Physics of Turbulence
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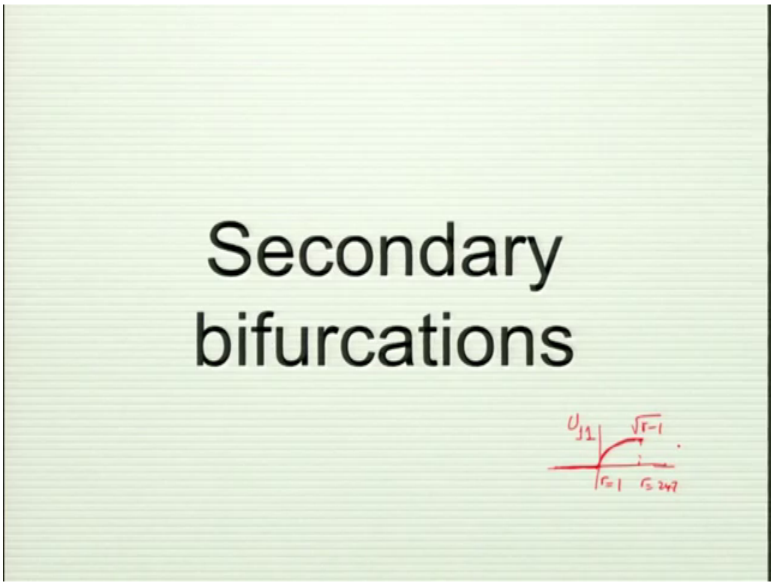
Lecture - 20
Rout to Turbulence
Patterns, chaos, and turbulence

In this lecture, I am going to conclude the *Route to Turbulence*, how turbulence arises. The route to turbulence is example specific. How turbulence arises in a pipe, in Rayleigh-Benard convection, in stars, etc., are all different, but there is a certain pattern. We first observe instability, then we get non-linear saturation. Instability will make the mode grow, but nonlinearity will come over and saturate it. Then patterns appear and chaos ensues.

In this lecture we focus on how patterns appear in Rayleigh-Benard convection, and how turbulence arises subsequently. After that, we will start studying the physics of turbulence - Kolmogorov theory, and so on. But for now, we will cover these three topics on Rayleigh-Benard convection and of course, it is a huge topic.

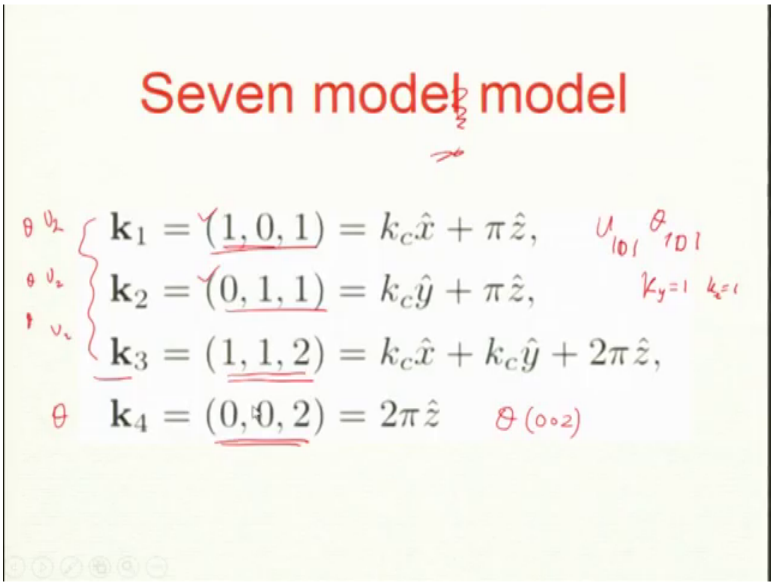
I am going to take simple examples of Rayleigh-Benard to demonstrate this. This is covered in-depth in the book, Physics of Buoyant Flows, in chapters 7 and 8. It can be used as a reference for these topics - instability, saturation and patterns, covered here.



Saturation is a critical point, and this is common in many systems. As we saw previously, non-linearity tends to saturate, and prevent modes from growing infinitely. Such unconstrained growth is unphysical. We will first look at something called secondary bifurcation. What is bifurcation? As mentioned previously, bifurcation is a change in behavior of a dynamical system. In some sense, it is like a road splitting into two separate roads.

