Physics of Turbulence

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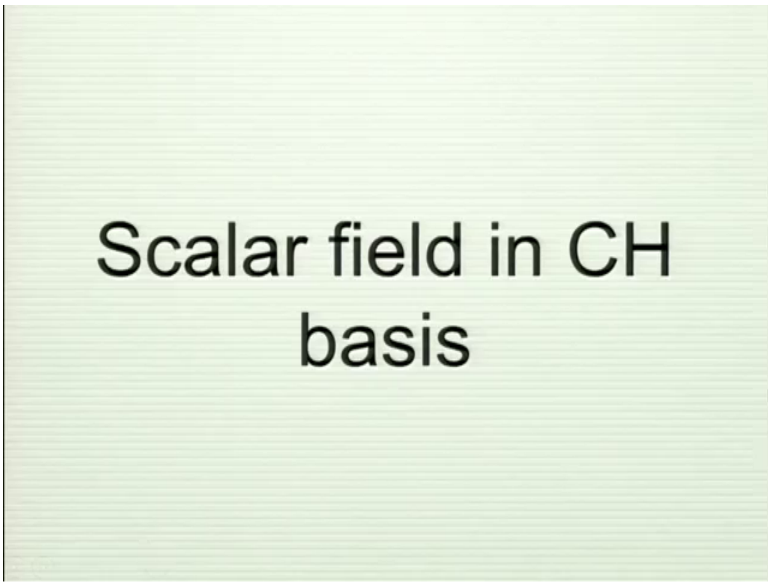
Lecture - 19
Route to Turbulence
Nonlinear Saturation Lorenz Equation

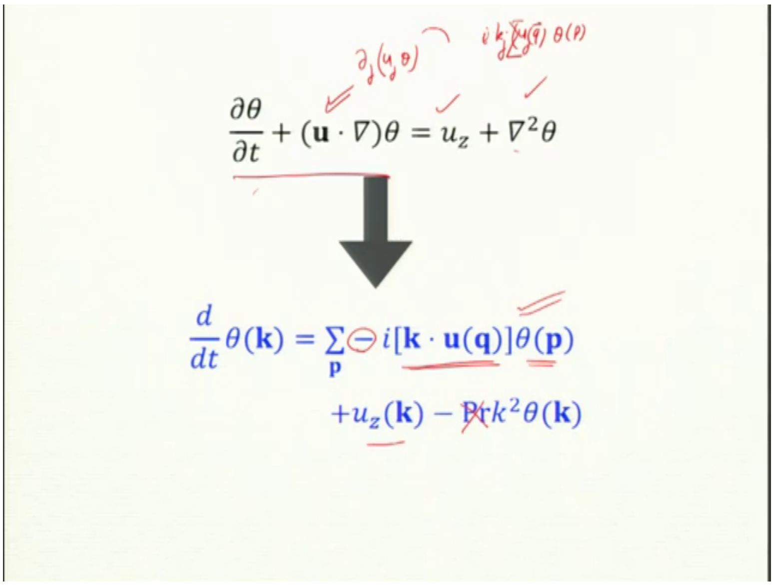
So, now we studied instability, but next is what happens when nonlinearity kicks in and when it starts taking over. So, one thing we discussed already is that nonlinearity will saturate the growth. So, things cannot keep growing exponentially, energy cannot keep coming in, and in fact temperature cannot keep growing, because the system will melt. So, nonlinearity comes and saturates.

We will see this through Lorenz equations. I am going to derive Lorenz equation, but my normalization is not like that of Lorenz. This derivation is quite different. Again, I will use Craya-Herring basis. But the equations are same, apart from the factors, since they are not normalized in the same manner as done by Lorenz.

We will study non-linear saturation using Lorenz equation, which is the simplest derivation for non-linear interactions. Moreover, this is for Rayleigh Benard convection. So, there is no rotation, and no magnetic field.



Before I discuss Lorenz equations, I need this interaction $u \cdot \nabla\theta$. This is the non-linear term and all non-linear interactions is via this term. So we need to write it in Fourier space, like how we wrote for $u \cdot \nabla u$.



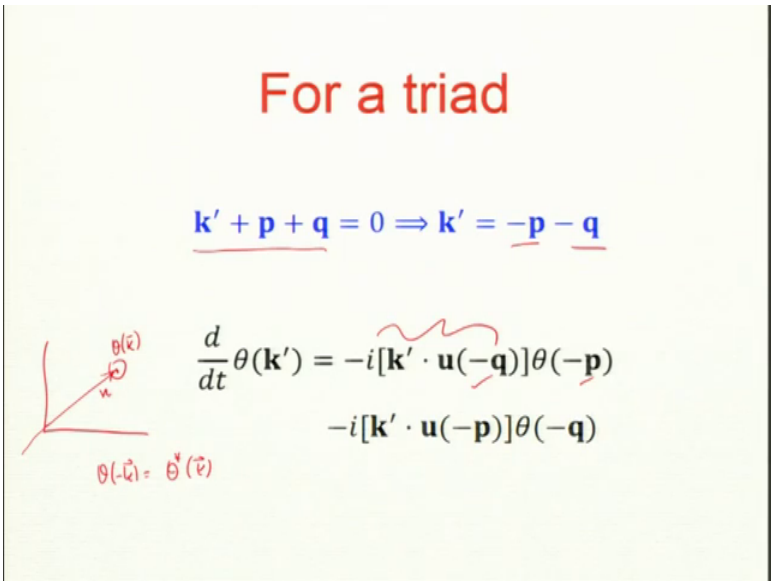
So, we had done exercises and homework on this. Now we need to do the same thing for $\boldsymbol{u} \cdot \nabla\theta$. So, I am just going to show you how to do this and I will do it right now for Lorenz. So, the $\boldsymbol{u}$ equation remains the same. There are new terms like $RaPr\theta\hat{z}$ viscous term, but non-linear term is the same as before. So, we need to worry about what happens for scalar.

