Introduction to Quantum Mechanics

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Lecture – 05

Black Body Radiation V- Wein's displacement law and analysis for spectral density

In this lecture we are going to derive Wien's formulas.

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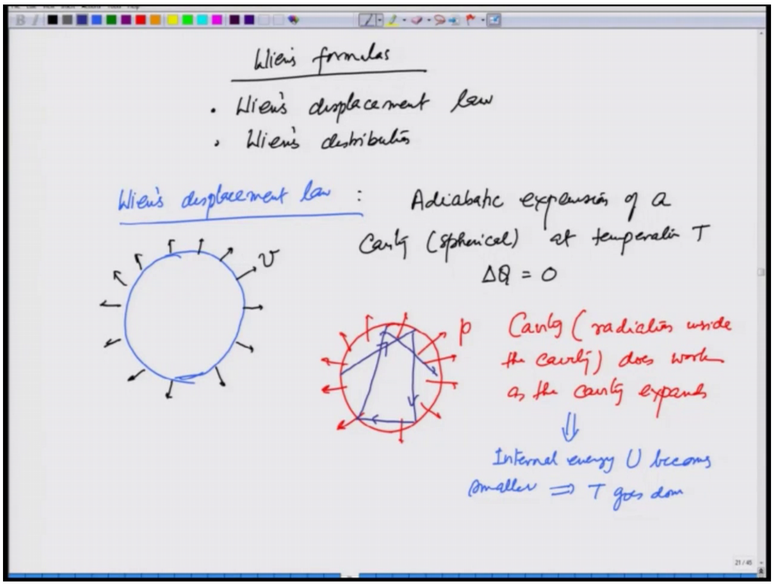

In particular we are going to look at Wien's displacement law and Wien's distribution and both are done using thermodynamics and as you must know, the thermodynamic does not really care about the details of interaction. It is a macroscopic theory and let us look at these beautiful arguments by Wien and he came very very close to obtaining the true formula for the distribution of spectral density and he was actually awarded Nobel Prize for this discovery.

So, first thing is Wien's displacement law, for this he considered a spherical cavity which; let us say this expanding or contracting by a very slow; very small velocity v, it could be either way by small velocity V and so he considered adiabatic expansion of a cavity which is spherical cavity (Refer Time: 01:59) adiabatic expansion of a spherical cavity; adiabatic means that will tuck you heat exchange was 0 at temperature T.

What we have seen earlier is that if I have a cavity like this, right, it applying a pressure which is applying pressure p, when it is allowed to expand adiabatically keeping it in equilibrium, right. So, I will be applying some force on it and therefore, it does work. So, the cavity or more appropriately radiation inside, the cavity does work as the cavity expands. So, as the cavity expands, it does work, if it does work, what should happen to its internal energy? This implies internal energy u becomes a smaller because the work is done at the cost of this energy because there is no heat being supplied and this implies the temperature goes down volume is increasing, temperature is going down.

At the same time, if I were to compress it if the radius was going down the temperature would go up because I will be pumping in energy something more happens as the radiation inside bounces around the cavity as it expands. So, you recall from a twelfth grade physics if a light or sound reflects from a moving wall its wavelength changes its frequency changes.

So, any light at lambda as it bounces s lambda is going to increase because frequency is going to go down. So, I can actually by studying this adiabatic expansion I can relate how much is the changes is lambda and how is that lambda related to T and that is what we ended and we are going to do it next.

