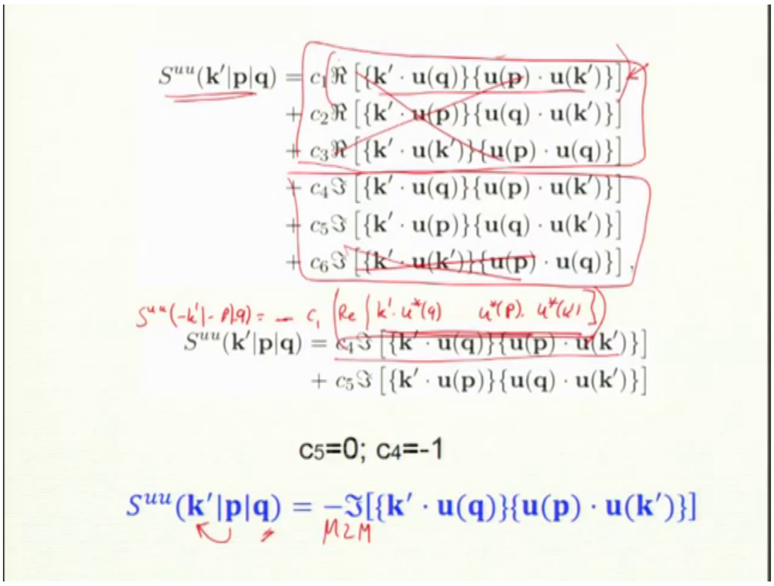

Now, if this is the case then I had to construct a general scalar quantity. This is a real number you know, and I have how many things, $\boldsymbol{u}(\boldsymbol{p})$, $\boldsymbol{u}(\boldsymbol{q})$, $\boldsymbol{u}(\boldsymbol{k'})$ and $\boldsymbol{k}$, I have to construct a scalar by 4 vectors how can you construct a scalar. So, dot product of pairs, right, you just take pairs and dot product, and either take a real part or imaginary part, ok. So, everybody happy with this. I mean these, in fact is done, it is not first time I am doing it, this is standard practice to construct quantities using tensor analysis and the c's are real numbers. So, these are 6 possibilities.

Now, among them some of them will become 0. So, these 3 are automatically 0. Why it is 0? Because of the 4th condition. So, let us say if the first term is nonzero, then what will I get for the negative part $S^{uu}(-\boldsymbol{k}|-\boldsymbol{p}|-\boldsymbol{q})$ will give you the first term. Please see the above slide.

Now, what is this? Is it related to the first object? So, this object inside this complex number is complex conjugate of this. Yes. So, real part of a complex conjugate is same thing, ok. So, this one is exactly same as this one, but we see because of this condition I got the minus sign. So, I cannot have -a = a, unless a is 0. So, this implies that these 3 are 0. So, the 4th condition was useful and so, this came after bit of algebra. So, this condition makes all the real part go to 0, so it must be imaginary. So, I got 3 of them; so, these gone. Now, what do I with c 4, c 5, c 6? One of them is 0. Which one is 0?

Last one is zero, because $\boldsymbol{k}.\boldsymbol{u}(\boldsymbol{k}) = 0$, incompressible condition. So, we left with two of them c 4 and c 5. Now, this must satisfy those 6 equations. Now, I will leave it for you to work out. Just substitute these in those 6 equations in fact, first equation and 4th equation, ok. You substitute in first and 4th equation you will find that c 5 is 0 and c 4 is -1, this follows, I mean this is algebra. So that means, I derived from this tensor analysis the c -1, it is this one. So, this is how I said well, using tensor analysis per phases we can show that this indeed is a formula for energy going from $\boldsymbol{p}$ to $\boldsymbol{k}'$ and $\boldsymbol{q}$ is a helper, ok.

So, we had a difficulty that this is not unique solution what to do with it, but now we show by additional arguments that indeed it is a solution, but we need to put some more, some more conditions. And of course, I think the physical argument is quite nice, that this guy advects the wave and I think is I am I like it most, so it is the physical argument.

