Physics of Turbulence
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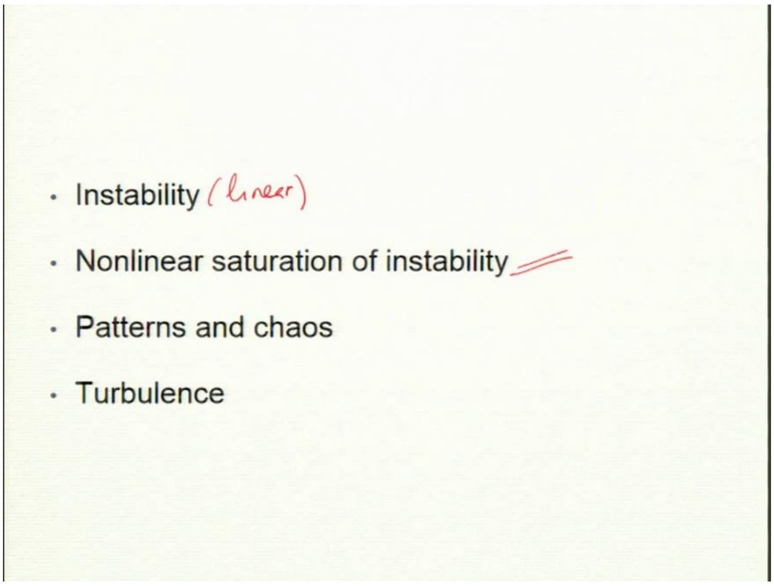
Lecture - 15
Route to Turbulence
Thermal Instability

How does turbulence occur in fluids? It is a very wide topic, but I will take some examples and show you how it happens for some systems. Some of these problems are still unsolved. For example, people really do not know how turbulence occurs in pipes – even for a simple pipe flow. However, the examples which I show you here are somewhat better understood - like convection.



So, first will be instability. In fact, the instability occurring here is very similar to what happens for a pendulum. As you know, a pendulum standing up is unstable - it just falls at the smallest perturbation.

So, the same thing happens for fluids. For some parameter below a certain value, it is stable - there is no fluid motion, or fluid motion is very ordered. However, if we increase the parameter a bit, then fluctuations start to grow, and that is instability. But in real life systems, things cannot grow to infinity, that is, there cannot be exponential growth. In a pendulum of course, theta grows, but it is bounded. Its value cannot exceed π. Similarly, there is a bound on the growth here as well, and that happens by non-linear saturation. In fact, this is a linear stability. In this course, I am going to work only on linear stability. There are non-linear stabilities too, but we will not deal with them here.

This is equation behaves linearly initially, but after some time, the non-linearity takes over and growth saturates. So, you could get a fixed-solution, or you could get an oscillatory solution. And beyond that, if you increase the non-linearity further, you will get more patterns, and then chaos. By the way, if you put more non-linearity, you may get a better pattern. In this example, putting in more non-linearity gives a stable solution. So, non-linearity does not imply that more non-linearity gives more disorder. Non-linearity is complicated. After we get patterns and chaos, we get turbulence.

So, first let us set up an equation. Let us set up a system which will act as a framework. I will not derive the full equation, but I will follow the example of convection.

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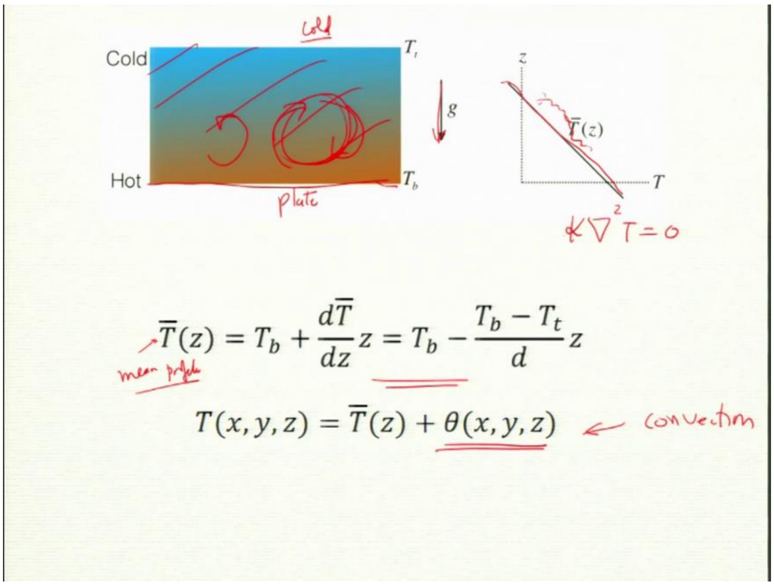

So, the idea is that we have a hot plate below and a cold plate above. It is not like heating in a kitchen, where the top lid is open typically. Here, the water is full, and the top lid is closed. This approximation avoids the complexity of an air-water mixture, with air bubbles. Gravity is acting downwards as usual. There is a mean temperature in this system.

