Introduction to Solid State Physics
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Lecture – 10
Understanding Thermal conductivity of a metal using Drude's Model Part – II

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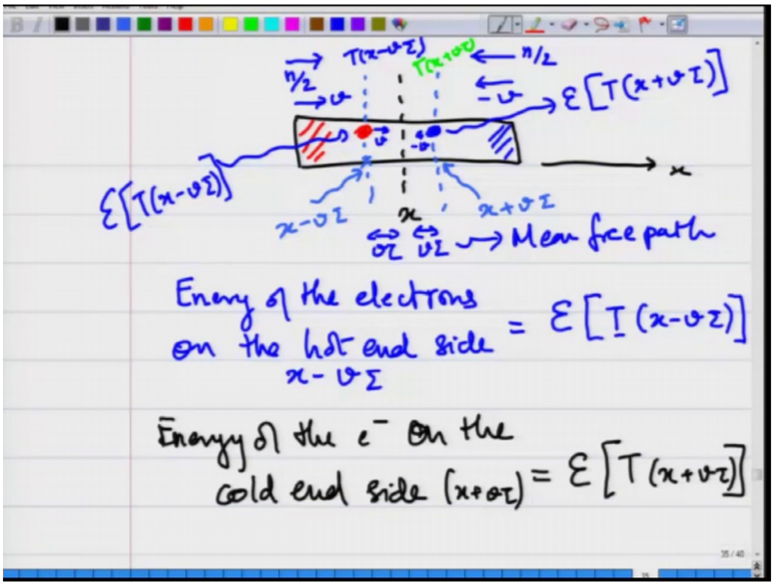

We had started to look at how to conduct the Thermal conductivity of a metal. And for that we need to find out how much is the net energy being transported across this point x, we had a one-dimensional type of a metal, we had a one-dimensional construction of a metal, where you have a hot end and you have a cold end. And electrons are flowing from both the hot end and the cold end. And we are going to consider that on the average n by 2 density of electrons are going to cross from the hot end to the cold end, and n by 2 density of electrons are going to cross from the cold end to the hot end.

So, let us consider from this point x at a distance of v times tau, this point x minus v times tau the location which is on the hot end side. And we look at the electrons which are flowing from the hot end side towards x. At this point x times v times tau, the electrons have the energy of the electrons on the hot end side at x minus v times tau, let us write it as the energy of these electrons which are on the hot end side, the energy of these electrons which are on the hot hand side is determined by the temperature at x minus v times tau.

It is one of the natural of assumptions of Drude's theory that the electron which is present out here, the energy of the electron is determined by the local temperature at that point. So, the energy of the electron at this hot end side will be E which is a function of the temperature T at x minus v times tau. Similarly, the energy of the electron which is on the cold end side will be determined by the temperature at x plus v times tau. And why are we using these v tau distance intervals, because whatever is the electron at x minus v times tau it will carry its energy and momentum undisturbed to reach up to point x.

Similarly, from x plus v times tau, the electron will move undisturbed without any scattering or collisions up to point x because v times tau is the mean free path. So, there is very low probability of collisions which are going to happen for these electrons which are within v times tau interval of x. However, the temperature on one side because there is a temperature gradient across the sample, this point is sitting at a temperature of x minus v times tau. And this is sitting at a temperature of x plus v times tau and that is the basic idea.

So, the energy of the electrons on the cold end side at x plus v times tau is governed by the temperature at x plus v times tau ok, because the sample has a temperature gradient ok, and then only there will be some flow of energy. So, now, with this built in let us calculate how much is the net heat flux that is moving from the sample from either ends.