So, scalar non-linear term is $\boldsymbol{u} \cdot \nabla\theta$. And then there are the linear terms - u_z and $\nabla^2\theta$. Using incompressibility of fluid, we can derive the term in Fourier space, for $\boldsymbol{u} \cdot \nabla\theta$:

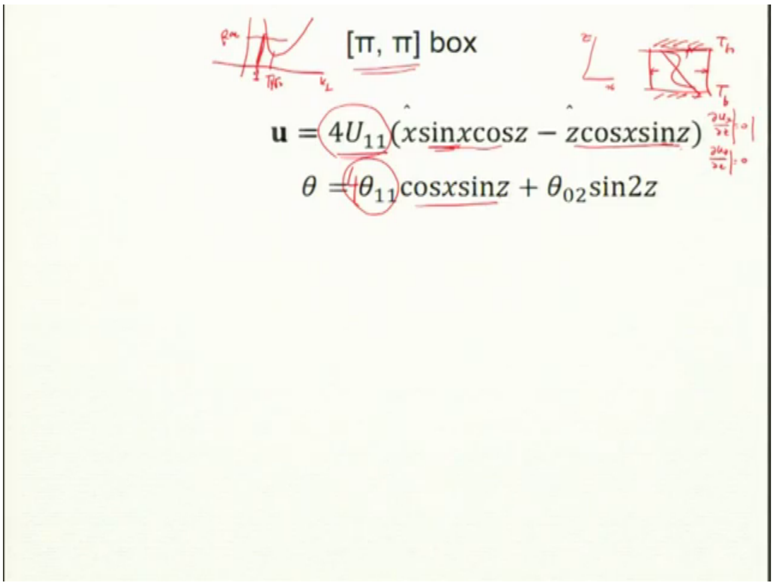
$$u \cdot \nabla\theta = u_j\partial_j\theta = \partial_j(u_j\theta) \Rightarrow ik_j \cdot \sum_p u_j(\boldsymbol{q})\theta(\boldsymbol{p}) = i\sum_p [\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{q})]\theta(\boldsymbol{p})$$

This is similar to the non-linear term that was derived for velocity equation in Fourier space. Instead of $\boldsymbol{u}(\boldsymbol{p})$, there is $\theta(\boldsymbol{p})$ here. In fact, this is simpler, since one of them is a scalar. So, this is my non-linear term and the linear terms are straight forward. u_z will give $u_z(\boldsymbol{k})$, and $\nabla^2\theta$ will give $-k^2\theta(\boldsymbol{k})$. So, we need to compute this for a triad interaction, the way we did for $\boldsymbol{u} \cdot \nabla\boldsymbol{u}$.

For the triad we write $\boldsymbol{k}' + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}$. So, they are symmetric. So, the convolution becomes a sum in Fourier space and the sum has only two terms now, because for a triad right hand side will be involving only two terms.



There is no need for Craya-Herring basis yet since $\theta(\boldsymbol{k})$ is a scalar. This is merely a number sitting at the wave number k. However, reality condition imposes that $\theta(-\boldsymbol{k}) = \theta^*(\boldsymbol{k})$. Although Craya-Herring basis is not used, we can still solve this numerically using a code.



We move on to derive the Lorenz equation. In 1963, Lorenz simplified the Rayleigh Bernard equation and retained 3 terms, or 3 Fourier modes. Velocity field must be incompressible. This in fact, we have seen it before, gives