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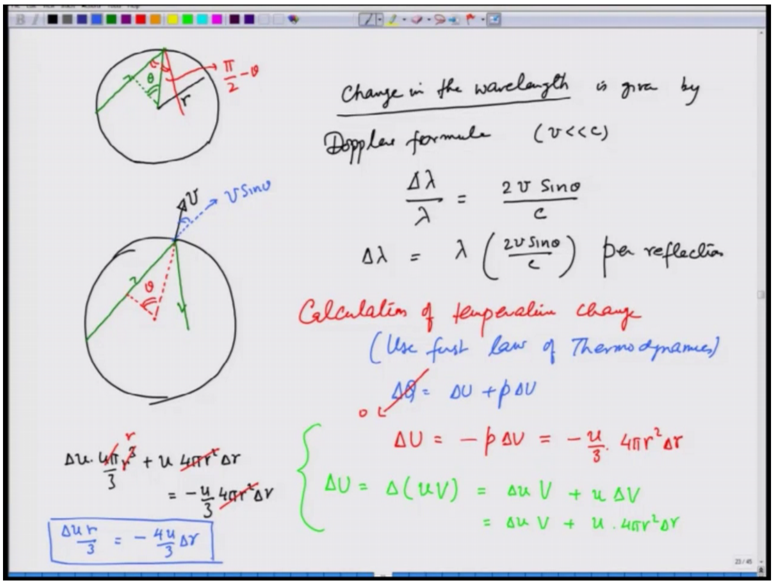

So, let us see now; how these changes occur when this cavity has radius r and light ray is going such that from the centre, if I draw a perpendicular to this the angle where it hits is theta and this slightly reflects again and this is going to be pi by 2 minus theta and so, is the (Refer Time: 05:00) let me make it here again cleanly. So, there is the light ray which is going this is a centre, this angle is theta, the light ray gets reflected like this and the cavity is expanding in this direction with velocity V.

Now change in the wavelength is given by Doppler's formula and V is much much less than c. So, delta lambda over lambda is given as 2 V the component of V is going to be taken in the direction of light which is V sine theta. So, this is going to be 2 V sin theta divided by c. So, this is going to be delta lambda is equal to lambda times 2 V sin theta over c per reflection.

So, during each reflection this is the change in the wavelength and I want to relate this 2 the temperature change; how do we calculate the temperature change? So, the calculations of temperature change. This is done using first law of thermodynamics which says that delta q is equal to delta u plus p delta V. Now I am taking an adiabatic expansion. So, this is 0. and therefore delta u is going to be equal to minus p delta V which I can write as minus u by 3 times 4 pie r square delta r where r is the radius of the cavity delta u is nothing, but delta of u times V which is going to be delta u V plus u delta V which is delta u V plus u times 4 pi r square delta r. So, let us collect this all together on the left hand side, we have delta u V is 4 pi by 3 r cubed plus u 4 pi r square delta r is equal to minus u by 3 4 pi r square delta r.

Now, I can write this 4 pi r square cancels from all through 4 pi r square 4 pi r square 4 pi r square. So, you get delta u r pi 3 is equal to minus 4 u by 3 delta r. So, this is a result which we have got which I will through take to the next slide.

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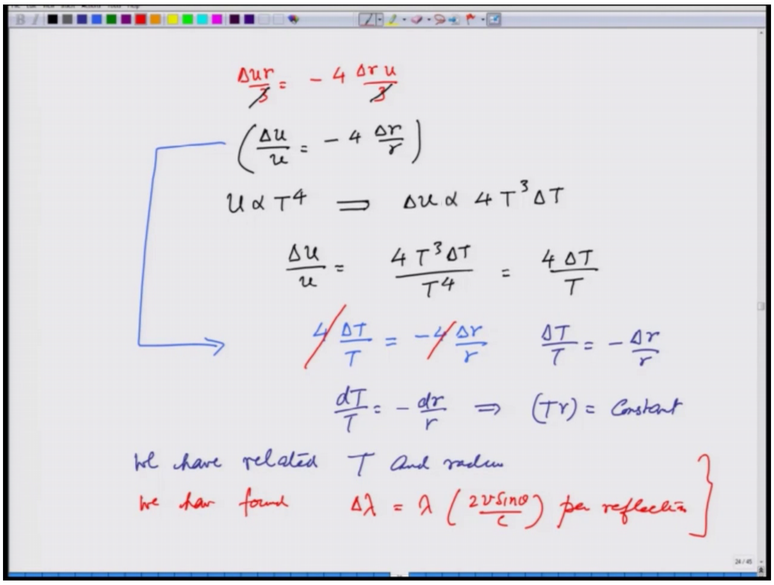

So, we have got delta u r by 3 is equal to minus 4 delta r u by 3; this 3 cancels and I get delta u over u equals minus 4 delta r by r. Now I know that u is proportional to T raise to 4 which implies delta u is proportional to 4 T raise to 3 delta T, if I substitute that I am going to get delta u over u is equal to 4 T cube delta T over T raise to 4 which is 4 delta T over T. So, from this combining all this, I get 4 delta T over T is equal to minus 4 delta r over r this 4 cancels and I get delta T over T is equal to minus delta r over r d T over T is equal to minus d r over r implies immediately that T times r is equal to a constant.