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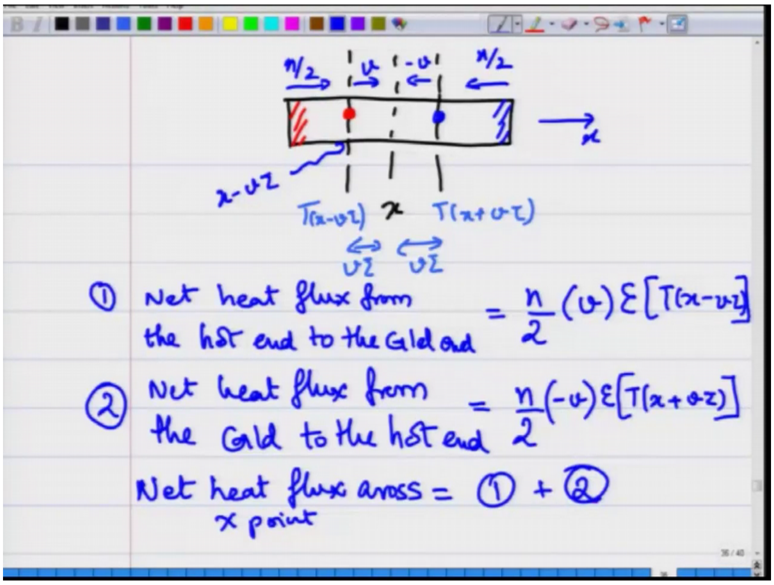

So, let us redraw this picture. You have x; you have the hot end side; you have the cold end side. This point is at temperature x minus v times tau, and this is at temperature x plus v times tau. These distances are of the order of mean free path. So, the electron which is coming from the hot end side, and the electron which is coming from the cold end side, this is moving with a velocity v. This is moving with a velocity minus v. And n by 2 is the density of electrons which are moving from the hot end to the cold end; and n by 2 is the density of electrons which are moving from the cold end to the hot end.

So, we can write down the net heat flux from the hot end to the cold end will be n by 2 into v into the energy which is determined at the point t x minus v times tau, n by 2 is the density of electrons which are moving from left to right, v is the mean velocity of the electrons mean speed of the electrons they are moving from left to right. And E is the energy of the electrons which are determined by the local temperature present at x minus v times tau - the hot end side. Because those are the ones which will carry the energy, those which are further away will scatter randomly and the net average momentum will go to 0.

So, there will be no net momentum for these electrons which are away from v times tau because they will undergo multiple scatterings, they will not carry any net momentum flux which will cross the point x. It is only for those electrons which are within v x star which have reached up to that point which whose a velocity is going to be determined or the momentum is going to be determined by the local temperature at this point. They are the ones which will continue on scattered up to point x. And therefore, contribute to a net momentum flow across x, and thereby give rise to a net flow of energy and heat current across that point and that is the basic idea.

So, you have a one term which is the net heat flux which is coming from left to right. Similarly, you have another heat flux which is coming from right to left. And so the net heat flux from the cold to the hot end is n by 2 into minus v, because now recall that the velocity of these electrons is in the opposite direction to this distance x, these are drifting in the opposite direction. However, their energy which will contribute to the net heat flux will be determined by the local temperature at the cold point which is x plus v times tau.

So, the energy is a function of the local temperature x plus v times tau. And as I said the electrons at these points whatever their momentum is governed by or determined by the local temperature at this point. And they are the ones which will continue undisturbed without scattering and carry a net momentum across x which will give rise to a net heat flow. So, the net heat flow, the net heat flux across x point is just the sum of 1 and 2, it is just sum of 1 and 2, you just take the summation of these two terms in this simple one-dimensional picture.

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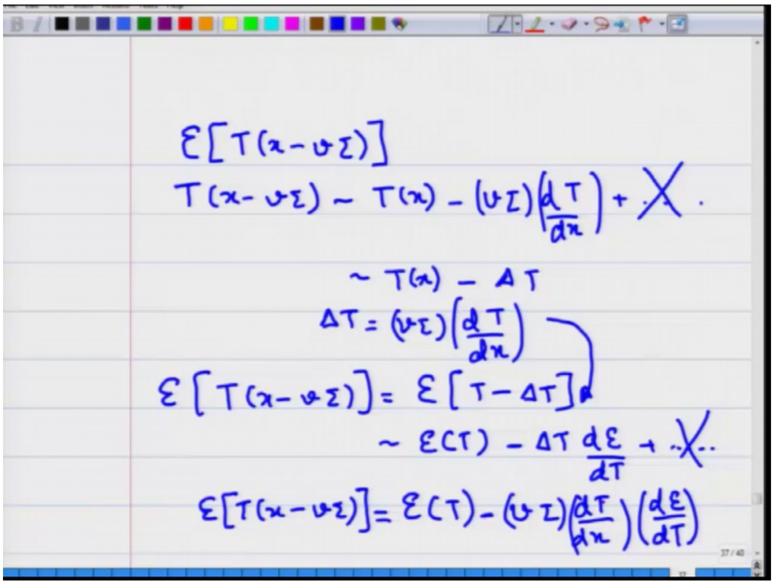

So, let us now calculate this term. The energy on the hot end side of the electron which are within v tau of the point x is determined by the local temperature at that point which is x minus v times tau. Now, this temperature x minus v times tau, I can easily Taylor expand it, because it is a very small interval around x. I can easily expand it to the temperature at point x minus v tau into dT by dx. And so I can write this as and I neglect higher order terms. So, I am going to neglect all the higher order terms and only consider the first order correction.