$$4U_{11}(\hat{x}\sin(x)\cos(z) - \hat{z}\cos(x)\sin(z))$$

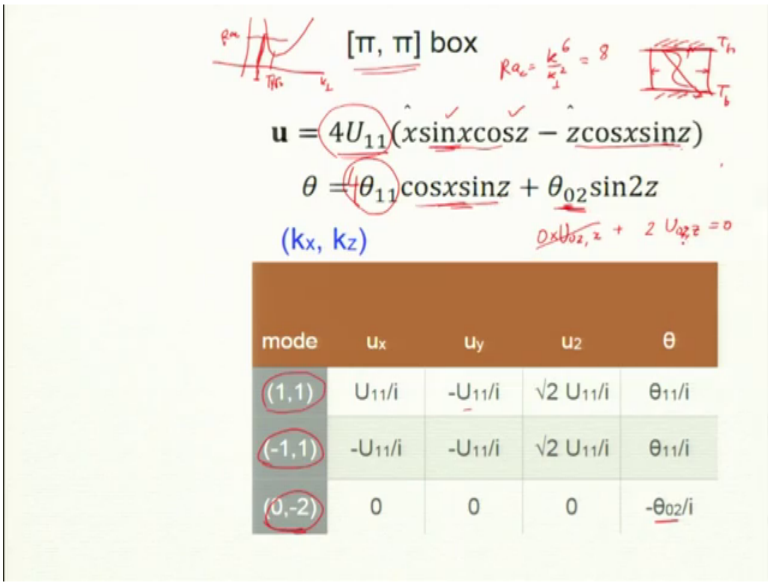
This satisfies the free slip boundary condition. So, there are free slip walls along z direction. So, is the same thing what I did for an instability. I do not want to disturb my earlier analysis. So, at the walls, the vertical velocity must be 0. Hence vertical velocity is 0 at $z = 0$, and at $z = \pi$. To make my calculation simpler I choose a box with dimensions $\pi \times \pi$. Otherwise there will be many terms with π in the equations.

So, $u_x = 0$ at both side walls. Now $\sin(x)$ is 0 at $x = 0$ and at $x = \pi$. Hence, it satisfies this boundary conditions $\frac{du_x}{dz} = 0$ and $\frac{du_z}{dx} = 0$ at the walls. This is a free slip boundary condition, and the equation satisfies the free slip boundary condition.

Now temperature fluctuation (θ) is 0 at the plates. Remember that I removed the conduction temperature profile, T_{cond}. So, the temperature at the walls are T_{bottom} and T_{top}. And there is a linear conduction profile for conduction state. So, θ is a fluctuation over it. Hence $\theta = 0$ at the plates for conducting plates.

The primary mode is $(1,1)$ mode. Only k_x and k_z are considered since $k_y = 0$ for a 2D system. This gives us U_{11} and θ_{11} modes like before. Earlier we had derived for $k_c = \frac{\pi}{\sqrt{2}}$ and $k_z = \pi$ but now I am not using those wave numbers. If you look at Rayleigh critical, it was a function of k_c. Here, $k_x = 1$ and $k_\perp = 1$. Earlier, Ra_c was $\frac{\pi}{\sqrt{2}} \approx 2.22$.

Now the formula for Ra_c is $\frac{k^6}{k_\perp^2}$ and we need to know k for the primary mode. Now since $k_x = 1$ and $k_z = 1$, we know that $k = \sqrt{2}$. Also, we know that $k_\perp = k_x = 1$. From this, we get $Ra_c = 8$



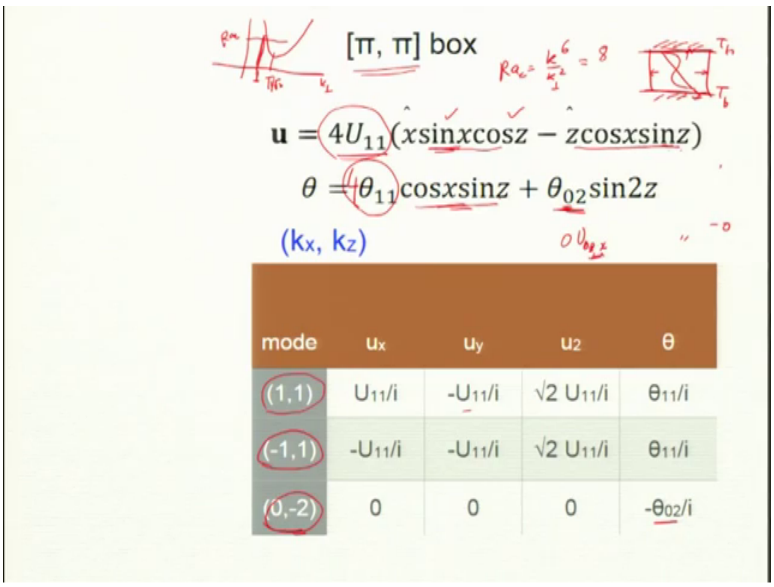
Now, these were the modes which would get excited for this box. This is also an illustration that Ra_c is not the same for all boxes. For this box, at Rayleigh critical of 8, convection will start.

Now, in addition to the primary mode which is (1,1), I am having another mode called θ_{02} and it will be $\sin(2z)$, because there is no x dependence. With $\sin(2z)$, it must be 0 at the walls. Now, what are the Fourier modes present in the system?

For mode (1,1), there will be 4 combinations of $(\pm 1, \pm 1)$. And for (0,2), there will be $(0, \pm 2)$. To understand why we use mode $(0, 2)$ and not any other number, we must look at the picture. Firstly, from the given equations, we can compute u_x, u_z and θ, as we did before. We can write $2\sin(x) = e^{ix} - e^{-ix}$. We focus on the modes $(1, 1)$, $(-1, 1)$ and $(0, -2)$, and compute the amplitudes for these modes. For instance, one can easily see from the formulae $u_x(1, 1) = \frac{U_{11}}{i}$.

$$u_x = 4U_{11}\sin(x)\cos(z) = 4U_{11}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{U_{11}}{i}$$

From the earlier derivations in the course, we can also see that u_z for this mode will be $-\frac{U_{11}}{i}$. That is how u_x and u_z are related. And this is because of the incompressibility condition, where $\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k}) = 0$. The $(0, 2)$ mode has no velocity component because there is no U_{02}. This is due to the incompressibility condition. So $\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k}) = 2U_{02} = 0$. Therefore $U_{02} = 0$.