From the previous lecture, we already have studied two behaviors - instability and saturation. However, we can have more complex behaviors. In the Lorenz equations, the mode U_{11} was 0 for $r < 1$. Later it started varying as $\sqrt{r-1}$. So, it was initially a constant value mode, but it did not remain constant throughout when I increased Ra. This parameter that we vary here denotes temperature difference between the plates. But the important thing to note here is that things start becoming time dependent as the parameter is varied.

However, we need a more sophisticated model. Lorenz equation does not give realistic bifurcation. There is a bifurcation of Lorenz equation. In fact, at around r equal to 24.7, there is a bifurcation called Hopf bifurcation. After this bifurcation, the dynamics become becomes chaotic. But we will look at a better model that has a more realistic bifurcation for large Prandtl numbers. We call it the seven-mode model.



What modes should we pick to derive the seven-mode model? Previously, in 2D, we had $(1,1)$ and $(0,2)$ modes, which can be written as $(1,0,1)$ and $(0,0,2)$ modes in 3D (since we had neglected y component to make out system 2D). These are in fact wavenumbers. Corresponding to $(1,0,1)$, we had 2 modes. We focused on U_{101} and θ_{101} modes.

Note that θ_{002} was brought in for non-linear saturation. Instability needed only U_{101} and θ_{101} modes. Now we introduce two more modes - wavenumbers $(0,1,1)$ and $(1,1,2)$. So, what is $(0,1,1)$ mode? Now $(1,0,1)$ mode has components along k_x and k_z. So, it has variations in xz plane. Similarly, $(0,1,1)$ has components along k_y and k_z. So, $k_y = 1$ and $k_z = 1$. It is therefore yz dependent.

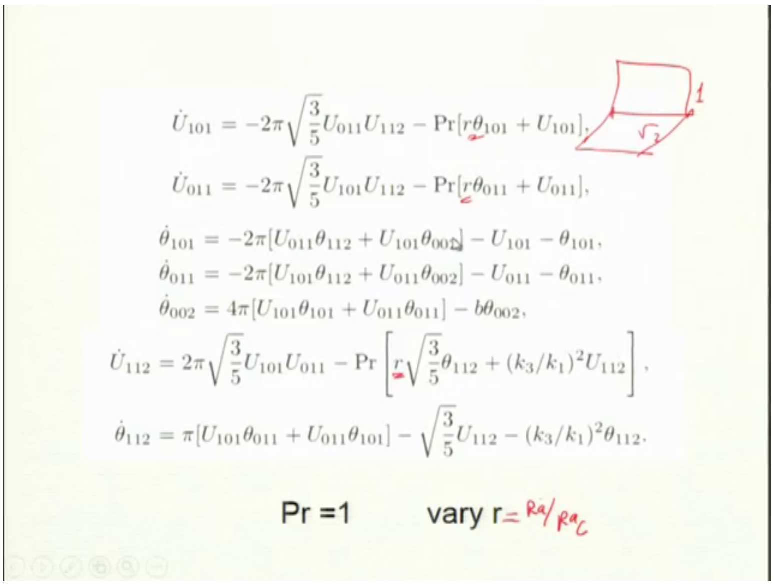
So, one mode depicts a roll in xz plane, aligned along y axis. The other mode is a similar roll aligned along x axis. Clearly, we now have a 3D structure, where 2 rolls are aligned along perpendicular directions. These modes will interact non-linearly. So, we include the seventh mode which mediates this interaction. This happens through mode $(1,1,2)$.

Note that these three modes form a triad:

$$(0,1,1) \oplus (1,0,1) = (1,1,2)$$

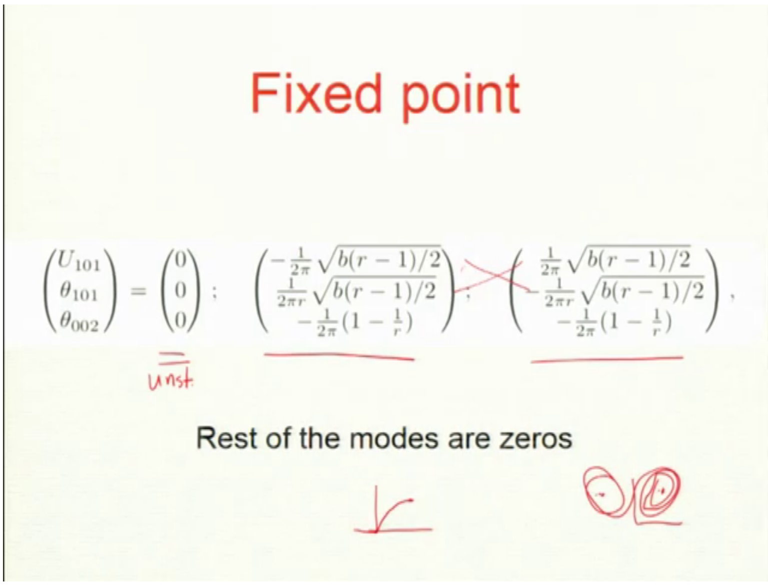
This non-linear coupling between k_1, k_2, and k_3 will give you a rich set of patterns. We now look at the results. More details on its derivation can be found in the book.