So, the temperature minus delta T, where delta T is v times tau dT by dx. And so the energy at T x minus v times tau is nothing else but the energy at T minus delta T. And this again I can Taylor series expand, energy at point T minus delta T into d E by dT. Again Taylor series expansion, this is the Taylor series expansion where I am going to neglect all the higher order corrections. And so if I put it this all of this here, the energy at x minus v times tau can be written as energy minus v times tau into dT by dx into d E by dT.

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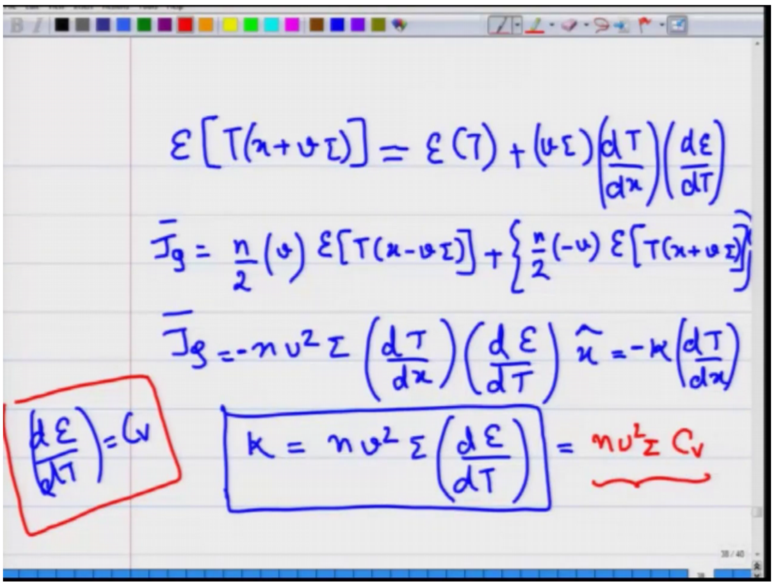

And you can write a similar expression for v plus for the temperature at v plus tau, for the energy at temperature x plus v tau. This will be energy E T plus v tau dT by dx into d E by dT. And the net heat flux J q, if you recall was the sum of two terms, one was n by 2 into v into energy of T x minus v times tau plus another term which was n by 2 into minus v into energy of T at x plus v times tau.

And now we have got these expressions. These are approximations you can put that in here and after you put those terms here and taking into account these negative signs which are present in the above expressions as well as here. You will end up with an expression which is n into v square tau dT by dx into d E by dT.

This is the net heat current which is flowing along the x direction. This is the heat current in this one-dimensional piece of a metal where you have maintained a gradient in temperature across the metal, and how much is the heat current which is flowing across a given point x and that you have a very general expression for this. And this if you recall J q will be I am sorry there is a negative sign here, there will be a negative sign which you will get out here. And this is nothing else but in the one dimension is the thermal conductivity into dT by dx ok. And so your thermal conductivity turns out to be n into v square tau d E by dT.

So, using this Drude's idea, you can get an expression for the thermal conductivity of the metal. The thermal conductivity itself as you see is modified by the presence of this scattering. The scattering actually changes the thermal conductivity of the material ok. And d E by dT is the change in the total energy of the particles as a function of temperature and that is nothing else but specific heat, you know this is the specific heat. So, k is n v square tau times the specific heat. This is for the thermal conductivity in the x direction for this one-dimensional system ok.

And so you have a nice expression where you can see that n is the density of electrons, v is the average speed of the electrons inside the metal, tau is the scattering time, and c v is the specific heat of the electrons inside the metal ok. You can also work it out for the three-dimensional case. And if you work it out for the three-dimensional case, now this is the thermal conductivity in the x direction; you will get a thermal conductivity in the y direction; you will get a thermal conductivity in the z direction.