Now convection means that the velocity field will start to turn around. Like the convection rolls (in above Figure). But if there is no convection, then the temperature will drop linearly. This is called a conduction solution. So, $\kappa\nabla^2 T = 0$ is the equation you get by turning off d/dt and the solution in 1D is T, and is linear in z. Now I need to set up my equations. To do this is, we make some assumptions.

So, we assume that the conduction profile or mean profile, $\overline{T}$, is linear. This mean temperature comes from the boundary condition - bottom plate is hot, top plate is cold. When convection is not present, the temperature is just $\overline{T}$. When convection starts, there will be fluctuations in temperature. So, there will be temperature fluctuation added to the mean temperature. Here, $\theta = 0$ means there is no convection, and $\theta \neq 0$ means convection. Instability will happen when θ grows exponentially in time. So, that is the solution we are looking for - instability.

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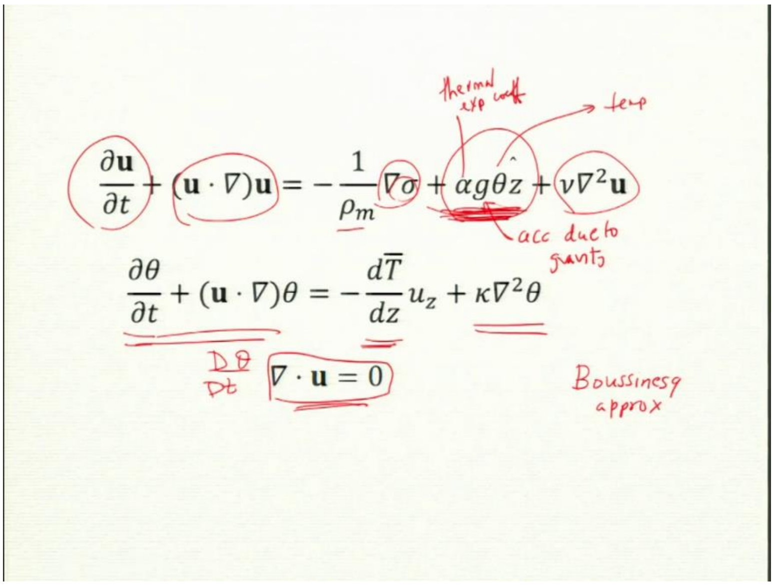

In the final equations, α is the thermal expansion coefficient, g is accelearation due to gravity and θ is the temperature fluctuation. Also, I have absorbed that mean pressure in σ. The $\bar{T}$ has also been absorbed here. More of this is covered in the book – Buoyancy Driven Flows.

Apart from the equation for velocity field, I need one more equation, for temperature. We still assume that $\nabla \cdot u = 0$. These is slightly surprising, because it implies that the density is constant. If density is constant, then buoyancy effects will stop. But I want hotter fluid to be lighter. So, it turns out that this is a first order approximation. So, variation in density is non-zero, but that occurs only here and that is higher order. This assumption is called Boussinesq approximation. This approximation is not good for sun. Also for Earth it is not excellent, but it is okay.

So, this equation is good for convection in the kitchen, but not for solar convection. For atmosphere this is a good equation. Buoyancy is feeding the velocity. The system will become unstable because something is giving energy. So, that is what this is going to happen - with significant energy, rolls start to form.