In total, we have 4 wavenumbers. Out of them, 3 have both velocity field and temperature fields associated with them. According to Craya-Herring basis, each wave number has 2 velocity modes – $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. But we consider only $\boldsymbol{u_2}$ for the current derivation. We take into account $\boldsymbol{u_2}$ and θ for the 3 modes – $(1,0,1)$, $(0,1,1)$, and $(1,1,2)$. This gives us 6 modes so far. Mode $(0,0,2)$ has no velocity component. Like how we saw in 2D, incompressibilty condition dictates that there is no velocity component for this, and we keep only θ. So, there are 7 modes, and they will have seven differential equations associated with them. These 7 equations are shown in the slide.



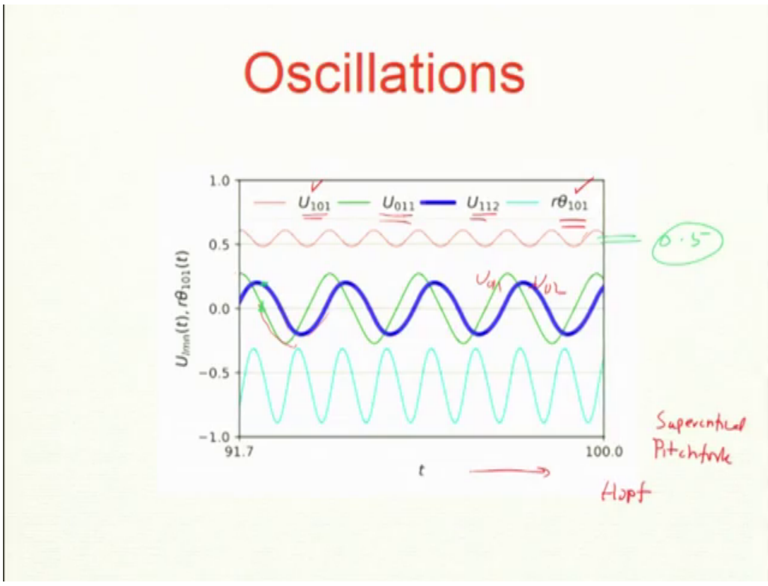
Now, these modes are used with a box dimension $\sqrt{2} \times \sqrt{2} \times 1$. Its length is $\sqrt{2}$ because if you remember, instability condition gives you $k_c = \frac{\pi}{\sqrt{2}}$. That is the lowest wave number associated with the lowest Rayleigh number for instability. Hence the dimensions of the box help simplify the equations by eliminating some factors.

To study the behavior of this model, we take Prandtl number unity and vary r. Remember that $r = \frac{Ra}{Ra_c}$. For $r < 1$, what is the solution of these equations? We see that the stable solution has all zeros. So, there is no convection, only conduction. As soon as r becomes greater than 1, we get these solutions shown on the slide. One of the 3 modes is $(0,0,0)$, which is unstable. But the other modes are stable. These were derived in the last lecture. Here, b is a parameter, and we are already familiar with the term $\sqrt{r-1}$.



Note that the first 2 components of the 2 stable modes are negative of each other. Perhaps the most famous aspect of the Lorenz equations is the butterfly pattern. We will see this pattern soon, but right now, we have fixed-point solutions.