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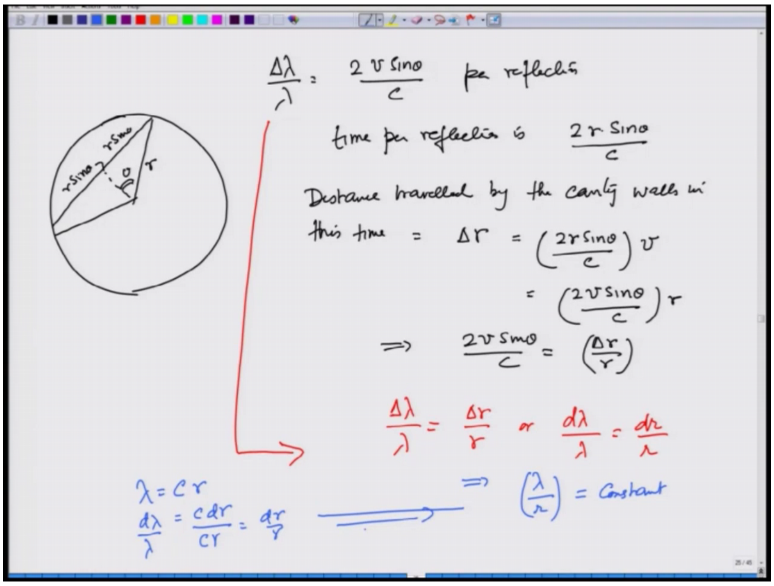

As the radius goes up, the temperature comes down. So, we have related temperature and radius and we have also found delta lambda is equal to lambda 2 V sin theta over c per reflection and we combine these 2 to relate lambda and T let us see how we do that delta lambda over lambda is 2 V sin theta over c per reflection. Now per each reflection the distance travelled by light is if the radius is r then time per reflection is 2 r sin theta because this is r this is theta. So, this is r sin theta this is r sin theta. So, 2 r sin theta divided by c time per reflection is 2 r sin theta divided by c.

So, the distance travelled by the cavity walls in this time is going to be equal to delta r which is equal to 2 r sin theta over c times V which is I can write as 2 V sin theta over c times r. This implies to V sin theta over c can be written as delta r over r. Now I substitute this in this formula and get delta lambda over lambda is equal to delta r over r or d lambda over lambda equals d r over r and this immediately gives you that lambda over r is a constant.

Let us check that lambda equals c times r d lambda is equal to c d r d lambda over lambda is equal to c d r over c r which is d r over r. So, this checks lambda over r is a constant.

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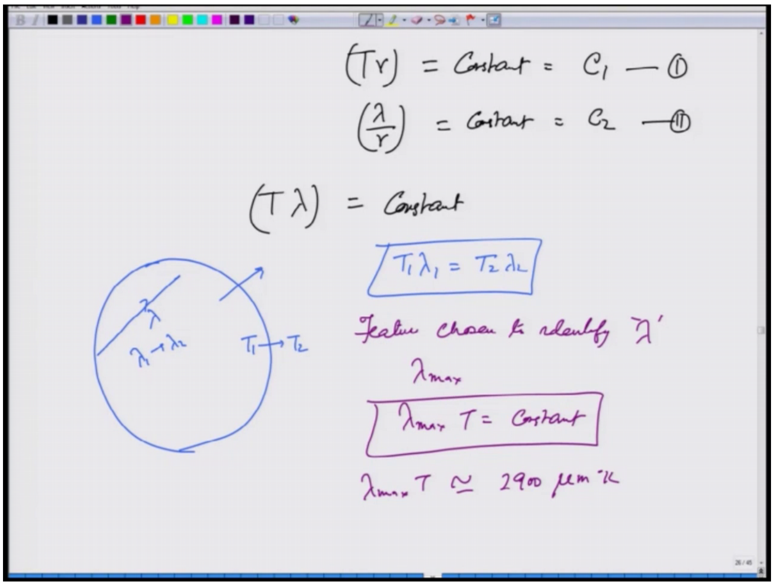