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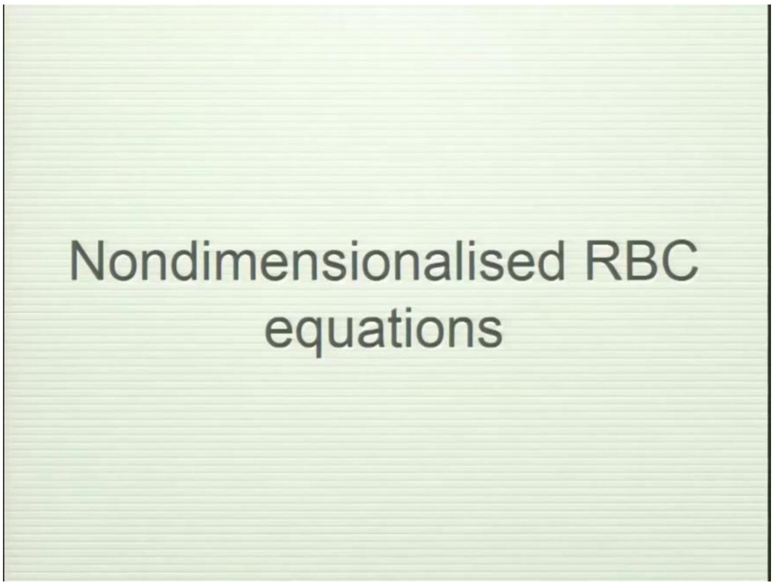

Now the non-dimensionalisation used here is slightly different from what was shown before. For Navier-Stokes, the standard velocity is mean velocity field that is used as velocity scale, and cylinder or box size is used as length scale. Here we make a slightly different assumption.

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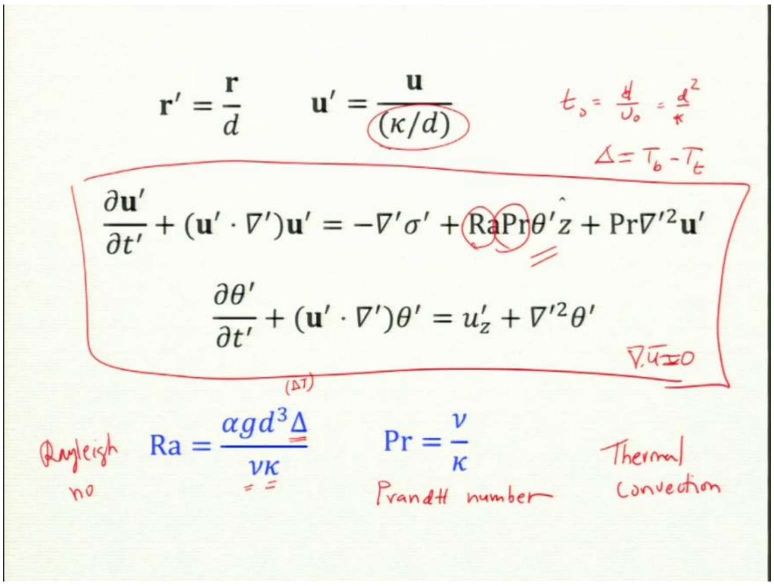

We take d, the box size, as the length scale, but the velocity scale is κ/d. Here, κ is thermal diffusivity. There are other types of scaling, but I am going to use this for my lectures as it is the simplest.

What is the time scale? $t_0 = \frac{d}{u_0} = \frac{d^2}{\kappa}$. Similarly, you can also find the pressure scale. Finally, plugging it all in, we get a non-dimensional equation with two parameters Rayleigh and Prandtl. Rayleigh number is $Ra = \alpha g d^3 \Delta / \nu\kappa$. Here, Δ is the temperature difference between the plates.

So, $\Delta = T_b - T_t$. And ν is the kinematic viscosity and κ is thermal diffusivity. Finally Prandtl number, $\Pr = \frac{\nu}{\kappa}$. So, we will work with these equations under the assumption that $\nabla \cdot u = 0$. So, it is still incompressible except for the buoyancy term. So, now we can use Craya-Herring to solve these equations. These are the equations for thermal convection.

We will work on two more systems. One will include effects of rotation in it - rotating convection. Second will have magnetic field – magneto-convection. Rotating convection has just Coriolis force in it, as centrifugal force is absorbed into the pressure term, but magnetic convection is more complicated. We will start the instability calculation for thermal convection. We will see how the rolls will start coming in and how that happens because of buoyancy.