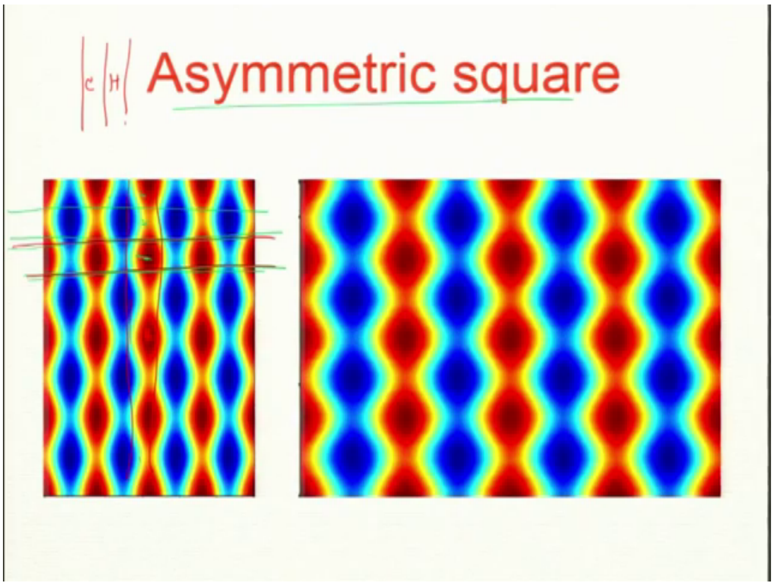
In the model, till $r = 13$, we get stable solutions. After that, the behavior of the system changes, and the modes start oscillating. This means that the modes are time dependent now. Now we can plot the modes in a time series, as a function of time. This way, when we plot the velocity mode U_{101}, we see that it is oscillating around the value of 0.5 units. Note that these are all non-dimensionalized.



However, the modes U_{011} (green line) and U_{112} (dark blue line) oscillate around 0. These two modes were 0 previously. But now it is like they are getting born and killed, born and killed.

I have also plotted one temperature mode, namely θ_{101}. It is like U_{101} except that it has a factor r multiplied to it. Like U_{101}, θ_{101} also oscillates around a finite value. So, all these modes are starting to oscillate, even θ_{002} oscillates.

This change in dynamics is called a Hopf bifurcation. The first one, where modes changed from zero to non-zero values, is called supercritical pitchfork bifurcation. Hopf bifurcation is named after the Austrian mathematician, Eberhard Hopf. If r is increased further, then the system becomes chaotic. Then the dynamics will display more rich features, but we will not explore them for now.



Before moving on to chaos, let us understand the physical interpretation of these oscillations. We take the physical domain that we are looking at, and take a mid-plane cut, perpendicular to the z axis. If we plot the temperature on this plane, we see the patterns shown on slide. Note that in this view, the hot plate is below the screen and the cold plate is above the screen.

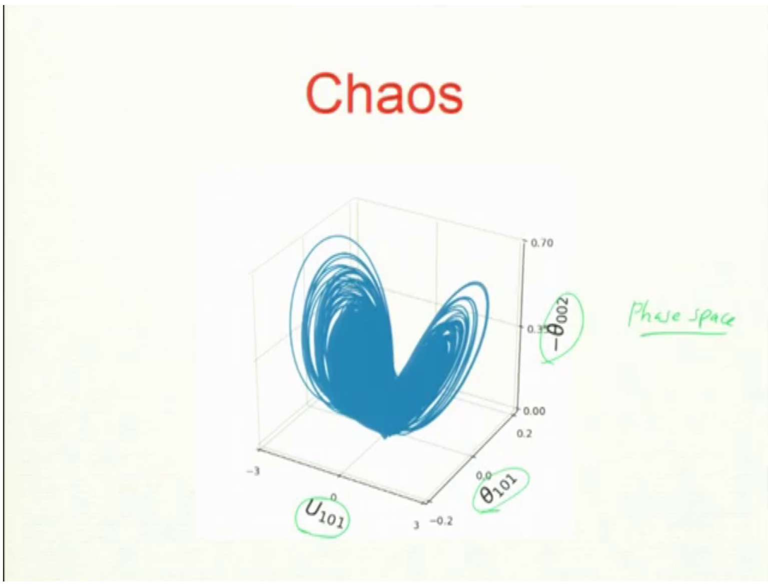
Now red regions are hot and blue regions are cold. In the red regions, the fluid is rising out of the screen, since the fluid is hot. Conversely, in the blue regions, the fluid is falling into the screen, since the fluid is cold. From this, we can infer that these strips of red and blue in the plot depict circulating rolls of fluid. These are 2D rolls, and if you look at them from the side view, you will see circular rolls. But from the top view, we see hot and cold strips of fluid.

Note that we do not see a clean hot strip, but there are variations. There are undulations along the length of the strips. What can you make out of this? It implies that there is another set of rolls aligned perpendicularly to them. So, there are two cross rolls, giving rise to a 3D structure.

The thick red and blue patches correspond to the regions where both the rolls reinforce each other. The thin regions are places where the two rolls counteract each other. This 3-dimensional structure of thin and fat strips is called asymmetric square. It is called asymmetric, because one roll is stronger than the other roll. Here we see that the xz roll is dominant, while the other roll is coming and going.