So, what have we got? We have got T times r is a constant, let us call this c 1 and we have also got just now that lambda over r is a constant which is some c 2, we can combine the 2 and multiply both equations; this equation 1 and this equation 2. This tells me that T times lambda is equal to some constant; what does this mean? This means in this cavity as it expands, the radiation remains at equilibrium. It goes from T 1 to T 2, if I look at a particular wavelength and keep following it, this lambda changes from lambda 1 to lambda 2; it will such that T 1 lambda 1 is equal to T 2 lambda 2.

This lambda has to correspond to certain feature of the radiation. So, the feature we choose; feature chosen to identify lambda is lambda max and then we write lambda max times T is equal to constant and this was confirmed by experiments carried out and they found that lambda max times T is of the order of 2900 micrometer k. So, this is confirmed and therefore, Wien's analysis was correct, this analysis was also used for the first time to estimate the temperature of distance stars, and things like those because you have to find in their spectrum where lambda max is and put that in this formula and find the temperature of that now that star.

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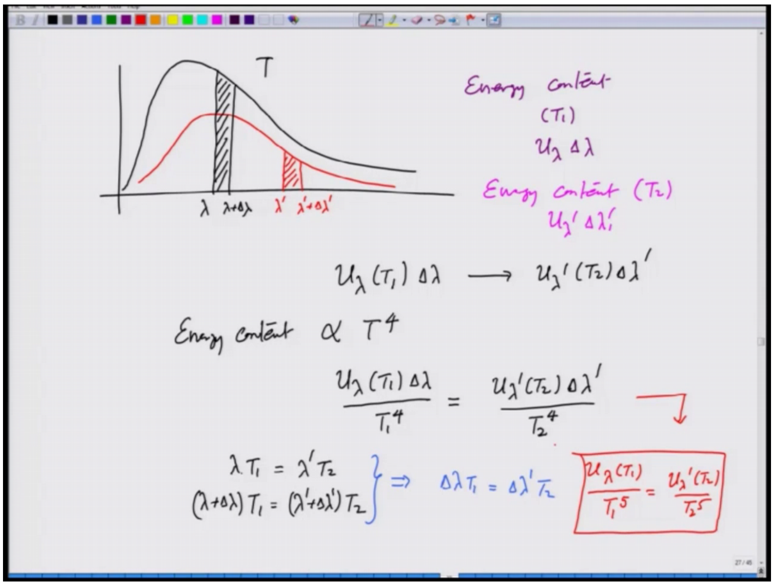

The second one is more important and for this what Wien considered is that when form a high temperature T 1, if I go to a lower temperature T 2 radiation which was contained between say lambda and lambda plus delta lambda when 2 radiation is shifted because of that displacement formula, some lambda prime; lambda prime plus delta lambda prime the energy content. In the first case, T 1 was u lambda delta lambda and this became energy content.

In the second case which is u lambda prime delta lambda prime and I should be careful this is at temperature T 2. So, what we have is that u lambda at T 1 delta lambda went to u lambda prime at T 2 delta lambda prime. However, I know what the temperature dependence is the energy content is proportional to T raise to 4, mind you, I am not going to take a ratio of energy density, but energy content. And therefore, I am going to have u lambda T 1 delta lambda over T 1 raise to 4 is equal to u lambda prime T 2 delta lambda prime over T 2 raise to 4.