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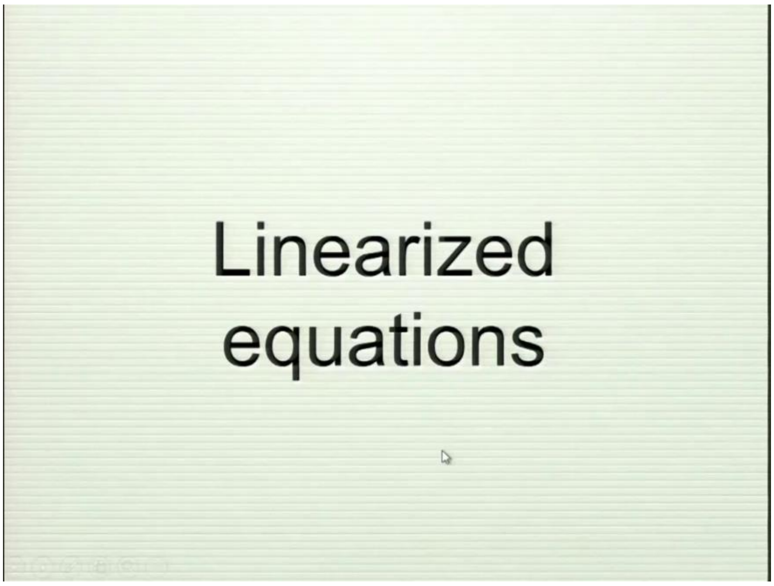

When the system is heated from the bottom, the light fluid tries to go up and heavy fluid tries to go down – this is how energy is fed into the system. However, viscosity tries to suppress this. The viscous term damps the motion, but when the temperature is strong enough, the viscous term can be overcome.

We are focusing on linear stability. So, you must turn-off the non-linear term and see whether the linear system gives you stability. By the way, not every system gives instability. For instance, in oceans, gravity tries to pull the fluid down. So, it is a stable system, and we get only wave solutions which are not unstable. In the linear regime, it only gives you wave solution – these are gravity waves.

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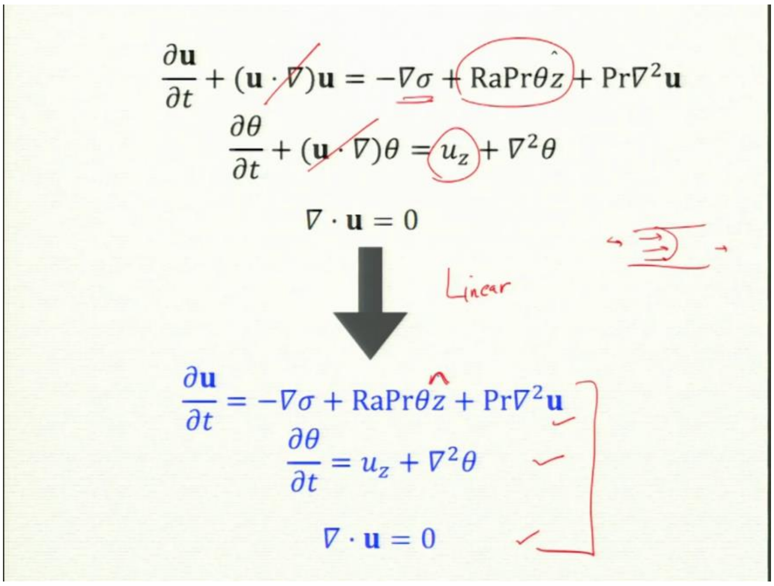

So, first we need to linearize our system. So, which are the non-linear terms in the equation in the above slide? Non-linear terms are the quadratic or higher order terms here - $(u \cdot \nabla)u$ and $(u \cdot \nabla)\theta$. We must drop these and retain rest of the terms. Now we have a vector equation, scalar equation and a constraint equation.

Now, we will work in Fourier space. What is the stationary state? It is the state where $u = 0$ and I want to see the perturbation over this state. Stationary state is different for different systems. For pipe flow, there is a mean velocity profile. This linear/viscous solution is the stationary state for pipe flow.

Now, if we increase the pressure gradient, we expect that at some point, the system may become unstable. Similarly, $\theta = 0$ is a stationary solution, $\theta \neq 0$ is the non-stationary solution for this problem. So, if $u = 0$ and $\theta = 0$, the whole equation is 0 equal to 0, pressure also will be 0, pressure gradient will be 0. We thus get a trivial solution.