This can be observed in the time series in the previous slide. The dominant roll corresponds to mode U_{101}, which is oscillating around 0.5. The other roll, U_{011} oscillates around 0, and hence keeps appearing and disappearing. Some effect of mode U_{112} is also there, but it is not discernible here.

This is the interpretation of the structure that we see from our simple 7 mode model, and this is also seen in experiments. Of course, physical experiments have more complexity, but the 7 modes capture these basic features. That is why these models are not just for the sake of constructing them. They have a physical meaning and shows the non-linear interaction of terms. Knowing how to compute this non-linearity is important and useful.

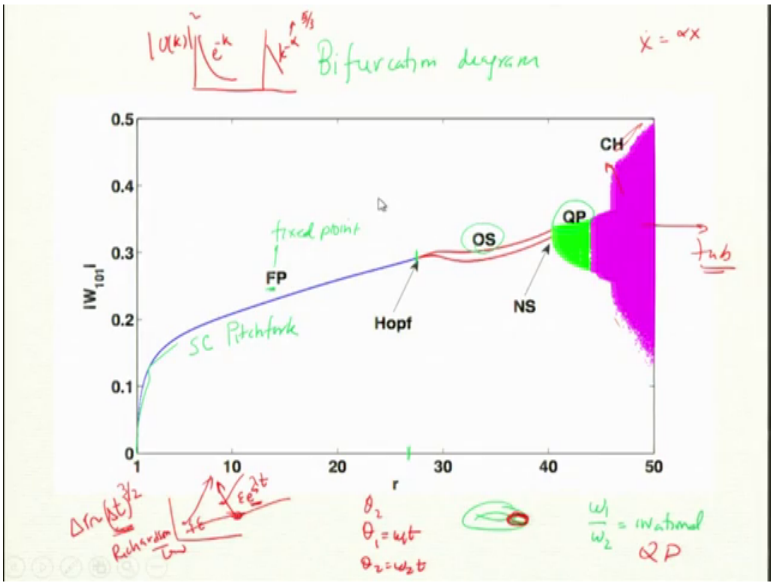


In the slide, we see a 3D picture of the interaction of modes. This is called a phase space. The earlier picture was real space picture. The 3 axes of this phase space are U_{101}, θ_{101} and $-\theta_{002}$.

The butterfly pattern is visible here, but it is not the same butterfly pattern as seen from Lorenz equation. This representation is more realistic since there are more modes, in real life there are many more modes, but seven modes is reasonably accurate. This structure will then differ, but qualitatively we will see same behavior. Now you have a good picture of the seven-mode model.

We now look at some results. We look at a paper written from our lab. Here we made a model which has more modes, and we also perform simulations for large Prandtl numbers. Now, Prandtl number is defined as $Pr = \frac{\nu}{\kappa}$.

For high Prandtl number (> 1), the system behaves one way. And for small Prandtl number (≈ 0.001), the system behaves differently. This is mainly because the viscosity becomes very small at low Pr. Of course, κ also has an influence. For $Pr \approx 10$ and so on, the model described in this paper presents a good picture. The model uses 20 modes. First, we observe the bifurcation diagram from this model.



As r is increased, we first see pitchfork bifurcation. Before that, FP stands for fixed-point, and this pitchfork bifurcation is supercritical. In the model, this happens at around $r \approx 25$.

Subsequently, we get Hopf bifurcation, after which there are oscillations (denoted as OS). Specifically, these are asymmetric square oscillations described earlier. Then the system becomes quasi-periodic (QP).

What is quasi-periodic behavior? If we consider a torus (which can be visualized a cycle tube) there are two frequencies associated with the motion on its surface. A circular motion along the outer periphery of the torus will have one frequency (ω_1). But there is also a secondary frequency (ω_2) associated with the motion encircling the band of the torus.

To elaborate further, for a tube, there are two radii - r_1 and r_2. Associated with each radius are the angles θ_1 and θ_2 respectively. So, every point on the tube has a unique pair of angles. Now imagine that a point will move on the surface of the tube as a function of time. If ω_2 is 0, so that θ_2 is fixed, while θ_1 varies with a constant ω_1, then the point will trace a circle around the outer periphery of the tube.

However, if ω_2 is non-zero but ω_1 is zero, then θ_2 varies while θ_1 is constant. In this case, the point will again trace a circle around a section of the tube. Moreover, if both ω_1 and ω_2 are non-zero and equal, then the point will trace a helical path around the tube.