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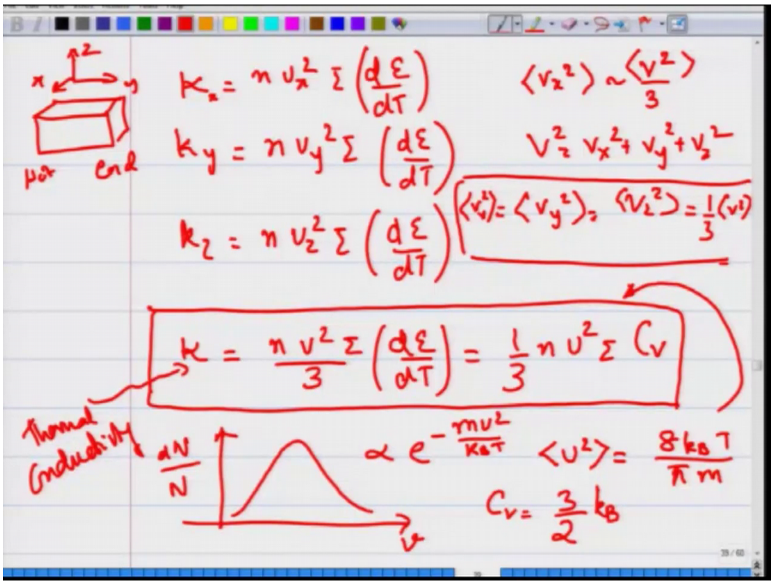

So, you will get terms like kappa is equal to n v x square tau d E by dT in the x direction, in the y direction nv y square tau d E by dT. And the kappa in the z direction is n vz square tau d E by dT. For a general three-dimensional material with you maintain a certain temperature gradient across the material and then in different directions how much is going to be the thermal conductivity will be governed by how much is the velocities that you have in the x direction the velocities or speeds in the y direction and the speeds in the z direction.

And here we are going to again go into our assumption of the free electron theory namely the kinetic theory of gases, where we will assume that if there is no other effects of anisotropy, there is no reason to assume that the velocity in the x direction or the y direction or the z direction will be very different from the mean velocities in any other direction. So, we will consider that the average velocity in the x direction is roughly one-third the total mean velocity where this square is v x square plus v y square plus v z square.

Similarly, the velocity in the y direction is going to be equal to the velocity in the z direction will be one-third of v square which is equal to v x square. So, this is a well-known thing that you do in the kinetic theory of gases. And so you can put that in here and you get the average conductivity, the thermal conductivity of the material is n v square tau by 3 d E by dT which is again the specific heat. So, it is one-third n v square tau times the specific heat ok. So, you can also find out what is the average thermal conductivity of the material for a three-dimensional material. Given that you have a three-dimensional material most a lot of our materials are three-dimensional. And what is the thermal conductivity of this metal in three dimensions. C is the specific heat of the material, and v is the mean square average velocity of the material, n is the density of electrons.

Now, with this let us look at a little bit more about the thermal conductivity. We have obtained a way to actually estimate the thermal conductivity of the material. Now, let us go into a little bit of the details of the Drude's model. And in Drude model you know that there is a Maxwellin-Boltzmann distribution, we have seen that we can write down an expression for a 3D material across which we maintain a hot and cold end. Suppose we have a hot end and you have a cold end, then there is going to be a heat flow there is a gradient in temperature across the material, and there is going to be a heat current. And from that heat current we found out what is the thermal conductivity of the material. And the thermal conductivity of the material was related to the mean square velocity average mean square velocity of the material as the specific heat, the scattering time constant, and the density of electrons.

Now, let us try to put in some numbers. And if you recall the Drude's theory is the kinetic theory of gases. And in the kinetic theory of gases, we know that the electrons are supposed to have a Maxwellian type distribution. The distribution the probability distribution of the electrons if d N is the number of electrons between v and v plus dv; and if v is the speed of the electrons, then they have a Maxwellian type distribution, where the distribution is proportional to e raise to minus m v square divided by k B T roughly this is the form of the distribution.