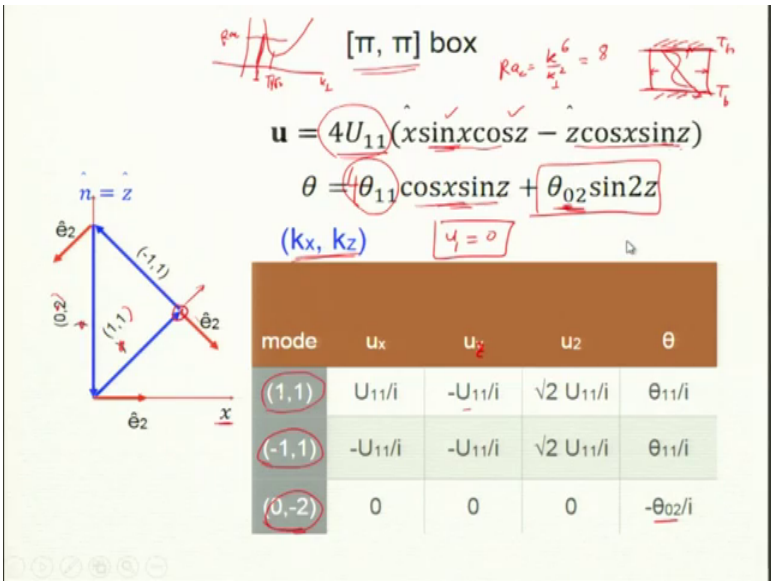


mode	ux	uy	u2	θ
(1,1)	U11/i	-U11/i	√2 U11/i	θ11/i
(-1,1)	-U11/i	-U11/i	√2 U11/i	θ11/i
(0,-2)	0	0	0	-θ02/i

Now why did we choose these 3 modes like this? Because they form a triad.

$$(1,1) \oplus (-1,1) \oplus (0,-2) = 0$$

A triad can interact and pass energy across its modes. And this is how new modes are generated.



These triads can be drawn to form a triangle as shown in the figure. Now, we can do Craya-Herring analysis. So, I need to get $\hat{n}$ and $\hat{z}$. Choose $\hat{n}$ to lie along z direction. So, I am not following the earlier notation that $\hat{n} = \boldsymbol{q} \times \boldsymbol{p}$. Instead we say $\hat{n} = \hat{z}$.

Now for mode $(1,1)$, what is $\boldsymbol{e_1}$? Well, $\boldsymbol{e_1} = \boldsymbol{k} \times \hat{n}$. So, it will be a unit vector pointing out from the plane as shown. We also know that $\boldsymbol{e_3}$ lies along $\boldsymbol{k}$. Finally, we get $\boldsymbol{e_2}$ as $\boldsymbol{e_2} = \boldsymbol{e_3} \times \boldsymbol{e_1}$.

For the mode $(-1,1)$, $\boldsymbol{e_1}$ points into the plane, and $\boldsymbol{e_2}$ lies to the left of $\boldsymbol{k}$ and not to its right as before. This can be seen in the figure. As a result, the vectors do not satisfy the circular dependence as before, and this poses a small difficulty. This is due to the change in $\hat{n}$.

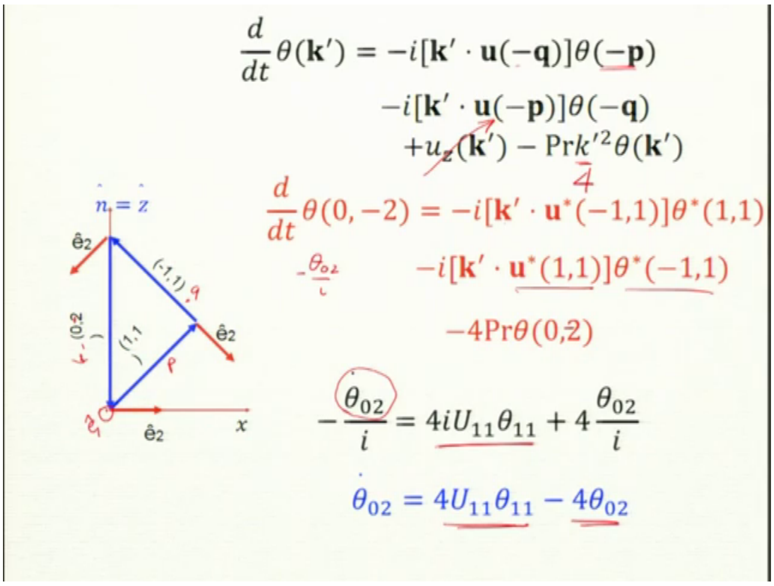
My computer program generates these vectors and performs all the calculations using SymPy module of Python. We need $\boldsymbol{e_2}$ because there is no component along $\boldsymbol{e_1}$ since the velocity field is in the xz plane.