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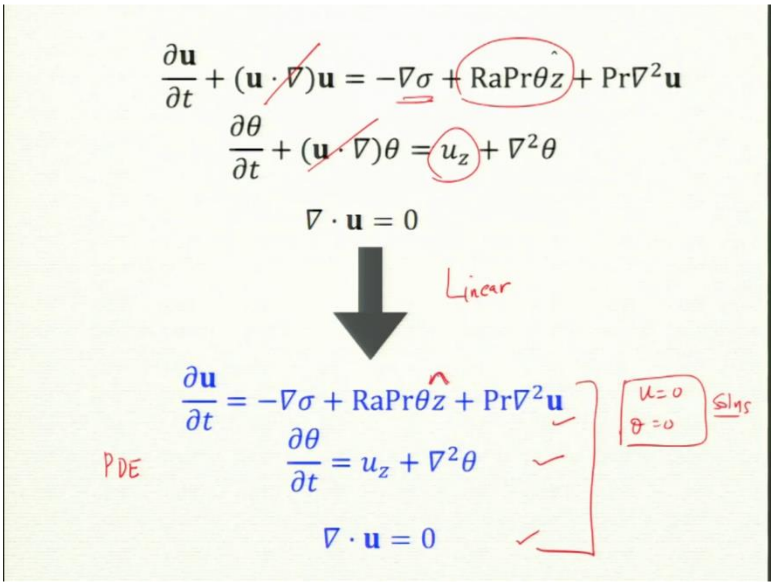

This solution implies that there is only conduction, no convection. These are stable solutions. One important idea in all instability calculations is to use Fourier modes. I want to convert these PDEs to ODEs. And ODEs are much easier to solve, and then we look for some growing solution in ODEs. One standard way to do convert PDE to ODE is, to use Fourier transform.

In fact, all my previous exercise was to use the Fourier mode to derive equation for the mode which are all ODEs. Now, there are too many Fourier modes. Which one will I choose? I look for the one which is most excitable, the most unstable mode. We will find which is the mode which gets excited fastest and that will set the criteria. So, when I heat the fluid, not everything is excited at the onset itself, some are getting settled later, but the one which is most unstable gets excited first, and that mode will come out of the calculation. So, I will keep a general k and we look for one k which will be coming out first when the instability sets in.

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So, for our calculation, so there are many different boundary conditions, but free-slip boundary condition is one of the simplest.

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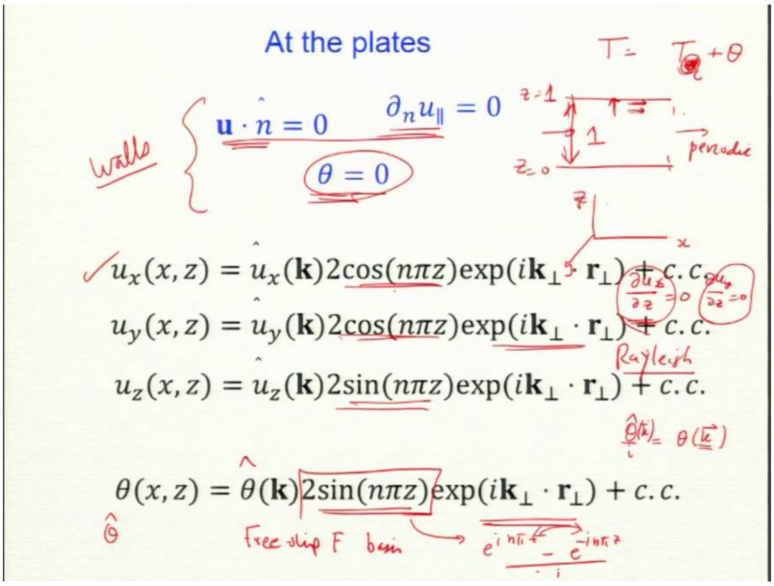

So, in the horizontal direction, I assume periodic BC. Remember, Fourier is good for periodic, we showed that, but in the top and bottom, I use a wall. Without that, I cannot set up convection. Then you can see that my vertical velocity should be 0, right? At the wall, you hit the wall and you must have 0 velocity. So, this means $u\cdot\hat{n} = 0$, for both top wall and bottom wall. But this is only in one component, what about other two components?

In a realistic boundary condition, for hard surfaces, all the 3 components must become 0. But the calculation for that is quite complicated in Fourier space. We use a simpler boundary condition called stress free BC. So, if I go along z direction, my horizontal velocity must be 0. This means $\frac{du_x}{dz} = \frac{du_y}{dz} = 0$. How does stress come? If the velocities are not same in two layers, then there is stress, but if velocities are same, there is no stress.