Now, in kinetic theory of gases, you have learned how to calculate the mean square velocity distribution. And if you calculate using this Maxwellian distribution the mean square velocity of the electrons, then this means where velocity of the electrons using the Maxwellian type distribution, you can show specifically is equal to 8 k B T by pi into m, where m is the mass of the particle which in this case is the mass of the electron because whether its gases particles in a gas or whether here it is a gas of electrons m is the mass of those particles constituting the gas. And in this Drude's model we have a gas of electrons, so m is the mass of the electron. So, the mean square velocity if you calculate using this Maxwellian distribution you will get this.

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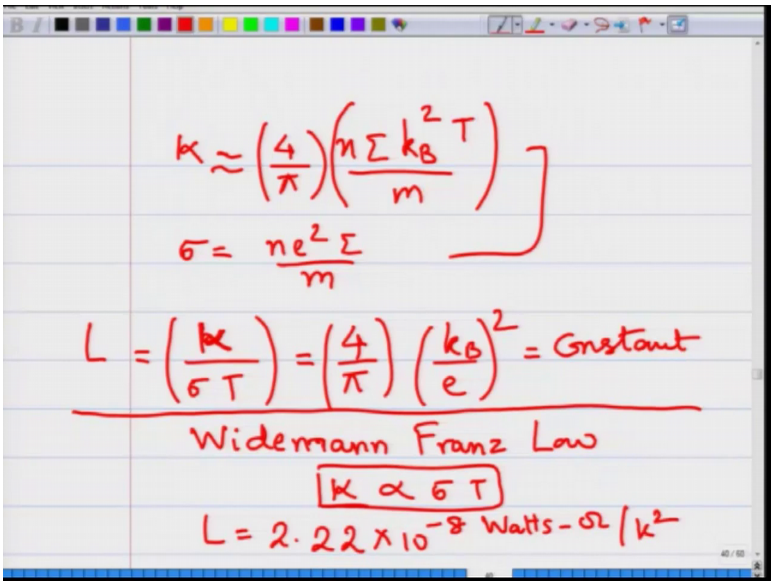

So, if you now put in this value into this expression, and you use the behavior of specific heat for this gas as 3 by 2 k B, then you will get that the thermal conductivity was shown to be equal to 4 by pi n times tau k B square T divided by m, where we have just put in from kinetic theory whatever is the velocity distribution that one can calculate from the Maxwellian distribution, mean square velocity distribution. And the specific heat which also we know how to calculate for an ideal gas of electrons. And these were the assumptions of the Drude's theory.

So, with that we get our expression for the thermal conductivity of the material. And this thermal conductivity of the material is basically related to the flow of momentum across a given cross sectional area. It measures the transport of momentum across a given cross sectional area. The conductivity measures the transport of charges across a given cross sectional area, are the two related. You have thermal conductivity which is related to the transport of energy or momentum, and you have electrical conductivity which is related to the transport of charges.

And if you recall the Drude's theory has an expression for the electrical conductivity, which is n e square tau divided by m and are the two in some way connected I mean they have to be connected because both of them basically have the electrons which are the charge carriers or the momentum carriers, both of the it is the electron which is transporting charge and it is the electron also which is transporting momentum. So, if we find out kappa divided by sigma times T, if we find out this expression, then this turns out to be a constant from these two above expressions 4 by pi k B by e the whole square, and these are nothing else but complete constants. So, this is a constant which is denoted by L.

And so what this states is that the ratio of thermal conductivity to the electrical conductivity divided by the temperature is a constant. And this goes by a very important law namely the Widemann-Franz law, which states that the thermal conductivity and the electrical conductivity are related to each other. The thermal conductivity is proportional to sigma times T. And this constant L which we have here is has a value which is about 2.22 into 10 to the power minus 8 watts ohm per Kelvin square.

So, this is an important property of the Widemann-Franz law that materials which have high electrical conductivity would also have high thermal conductivity. If you are maintaining them at the same temperature T, then materials which have reasonably high electrical conductivity would also show up with high thermal conductivity and that is sort of understandable because scattering is low. Because the electrons are experiencing weak scattering, you will find that they also not only the charge transport is much more, but the momentum transport or the energy transport which is carried by the electrons is also much better, and so the thermal conductivity becomes much more. And so you see that there is an important law which relates thermal conductivity to the electrical conductivity, which is the Widemann-Franz law.