So u_1 is along $-y$ direction. However, there is no component of velocity field along y. So, u_1 is 0. For all these fields I can say $u_1 = 0$. After all we have only two modes - $(1,1)$ and $(-1,1)$, and their negative counterparts - $(-1,-1)$ and $(1,-1)$. We need to focus on only one of these triads to get the equation of motion.

So, you will work for this triad and solve the full non-linear equation. We will need to do that because there is a coupling now. In linear stability analysis, there was no other mode to couple. So, there was $(1,1)$ mode, and there was $(-1,-1)$ mode as well, but they were not coupled. It was not able to pass energy, but now it can pass energy.

So, the triad is clear? That is why Lorenz chooses this triad. This is by necessity. Now let us proceed. So, I am going to work out the equation for θ_{02}. So, we have the same triad again. So, I have this equation.

$$\frac{d}{dt}\theta(\boldsymbol{k}') = -i[\boldsymbol{k}' \cdot \boldsymbol{u}(-\boldsymbol{q})]\theta(-\boldsymbol{p}) - i[\boldsymbol{k}' \cdot \boldsymbol{u}(-\boldsymbol{p})]\theta(-\boldsymbol{q}) + u_z(\boldsymbol{k}') - Prk'^2\theta(\boldsymbol{k}')$$



I am using $\boldsymbol{k}'$ since we are using cyclic notation between $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{k}'$. We first focus on mode $(0,2)$. For this mode u_z term is 0 since there is no velocity field for mode $(0,2)$. Now $k'^2 = 4$ for the mode. So, the linear term is $4Pr\theta(0,2)$. Now the non-linear term will be given by my computer program.

Instead of using terms with $(-\boldsymbol{q})$, we can use their complex conjugates. So, $\boldsymbol{u}(-\boldsymbol{q}) = \boldsymbol{u}^*(\boldsymbol{q})$. Similarly, $\boldsymbol{u}(-\boldsymbol{p}) = \boldsymbol{u}^*(\boldsymbol{p})$, $\theta(-\boldsymbol{p}) = \theta^*(\boldsymbol{p})$ and so on. We can use the table to get values of $\boldsymbol{u}(1,1), \boldsymbol{u}(-1,1)$ and so on.

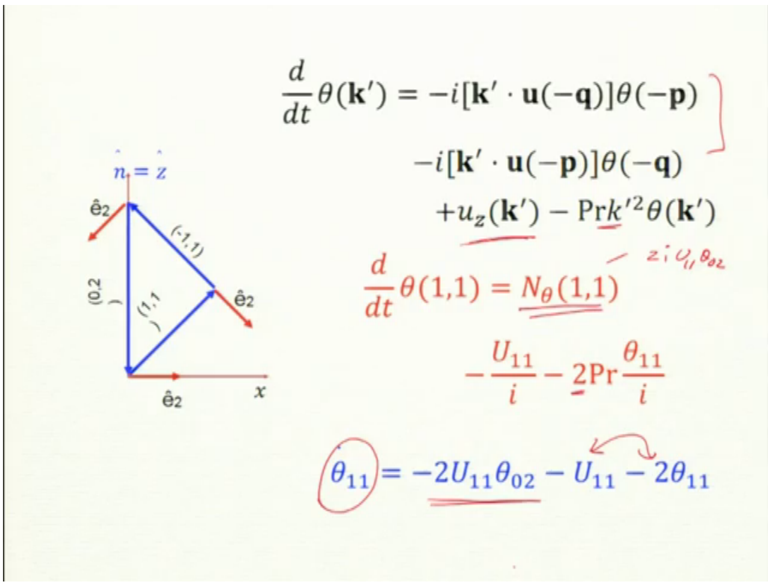
So, now from the table, I can just plug in the numbers. In fact, you can do this problem by hand. So, there is a linear term which is $-4Pr\theta_{02}$. So, you now start plugging in from the table. There will be cancellation of i, then we will get a real equation for θ_{02}. It must be real because my temperature field was $\theta_{02}\sin(z)$ and temperature is a real field.

Similarly, U_{11} and θ_{11} are real. So, if I plug that in, then I will get the required equation. This is also what my computer program gives you, but we can do it by hand, and this is what I get:

$$\dot{\theta}_{02} = 4U_{11}\theta_{11} - 4\theta_{02}$$

This has a diffusion term and a non-linear term. If the non-linear term was absent, then what happens with θ_{02}? It will decay to 0.

So that is one thing that when non-linearity is off, this mode will go to 0. Now this is for one mode, now we must do the same for three modes. The remaining 2 modes are θ_{11} and U_{11}.