If two runners are just running in the same direction, and you tie them with a rope, they would just run. Only when there is a difference in velocities, they tug on the rope. This implies that horizontal velocities when I go near the wall, I get the same velocity. Horizontal velocity is still non-zero.

So, this is stress free condition. And my conducting wall means temperature of the full plate must be constant, like copper plate, temperature is constant. So, temperature fluctuation on the plate will be 0. So, temperature at the top plate is T_t. If you recall, $T = T_c + \theta$. So, $T_t = T_c$ at the top wall. So, $\theta = 0$ at the top wall. So, these are the conditions on the walls, and of course, the sidewalls are periodic.

If the plates are let us say, 1 centimetre apart, but in the horizontal direction, I have, let us say 20 centimetres, periodicity is a good assumption. These calculations were done by Rayleigh first, then the standard reference given book is Chandrasekhar's book, but my calculation is simpler than Chandrasekhar's book.

So, I need to choose the velocity field, I am choosing a Fourier mode. We should satisfy this boundary condition. Now, you choose your basis function. So, u_x, I want this u_x, $\frac{\partial u_x}{\partial z} = 0$. So, if I take partial u_x what will I get? I will get $sin\, n\pi z$. After non-dimensionalisation box height is 1, because my unit is d. So, in the unit of d, my box height is 1, I choose these to be $z = 0$ and $z = 1$.

So, if I take the derivative, I get sin, a sin = 0 at both the walls, so straightforward. So, it is $z = 0$ and $z = 1$. I need to satisfy these boundary conditions. Same thing for u_y, what should I use z component for u_y? Sine or cos? This one, $\frac{\partial u_y}{\partial z} = 0$ again. I should use cosine. Function in z is cos and this is exponential because of periodicity - $e^{ik_y y} + e^{ik_z z}$.

Now, u_z is sine, because I want to make $u_z = 0$ at the wall. This will satisfy the boundary condition, that is the beauty of this. You can use sine cos to work with free-slip basis function. If all 3 components are 0, you cannot use sine cos. That is why free-slip is easier. You must use more complex special functions like Chebychev functions. What about theta? What should I use for z direction?

Sine, because I need 0. So, theta has same as this one. And along x and y I use this, ok? Now, I use a hat here. Hat is when it is a mixed basis. This is sine and this is exponential. So, I used a special name, hat, without hat, it is exponential for all 3 directions. I hope this point is clear. These are called free-slip Fourier basis, but I can always convert this to exponential.

So, what will I do for converting to exponential? I say $(e^{in\pi z} - e^{-in\pi z})/2i$ and 2 will cancel, so i. So, I had absorbed this i. So, what should I do? $\frac{\hat{\theta}}{i}$. So, how do I convert from $\hat{\theta}$ to normal Fourier basis function? So, I write it here, it will be $\theta(k)$. So, this is k_x, k_y, k_z but I want exponential in all 3 directions. So, I have to divide by i, I have to absorb i along with $\hat{\theta}$ to get θ, because I have this.. Craya-Herring requires Fourier, ok?

Do not get mixed, with mixed basis, Craya-Herring may have problems. So I want to convert it to Fourier, ok? And it is straightforward. This is the technology. Cosine, you do not need to do anything, for sine you need to divide by i, and of course, there is a symmetry you must keep. So, this has these two modes connected by minus sign. So, this bookkeeping must be kept in mind. So, that is why I do $\sin(x)\cos(z)$. So, all the modes have certain relations among them. It is imposed by the boundary condition.

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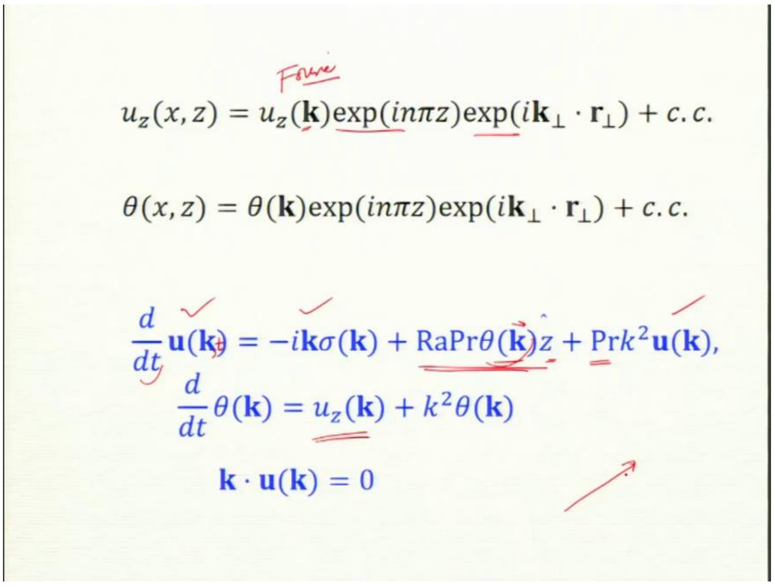