So, let us see what happens to this Widemann-Franz law in different metals, because I had told you in few earlier lectures that the Drude's model has certain limitations. So, is it still correct, this ratio of the thermal conductivity to the electrical conductivity and temperature, is it still how does this value compared to when you actually measure it in solids?

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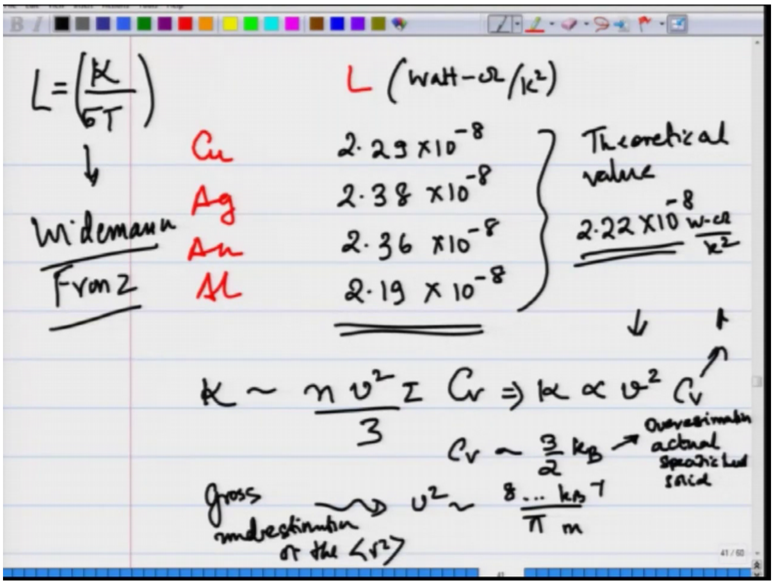

So, let us see the typical values in some materials like copper, silver, gold, aluminum. And let us look at the value of L. The value of L that you get for copper is 2.29 into 10 to the power of minus 8. It is watt, units are watt ohm per Kelvin square ok, 2.29, 2.38 into 10 to the power of minus 8, 2.36 into 10 to the power of minus 8, 2.19 into 10 to the power of minus 8. And the theoretical value which one has calculated is 2.22 into 10 to the power of minus 8 watt ohm per Kelvin square. So, you can see that there is a very close match across different materials; you can get a very close match of this constant.

The Widemann-Franz law is valid over all these different class of materials where we had found that there were some inadequacies, that if you look at the Hall effect then certain aspects of the Hall effect were not consistent across these different metals which could not be explained by Drude's theory. But here you have a quantity L and the Widemann-Franz law which seems to be valid across all these materials. And this is a surprising outcome of the Drude's theory that although there are some limitations and there are limitations serious limitations associated with the Drude's theory, it works quite well when it is trying to explain something like the Widemann-Franz law.

And the reason for why it works is actually a surprising confluence of two different types of things which happens. If you recall the thermal conductivity that we had obtained was related to the density square of the velocity tau specific heat, and there was a factor 3 which is coming. Now, the thermal conductivity from here is proportional to the velocity the mean square velocity times the specific heat. And in the kinetic theory as I said you consider that the electrons are completely free which is one of the problems of Drude's theory, and because of that we got a temperature dependent specific heat which was used which was 3 by 2 k B per electron ok, the specific heat per electron 3 by 2 k B.

So, this was assumed to be the specific heat of the electron and using the kinetic theory of gases we obtained a velocity. We are using Maxwell's distribution, we obtained a velocity which was whatever 8 by k B T divided by pi m and so on ok. So, we had obtained this velocity, the mean square velocity using Maxwell's distribution. Now, it turns out that this is an overestimation of actual specific heat of the solid. And this is a gross underestimation of the mean square velocity. So, here you have something which you have overestimated, and here you have something which you have underestimated, you have taken the product of the two and actually low and behold, it turns out to be the right value ok.

So, although there were these problems in the Drude's theory, where you had a gross underestimate of the specific heat which you have already seen earlier, and there is a gross overestimation of the specific heat, and there is an underestimation of the mean square velocity, you ultimately land up with a constant value. But be that as it may this value L which is a ratio of the specific heat to the thermal conductivity into temperature is a very important quantity and it is associated with this law which is called the Widemann-Franz law, which describes the relationship between thermal and electrical conductivity. And we get a number which is reasonable.