We move on to repeat the steps for θ_{11}. In this case, the $u_z(\boldsymbol{k}')$ term is not 0. So, we get the following equation:

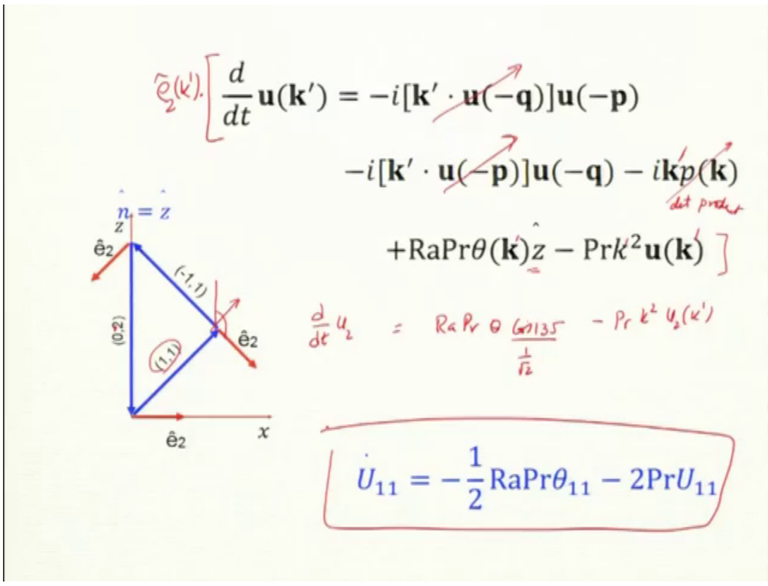
$$\frac{d}{dt}\theta(1,1) = N_\theta(1,1) - \frac{U_{11}}{i} - 2\,\mathrm{Pr}\frac{\theta_{11}}{i}$$

Here N_θ is the non-linear term. Like before, we can derive that $N_\theta(1,1) = 2iU_{11}\theta_{02}$. Substituting and cancelling i, we get:

$$\dot{\theta}_{11} = -2U_{11}\theta_{02} - U_{11} - 2\theta_{11}$$

So, this is my real equation which tells you the time dependence of $\dot{\theta}_{11}$. If the non-linear term is off, then I will get $-U_{11} - 2\theta_{11}$. Now that we have equations for θ_{02} and θ_{11}, we derive the third equation for U_{11}.

To derive this, I take dot product of velocity equation with $\widehat{e_2}(\boldsymbol{k'})$. This will yield $\frac{d}{dt}(u_2)$. Non-linear term can be worked out like before.



Note that velocity field for mode $(0,2)$ is 0. As a result, $\boldsymbol{u}(-\boldsymbol{q})$ (or $\boldsymbol{u}(-\boldsymbol{p})$ depending on choice) will be 0, and the non-linear terms will vanish. Now $\widehat{e_2}(\boldsymbol{k'})\cdot\boldsymbol{k'} = 0$. Hence the pressure term goes away.

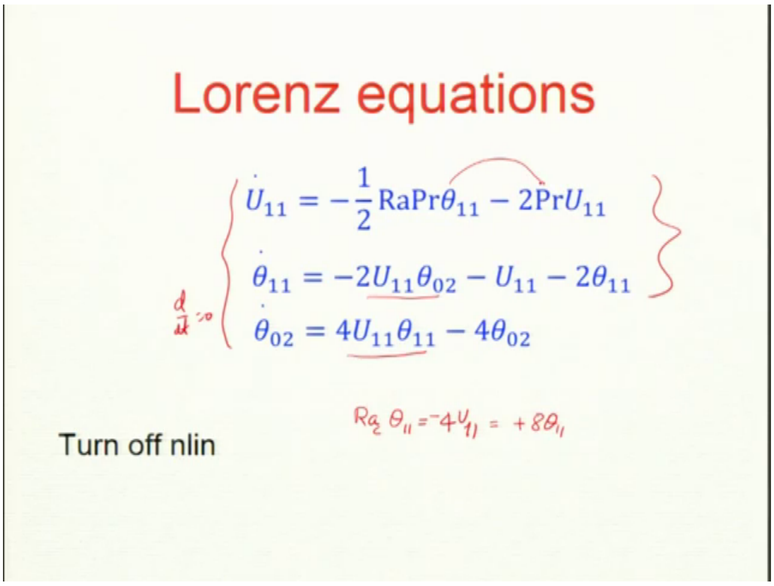
Keeping in mind that mode $(1,1)$ makes an angle of 45° with $\hat{z}$, we get $\widehat{e_2}(\boldsymbol{k'})\cdot\hat{z} = \cos(135°)$. Note also that both are unit vectors with magnitude 1. The final term on right-hand side is $-\mathrm{Pr}\,k'^2\,u_2(\boldsymbol{k'})$. Since $k'^2 = 2$, this term becomes $-\,2\mathrm{Pr}\,u_2(\boldsymbol{k'})$. This leaves us with the equation:

$$\frac{d}{dt}u_2(1,1) = -\frac{RaPr}{\sqrt{2}}\theta(1,1) - 2Pr\,u_2(1,1).$$

Finally from the table, we use the values $\theta(1,1) = \frac{\theta_{11}}{i}$, and $u_2(1,1) = \frac{\sqrt{2}U_{11}}{i}$. This gives us the required equation for evolution of mode U_{11}.

$$\dot{U}_{11} = -\frac{1}{2}RaPr\theta_{11} - 2PrU_{11}.$$

We now have 3 equations – two equations for temperature involving θ_{02} and θ_{11}, and one equation for velocity, involving U_{11}.