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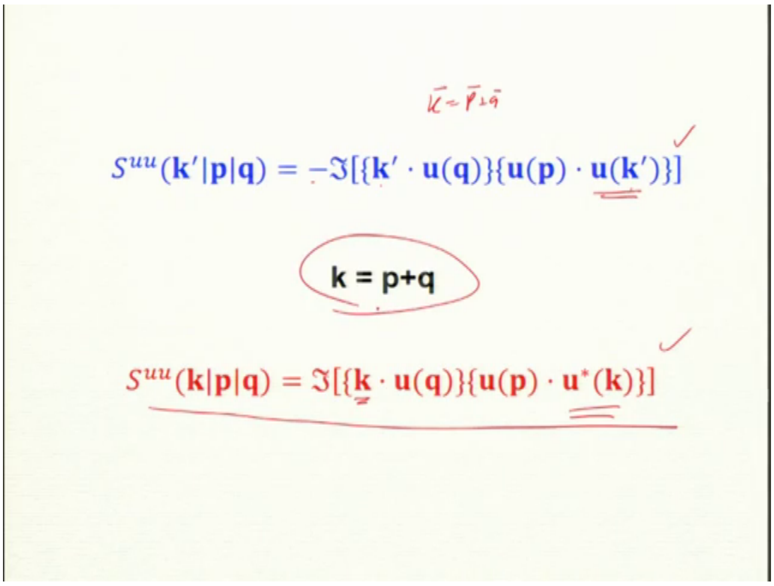
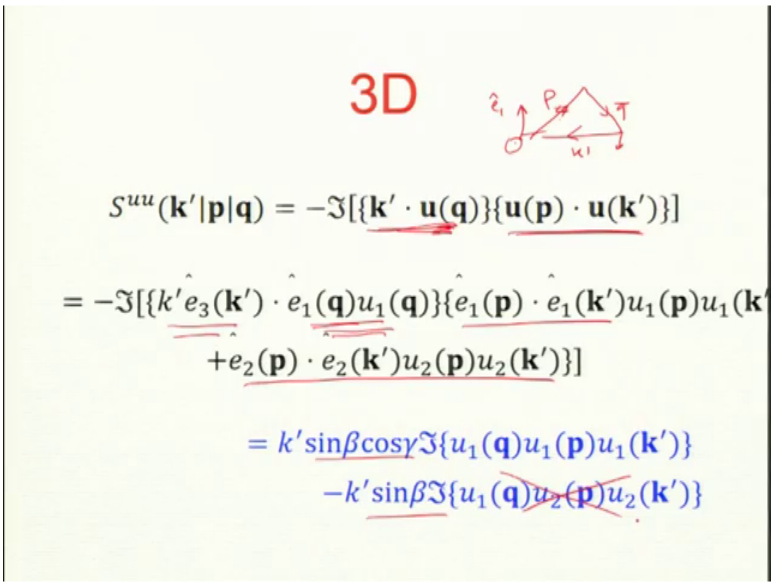


I do want to show one more important point. Now, many times if we use condition $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$, if that is the case then we can rewrite this formula. It is useful to have various versions of the formula, so I just put $\boldsymbol{k}' = -\boldsymbol{k}$. So, this becomes, so this $-\boldsymbol{k}' = \boldsymbol{k}$ and this$\boldsymbol{u}(\boldsymbol{k}') = \boldsymbol{u}^*(\boldsymbol{k})$**.** So, you remember both the formulas know, but they are once you remember you know this you can easily convert this. So, many times we use this second one, ok. So, this is a how your triad is represented then use that, you otherwise you use this.



So, Craya-Herring, now I am just going to show you in; so, Craya-Herring we did in the class. So, how does the formula look like? So, Craya-Herring was very specific $\boldsymbol{k}' + \boldsymbol{p} + \boldsymbol{q}$ and that forms a triangle and we had, it is very useful formula, we did derive our modes, but now we can also write down formula for energy transfer. So, it is you just follow the same technique. Please see the above slice. So, $\boldsymbol{k}'$ is along e 3 direction.

Now, u_1 and u_2, and u_2 is perpendicular to e 3 the plane. So, u_2 does not contribute at all. So, they form a plane and my e_1 was in the plane, but u_2 was perpendicular to the plane. So, u_2 will not give anything from the dot product, in fact 0. Now, this dot product has two parts, one is $e_1 . e_1$ because e_1 and e_1 in the same plane and $e_2 . e_2$, but the $e_1 . e_2 = 0$ because they are perpendicular to each other. So, this is what we get.

Now, from the table which I showed you before and these are two terms, ok. Now, this is coming from the dot products and you just take from the table. the dot products become complicated sometimes.

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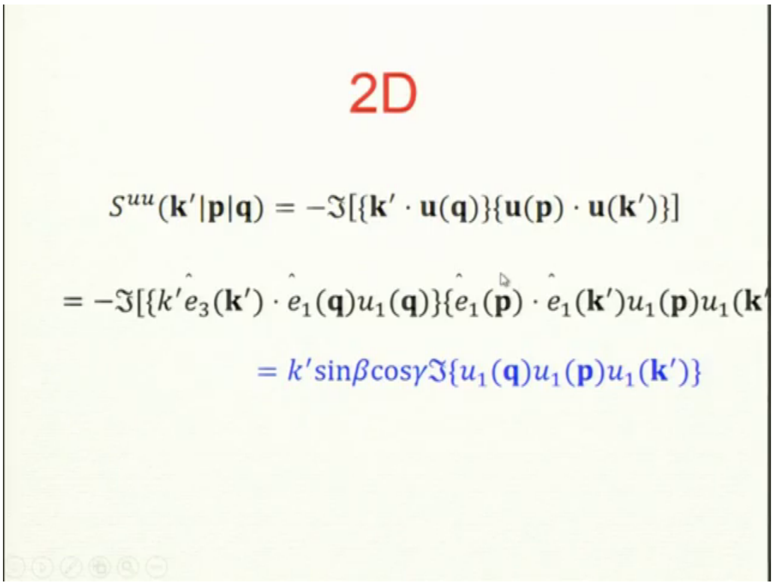

So, 2D is much simpler, is just does that. The first one exists, u_2 does not exist, ok. So, I have derived mode to mode now in Craya-Herring as well. So, this for completeness.