So, now, I put exponential, without hat. This, I am going to apply with Craya-Herring on that. So, $\theta(k)$, I take a Fourier mode, of course, I understand that the minus k is. This has k_x, k_y, k_z, but $-\widehat{k_z}$ is related with k_z.

Now, I want to plug this in. So, we already know how to write the Fourier equation. I do not need to do the full algebra. Just write down equation for the Fourier. So, this my equation in Fourier. So, these terms, you are familiar with.. viscous term is.. so there is Pr sitting in front, like νk^2, you know? That is $\hat{\theta}(k_z)$. So, $\theta(r)$, becomes $\theta(k)$, and theta is scalar.

So, from $RaPr\theta(\hat{z})$ I get $RaPr\theta(\widehat{k_z})$. So, here this $\frac{d}{dt}$ is acting on $u(k,t)$. In Fourier space, there is nothing called grad. So, non-linear term is off, and hence convolution is off, because I linearized my system. So, I write down for θ. So, the term which has come is u_z, and I have $k \cdot u(k) = 0$. This is the set of equations.. After that.. So, I focus on single k, there are many, many k's but I focus a single k, ok.

So, let us do Creya-Herring. Are you ready for Creya-Herring? So, this k. So, I fix the k which is in let us say a particular direction.

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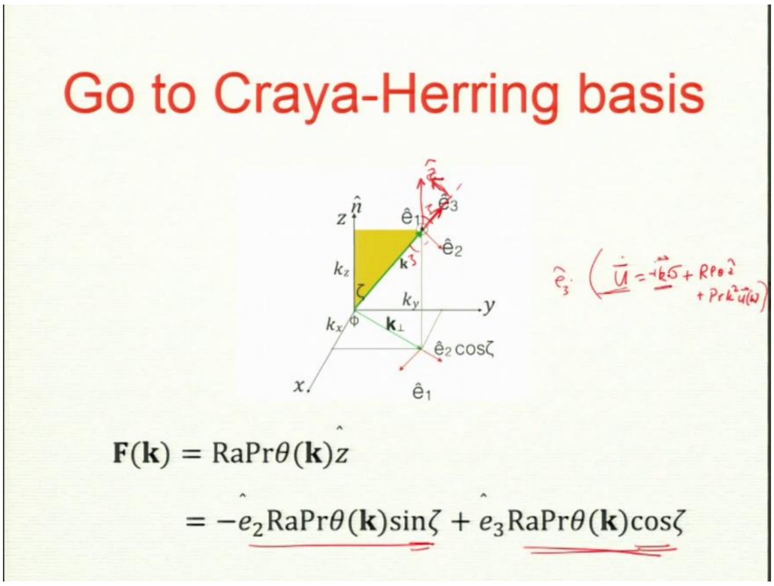

So, here.. my k vector is here, but my only problem is that new term, buoyancy term, otherwise I know. All the algebra I did in the last example is exactly same algebra. In fact, I do not have non-linear term, this is straightforward. I need to worry about this buoyancy term. So, buoyancy term is $\hat{z}$ which is in that direction. I need to resolve the $\hat{z}$ along e_1, e_2, e_3. When I use polar coordinate, then I need to resolve along the three unit vectors. So, what is $\hat{z}$ along e_3? So, this is ζ. This is also ζ, so $90 - \zeta$. (Refer figure)

So, $\hat{z}$. So, I just take the component, like this, e_3, and this is.. this is along e_3, and this is along $-e_2$. So, basically $\hat{z}$ is this plus this, right? Vector addition. So, there is a component along $-e_2$ and component along e_3, and this is related by cos and sine. So, this straightforward exactly; so, along e_2 it is $-\sin(\zeta)$ and along e_3 it is $\cos(\zeta)$.