Finally, if the ratio between ω_1 and ω_2 is an irrational number, then the point never returns to the starting point. The path will just cover up the whole surface of the tube if we wait long enough. Such a motion is called quasi periodic. It is not periodic, but it looks periodic.

So, with two frequencies (ω_1 and ω_2) we get quasi periodic dynamics. However, the Lorenz butterfly is chaotic. Even in chaos there is filling up of a region of the phase space. However, in chaos, a third frequency is required in the system. This is the reason behind the quote "*period three implies chaos*". Those who have read the book *Chaos* by James Gleick might be familiar with this phrase.

We are focusing on turbulence, which is in fact more complex than chaos. We need to increase out parameter further to see turbulence. One commonly asked question is, "What is difference between chaos and turbulence?" There is no one answer. There are several answers, and here we look at some of them.

The number of modes in a chaotic system will be large, but generally small. Even 3 modes are enough to produce chaos. But chaotic systems may have 10, 20, or more modes. But we need more than 2 modes to observe chaos, and this was proved by York. It is not necessary that a 3D system will be chaotic. It can be quasi-periodic or periodic, but if there is sufficient nonlinearity, and you satisfy the properties for chaos, then the system becomes chaotic.

One common definition for chaos is the system's sensitivity to initial conditions. This means that if I take 2 points in phase space which are close to each other, say separated by a distance ϵ. If the system starts evolving with this initial condition and if you wait for some time, then the points will start diverging and this gap between them will increase exponentially. So, it will be some form of $e^{\lambda t}$, but they simply do not diverge towards infinity due to saturation.

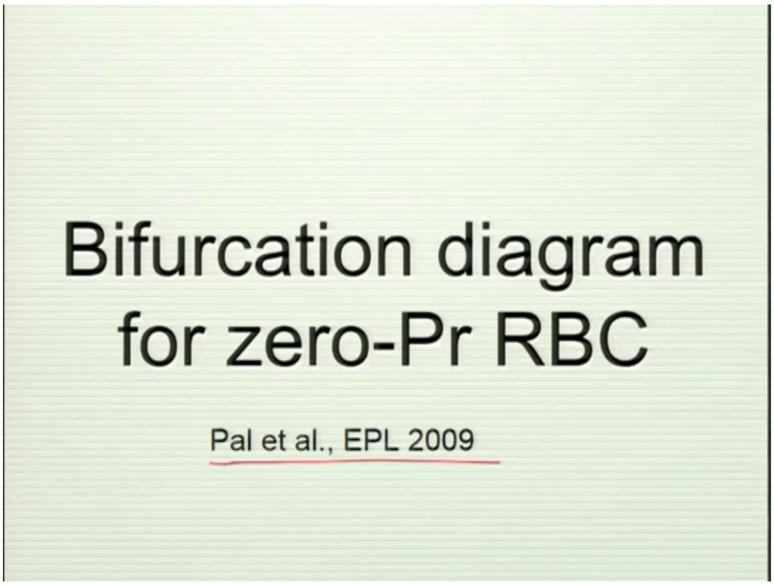
So, the region of phase space where they move around is a finite region. If it is diverging and going to infinity, then that will not show interesting physics. It is merely a divergent system, with no chaos. Nonlinear saturation prevents this from happening. But the distance is on the average expanding for small times as an exponential function.

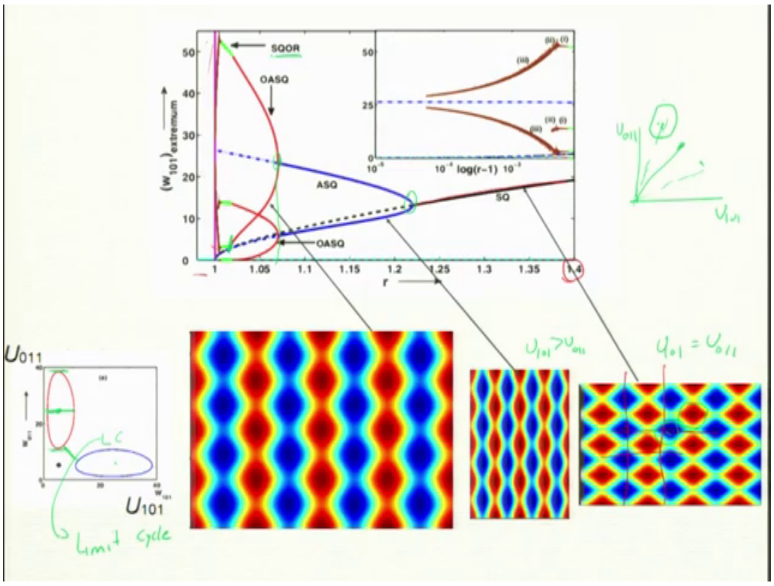
In turbulent systems however, the number of degrees of freedom is huge. So, typically you can have 1 million degrees of freedom. People do not even think of the phase space picture for turbulent systems. But you can take two particles in a turbulent system and see their divergence. I mean two smoke particles in a turbulent system move such that the distance between them, Δr, goes as $\Delta t^{3/2}$.