That should be the ratio by a Stefan Boltzmann law; what we are assuming is the Stefan Boltzmann law holds even for energy content in small intervals, all right, but you know you get your answer and if they matches you are on the right track. So now you have therefore, that this relationship. Now notice that lambda times T 1 is equal to lambda prime times T 2 lambda plus delta lambda times T 1 is equal to lambda prime plus delta lambda prime T 2 and this immediately gives you; this together gives you delta lambda T 1 is equal to delta lambda prime T 2 and we will substitute this in this equation and I am going to get u lambda T 1 delta lambda is going to be proportional to 1 over T 1 over T 1 raise to 5 is going to be equal to u lambda prime T 2 over T 2 raise to 5.

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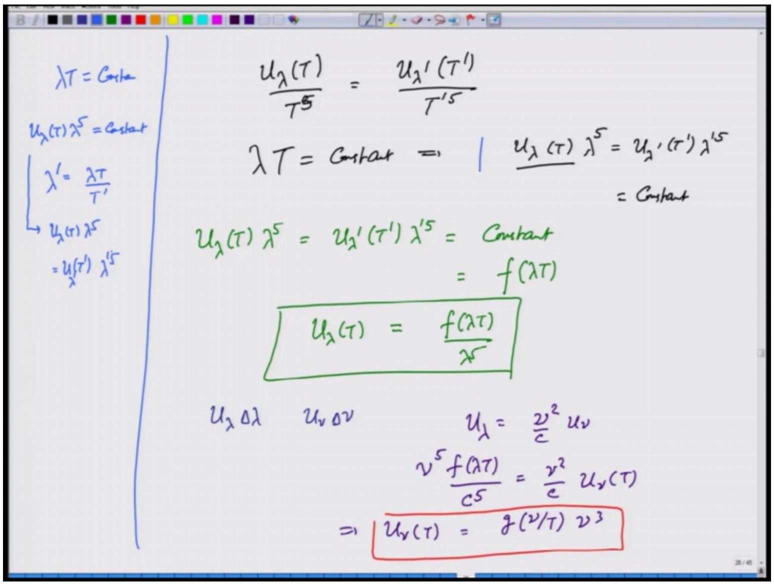

So, through the analysis of this curve and the way lambda shifts, we have found that u lambda T over T raise to 5 is same as u, let me call this primes u lambda prime T prime over T prime raise to 5. Now, I also know that lambda T remains constant. So, this implies that u lambda T over that constant raise to 5; lambda raise to 5 is equal to u lambda prime T prime lambda prime raise to 5 and this is a constant, right. So, what we have got through this analysis that u lambda T lambda raise to 5 is equal to u lambda prime T prime times lambda prime raise to 5 is a constant. What kind of constant is going to be I know; obviously, that u lambda T depends both on lambda and temperature.

So, this constant; I am going to write as some function f of lambda T because lambda T does not change as you go from one place to the other and therefore, as you increase this cavity size or if you change the temperature from one black body radiation to the other. So, this is going to be a function of lambda T in most general form, otherwise it would have been a constant throughout, but that would not give you u as a function of lambda and T both. And therefore, we put this and this immediately tells you that u lambda T is of the form some function of lambda T divided by lambda raise to 5.

This is Wien's distribution, this also tells you that since u lambda delta lambda be equated to u nu delta nu and if you recall from my previous lecture, one of the previous lectures, we have obtained that u lambda was nu square over c u nu. So, u lambda is f lambda T over lambda raise to 5 which can be written as c raise to 5 nu raise to 5 is equal to nu square over c u nu T and this implies u nu T is of the form some other function g nu over T, I have written lambda as c over lambda which is T over nu which can also write as nu over T times nu cubed.

So, these are 2 forms that have been derived by Wien that were derived by Wien, important thing that I am going to write on the side is our; what we have obtained is lambda T is a constant and u lambda T times lambda raise to 5 is a constant for each given lambda. So, if I am given the experimental curve as a particular temperature for different lambdas, I can always find the curve at some other temperature because as the temperature changes, I will find the corresponding lambda prime which will be equal to lambda T over T prime and then using this formula which says that u lambda T lambda 5 is equal to u lambda prime T prime lambda prime raise to 5, I can find u lambda prime T prime.

So, given experimental curve at one particular temperature, the experimental curve for spectral density, I can find these spectral density at any other temperature that is the utility of Wien's formula, we will analyze it further vis-a-vis, we experimental curves in the next few lectures.