So, I take the component of my velocity field along these two. Like exactly like what I did in the last example. So, that will give me u_1 and u_2, but does the force have component along u_1? Along e_1? It has no component along e_1, it has only along e_2 and e_3. So, what does e_3 component match with? Non-linear term is off. So, $-ik\sigma(k) + RaPr\theta(k)\hat{z} + Prk^2u(k)$. This is what I am writing slightly small, but $\sigma(k)$. This has component along e_1 and e_2. Along e_3, it has no component along e_3. So, if I take a dot product of this with e_3, then only thing will give contribution is pressure. So, pressure will balance with this one, e_3, which I told you in the linear example, pressure will balance with the force in the z direction, forcing e_3 direction not z, e_3 direction, ok?

So, I do not need to compute pressure, that is one interesting part. Pressure is off, gone, and pressure is managed by the.. this part, ok?

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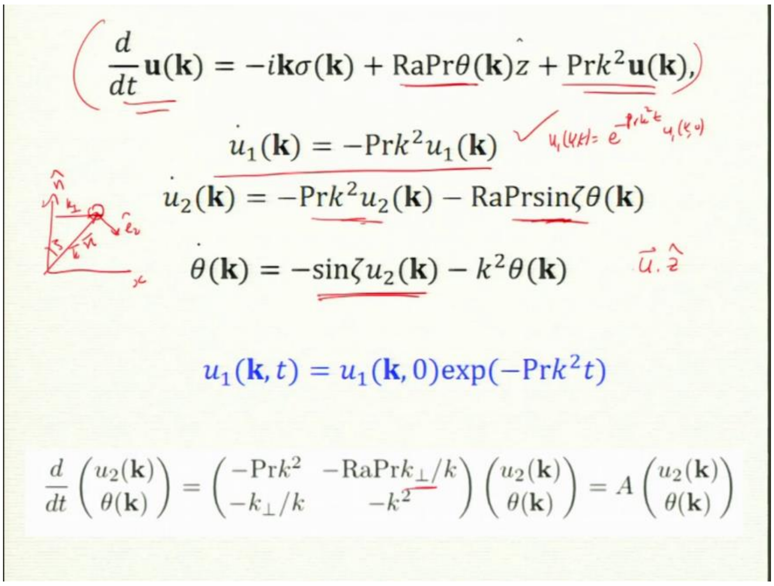

So, now we are ready for writing for.. I take this dot product with e_1 and e_2. So, u_1 has no component along force. No, u_1 has no contribution coming from force. So, in fact, it is a decaying equation, right? This is coming from here and this coming from here, so u_1 will decay in time. $\dot{u}_1 = -u_1$, u_2 gets contribution from here.

So, e_2 component, which was in the force, so this is the sum, and this is coming.. This is coming from the force u_2 component and this is coming from viscous term. And $\dot{\theta}$ is u_z. Now, I write u_z as $u \cdot \hat{z}$ because $\hat{z}$ has component along e_2 and e_3. So, e_3 will give you 0. So, u_2 will give you this, ok? So, $u \cdot \hat{z}$ that is it you just compute then this. So, I got 3 equations, and I do not need to worry about $u(k)$, $k \cdot u(k) = 0$. So, I need to solve this.

So, what does the first equation tell you? I look for asymptotic solution. Wait for some time, ok? u_1 will go to 0. What is solution for this? $u_1(k, t)$ is $u_1(k, 0)$exp $(-Prk^2 t)$. So, this goes to 0. So, do not worry about. In fact, my field becomes two-dimensional, because only u_2 is present, u_1 is 0. So, if I look at this k vector, remember u_2 is in the same plane, as $\hat{n}$ which is in z direction. So, this is x direction and this is z direction. So, my e_2 is like this. So, my velocity field has component along u_2.

The azimuthal component is 0. Azimuthal component is in that direction. Remember, this one, that goes to 0. And this, I can predict from here, like in Chandrasekhar's book he will not.. He will work with stream function and he will not talk about this, assumes 2D field. But here you can say that 2D will come by derivation.

Now, how do I solve this equation? Now I have two variables. By the way, u2 k is one variable, it is not.. I have fixed k. So, I have two ordinary differential equations. So, I can write them as a differential equation, like this. This is my equation. Now, what is sin (ζ)? This angle is zeta. So, $k_{\perp}/k$. So, this is a matrix equation, u_2, θ. So, matrix equation. So I solve the matrix equation, ok? So, this is where we will start in the next class, ok? So, we stop.

Thank you.