One of them is exponential, other one is a power law. One is x^n, other one is e^x. Which is faster? This can be judged from Richardson law. So, turbulence typically has more degrees of freedom, and obeys power laws. Energy spectrum of turbulent flows, which will be covered later, is a power law in turbulence. In chaos however, you should use exponential laws.

To see this, we can look at $u(k)$ for some Fourier mode, U_{11} or U_{22}, If we plot the energy of this mode against k, the energy will typically decrease as you go to higher wave numbers, exponentially. In turbulence also the energy will decay, but according to a power law like $k^{-\alpha}$. We know from Kolmogorov theory that alpha is $5/3$. We will investigate this in future lectures.

To summarize, we typically get chaos, then turbulence. The system keeps generating more and more modes, tending towards more and more disorder, and stronger interactions.





Now, we can look at small Prandtl numbers. We will first look at what happens with zero Prandtl number. This is again a work done at our lab around 10 years back and is quite interesting.

Previously in the bifurcation diagram we were moving from left to right. First came fixed point, then periodic, then quasi-periodic, and so on. But for zero Prandtl number, it was wiser to go from right to left. So, we start from a large Rayleigh number and decrease it.

The critical Ra here is 657. We start at an Ra of 1.4 times 657, that is $r = 1.4$. This is because zero Prandtl number flow is extremely chaotic. Viscosity is basically very small, it is not 0, but very small. And with small viscosity there is fluid turbulence. Zero viscosity means if you give the system any energy, it just becomes very violent.

So, the system is chaotic at the onset itself as we started from the right. It starts with a square pattern (SQ). So, in the first picture, at $r \approx 1.35$ you can see the square patterns. As before there are two rolls, and both have equal amplitude. Remember that asymmetric square had unequal amplitude. Here we can see the difference from asymmetric square pattern.

This happens when U_{101} is equal to U_{011}. Then the system bifurcates into ASQ - asymmetric square, but this asymmetric square is not a function of time. It does not change with time and is fixed in time. In ASQ, one mode is bigger than the other and remains that way.

If you look at the fixed point, initially the modes U_{101} and U_{011} were symmetric. Then they became asymmetric by bifurcating. Going further to the left, we get closer to the next onset (the red lines). At this point, the pattern starts to oscillate in time. So initially the patterns were not oscillating, but now they start oscillating and they oscillate exactly like what we saw in the previous case.

In the phase space picture what was initially two points will become a pair of closed curves. This can be seen in the slide. As you can see, the phase space is constructed from U_{011} and U_{101}. Note that they oscillate around a mean value and not around 0.

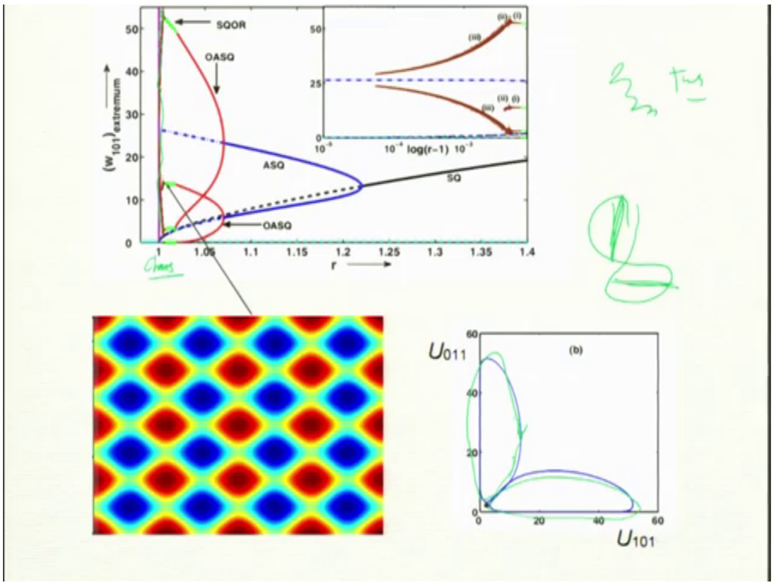
So, this is bit different than what we were doing before, it oscillates around this mean value. By looking at which of the two modes is bigger in the phase space, we can say which roll will be stronger. So, the strengths of the xz roll and yz roll varies according to what we see in the phase space. Like before, the fixed point also exists, but it is unstable.