Note that while $\dot{\theta}_{11}$ and $\dot{\theta}_{02}$ have non-linear terms, $\dot{U}_{11}$ does not. If non-linearity is turned off, θ_{02} decays to 0, while the other 2 remain coupled. Now can we estimate Rayleigh critical as 8 from linear approximation of these equations? For linear stability, $\dot{U}_{11}$ and $\dot{\theta}_{11}$ both should be 0 – no growth in neutral mode means no growth.

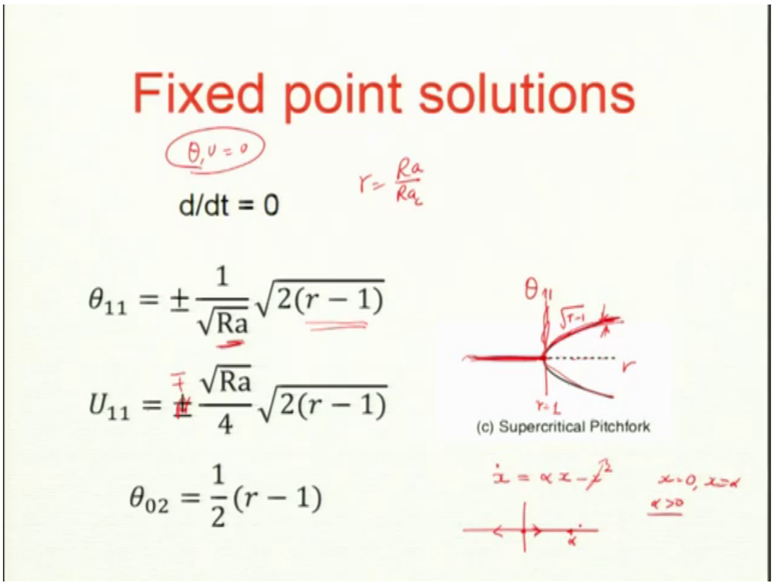
So, from the first equation it gives $Ra_c\theta_{11} = -4U_{11}$, where Ra_c is the critical Rayleigh number. From the second equation, we get $U_{11} = -2\theta_{11}$ (since non-linear term is off). Substituting this in the first equation, we get $Ra_c = 8$.

We see that this is consistent. Next, we look at the fixed points of the full non-linear equation. What is fixed point?

The solutions to the equations when I set their time derivatives to 0 are called fixed points. So, fixed point is a point where things do not change with time. It could be stable or unstable. For instance, one fixed point is when all the modes are 0. So, if I set θ_{11}, θ_{02} and U_{11} all to 0, that is a fixed point. That is when there is a conduction state and no convection. So that is a solution, but there are in fact, two more solutions.

What are the two solutions? You have three equations and three unknowns. You can easily solve them. Their solutions are:

$$\theta_{11} = \pm\frac{1}{\sqrt{Ra}}\sqrt{2(r-1)}; \qquad U_{11} = \mp\frac{\sqrt{Ra}}{4}\sqrt{2(r-1)}; \qquad \theta_{02} = \frac{1}{2}(r-1)$$



Here, $r = \frac{Ra}{Ra_c}$. Note that, these solutions are meaningful only if $r > 1$. For $r = 0$, the solutions are all 0. Hence $r = 1$ is a transition point, since $Ra = Ra_c$ at that point.

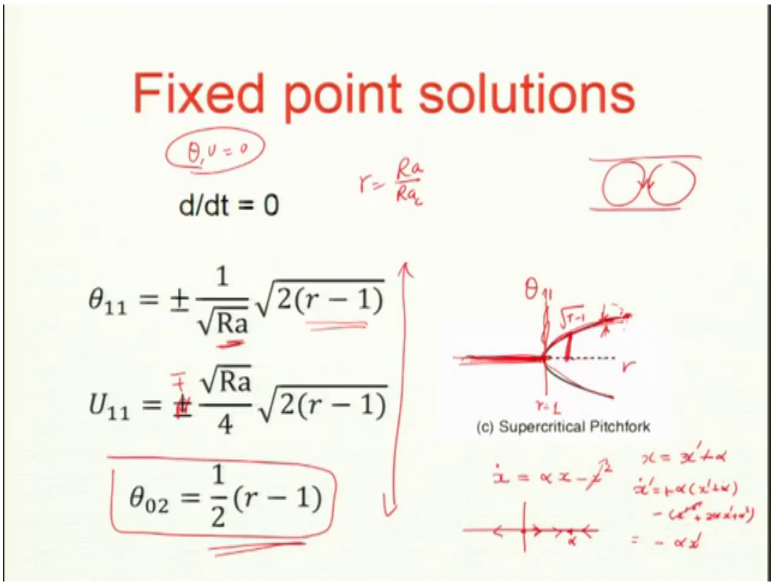
I plot them as function of r with r on x-axis, and one of the modes along y-axis, for instance, θ_{11}. We get a parabolic curve defined as $\sqrt{r-1}$, multiplied with some factor, as seen in slide. There are two solutions and before the onset, there is 0 solution. Before onset, there is conduction solution. Once convection starts, there two solutions with a change in sign.