In the language of non-linear dynamics, the two closed curves in the phase space of U_{101} and U_{011} are called limit cycles. One should keep in mind that these limit cycles are in fact projection of a higher dimensional phase space onto the space spanned by modes U_{101} and U_{011}. The full phase space structure is bigger, and it is not only a circle. In fact, the full phase space has dimensions since this is a 13-mode model.

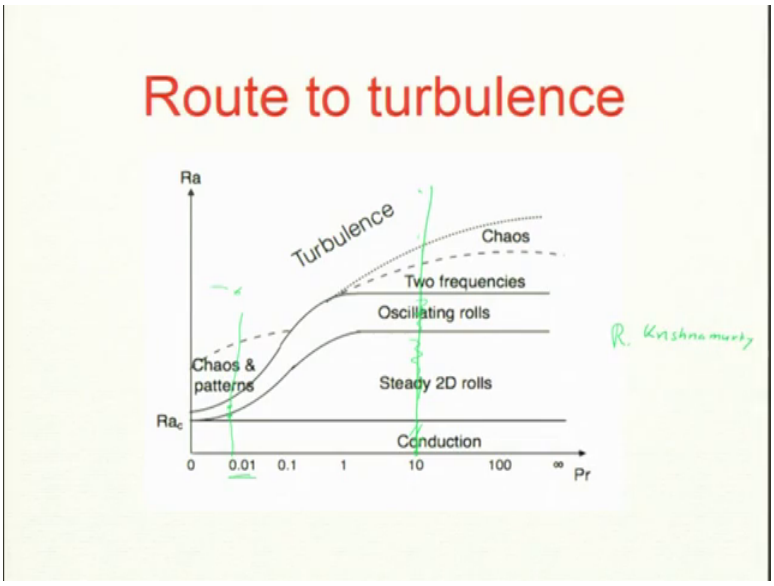
This dynamic with two limit cycles touching each other, is called oscillating square (SQOR). In fact, the two cycles are touching at the black dot shown in figure. This is a fixed point, and there is a lot of interesting things happening there. It is called a homoclinic tangle. These are all non-linear dynamics terms. If you do a course on it, you will be familiarized with them.

You can also see these two limit cycles merging to become one limit cycle. So as the two rolls get dominant and diminished alternatively, in between they are symmetric. This gives a square pattern in between. This symmetric back and forth between the two rolls is called relaxation oscillation square pattern (SQOR).

This can be seen in the movie shown in the video lecture. Note that the domain has a periodic boundary condition. Hence these rolls extend infinitely. Once we go beyond SQOR we get chaos. The thin region to the extreme left of the bifurcation diagram shows chaos, which has some different types of behavior. The dashed line denotes a fixed-point which is important.



Again, this interaction between fixed-point and chaotic dynamics is called a homoclinic tangle. So, we have seen the pictures for large Prandtl number, and small Prandtl number now. We have studied how the system becomes chaotic in both cases.



Now, we summarize all this in the next slide - Route to Turbulence. This is a plot of Ra vs Pr which was made by Ruby Krishnamurty. For a large Prandtl number, say 10, as we increase the Rayleigh number, we have conduction state till Rayleigh critical. No convection, then convection, then steady roll. Steady rolls have no time dependence. Then comes oscillating rolls. That is where we see the asymmetric square patterns. Then we have two frequencies, which leads to quasi periodicity. After that we see chaos, and after that, turbulence. So, it gets more and more complex with more modes.

Now if we look at small Prandtl number we see that the region has shrunk. If we trace the evolution for $Pr = 0.01$, we get turbulence at a relatively smaller Rayleigh number.

For larger Prandtl numbers, we need to go to a higher Rayleigh number to see action. This is because of the higher viscosity - there is too much friction. So, you need more power to overcome it. At smaller Prandtl number, the fluid is less viscous and there is less friction. So, for low Prandtl number systems the flow is more turbulent typically for a given Rayleigh number.

Thank you.