Note that the 0 solution still exists when the $r > 1$. This conduction solution always exists, but through eigenvalues analysis you can show that they are unstable. If you substitute the values for $r > 0$ into the eigenvalue corresponding to conduction solution, it will have a positive real part, implying that the solution is unstable. The other 2 modes are stable, however.

I will give you a simple example. Let us consider the equation $\dot{x} = \alpha x$. It has a fixed-point solution $x = 0$. Is it stable or unstable? It depends on α. If $\alpha < 0$, then the solution comes to $x = 0$, if $\alpha > 0$, the solution grows. In fact, α is an eigenvalue. So, the eigenvalue of the matrix of your linearized equation can predict the stability of the solution.

From linearized equations, we can get a matrix, but suppose I have non-linear equation, then what do I do? For instance, with $\dot{x} = \alpha x - x^2$, there are two solutions, $x = \alpha$ and $x = 0$.

Now which is stable, and which is unstable? If $\alpha > 0$, then would $x = 0$ be a stable solution or unstable solution? First you linearize around $x = 0$. Then check if a small perturbation around $x = 0$ will grow or not. We can see that x will indeed grow. But what happens for this solution? So, you must now do perturbation. So, how will I do the perturbation?



The best thing to do first time is just put in $x = x' + \alpha$, where x' is a small number. Substituting this into the equation, we get $\dot{x}' = \alpha(x' + \alpha) - (x' + \alpha)^2 = x'^2 - \alpha x'$. Since x' is a small number, x'^2 can be dropped. Thus, we get $\dot{x}' = -\alpha x'$. Now, if $\alpha > 0$, the solution is stable and x' will decay to 0. That is how you linearize around the solution and this is for a 1D equation, but you can also do the same for matrices. For matrices, you need to de look at the eigenvalues.

So, the summary of the story is that for $r < 1$, the stable solution is that all modes, U_{11}, θ_{11}, θ_{02} all decay to 0. For $r > 1$, I get these non-zero solutions, where U_{11} and θ_{11} vary as $\sqrt{r-1}$, and this is called a pitchfork bifurcation.

Now the solution for θ_{02} is interesting as it varies as $r-1$ and not $\sqrt{r-1}$. So we know that for mode θ_{11}, the growth stops instead of growing indefinitely as predicted by linear stability. For a given r, I know what value θ_{11} will tend to. It will grow in the beginning, but it will saturate to a value. So, in convection, there are rolls that are circulating around, but their amplitude gets fixed to a value. It is not increasing anymore. So, the velocity field is a function of x, and not time. It is a steady roll, and temperature field is also steady.

So, what we have achieved is that we suppressed the growth by non-linear interaction. So, θ_{02} took that energy away.

Now we will study how the energy is transferred. You can compute how much energy is being transferred at any time. So, a given mode can get energy from thermal convection. Meanwhile, it can lose energy by non-linear transfer, where it gives energy to another mode, and it can lose energy by dissipation as well. So, a fixed stable solution means they all balanced. So, without nonlinearity, a mode was getting more energy than it lost through dissipation. So, it was trying to grow fast, but because of the non-linear interaction, there is a balance. So, that is the fixed-point solution.

In the Lorenz equations, if you increase r further, the solution becomes chaotic, at some point. For a value of r 24 or 25 it becomes chaotic. So, there is another bifurcation, called Hopf bifurcation and that causes this. So, if you increase the parameters, there will be more dynamics.

So, we have seen the Lorenz equation which is the simplest equation that gives you saturation. I had to include only one mode, $(0, 2)$ here. I could include more modes. In fact, there are many people who have worked with more modes and with more modes, you get different patterns.

In the book *Physics of Buoyant Flows*, you will find an example where there is a 3D structure which has $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 2)$ modes. You can see that they form a triad. There is also an additional $(0, 0, 2)$ mode for temperature. With three velocity modes, and four temperature modes, it is called the seven-mode model, and it has a 3D structure comprised of rolls. These rolls compete with a complex set of dynamics and form interesting patterns, which we will study later.

Also note that Lorenz equation is valid for large Prandtl number. Because the viscous term is dominant and non-linear term for velocity field is 0. When does the non-linear term become 0 for velocity field? When the flow is laminar.

So, non-linear term is dropped. Hence, it is basically valid for large Prandtl number where the viscosity is high. So, these are basically good for $\text{Pr} > 10$ near the onset, but if Prandtl number is small, $\text{Pr} \cong 0.001$, then non-linear term will grow and dominate the velocity field.

So, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ term cannot be dropped. If there is some perturbation this term will become important. So, that is why Lorenz equation is not very good model for small Prandtl number flows.

Here we suppressed the non-linear term in the velocity field and kept the non-linear term in the temperature equation. However, we will have to go the other way round for small Prandtl number flows – you keep the non-linear term in the velocity field and suppress the other.

So, we can really do a lot of stuff once you consider the non-linear term. Then, interaction starts coming into play – patterns, chaos, and so on. We will look into them in future lectures.

Thank you.