Introduction to Quantum Mechanics
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Lecture - 06
Solution for the radial component of wavefunction for the hydrogen atom

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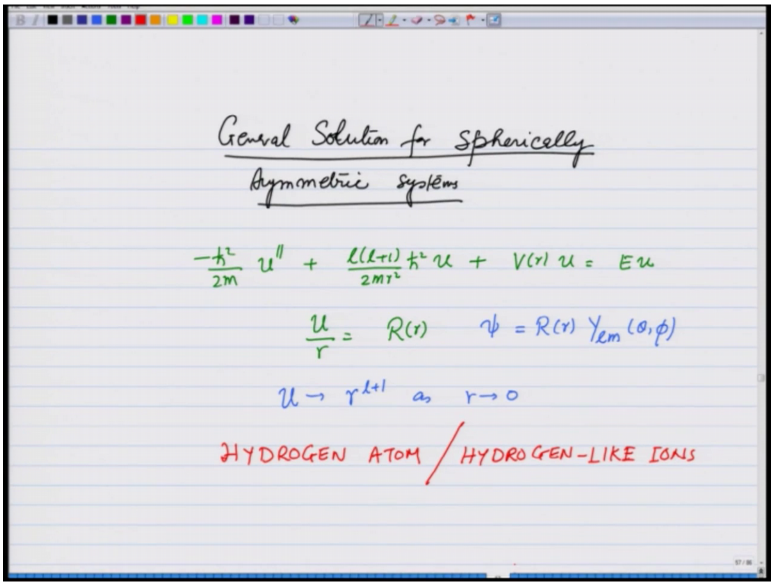

In the previous lecture we looked at the general solution for spherically symmetric systems, where we looked at the Schrodinger equation minus h cross square over 2 m u double prime plus l l plus 1 over 2 m r square h cross square u plus v r u equals E u; where u over r gives me the radial component of the wave function and the wave function psi itself is R as the function of r times Y lm theta and phi. And found that u goes to r raise to l plus 1 as r tends to 0.

In this lecture we want to focus on a specific problem of the hydrogen atom or hydrogen-like ions.

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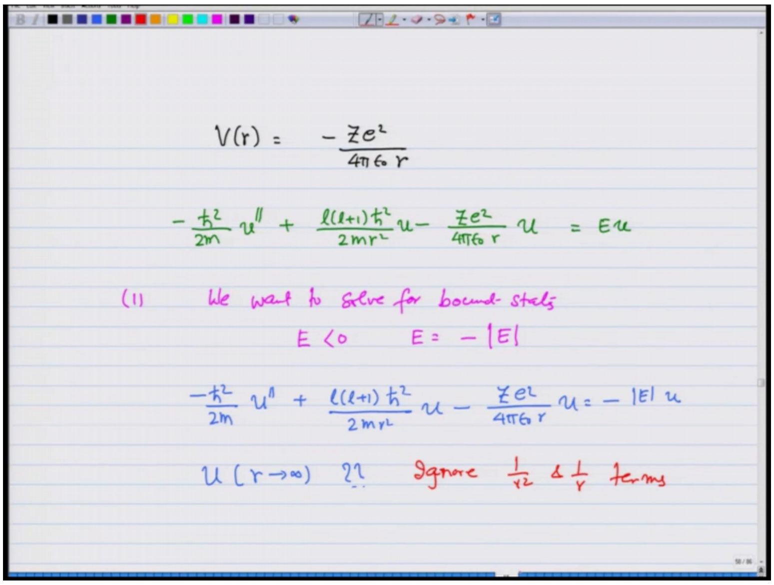

In such cases the potential energy v r of the electron is given as minus Z e square over 4 pi epsilon 0 r. And therefore, the equation for u is going to be minus h cross square over 2 m u double prime plus l l plus 1 h cross square over 2 m r square where m is the mass of the electron minus Z e square over 4 pi epsilon 0 r u equals E u; I missed a u here. And this is the equation we want to solve.

What I am going to do in this lecture is not give you very rigorous solutions, I will give solutions for one or two values of l. And enable you enough to follow this in books if you wish too, because this is the first course so I do not really give a many details here.

So, now first thing we notice is we want to solve for bound states, and therefore E is less than 0 I am going to write E as minus modulus of E. And therefore, my equation now becomes minus h cross square over 2 m u double dash plus l l plus 1 h cross square over 2 m r square u minus Z e square over 4 pi epsilon 0 r u equals minus modulus of E u. So, first thing as we did in harmonic oscillators and all that we wish to see how u behaves as r tends to infinity; behaviour of u as r tends to infinity. In such cases I can ignore 1 over r square and 1 over r terms because they will become nearly 0.

And therefore, I can write the equation as.

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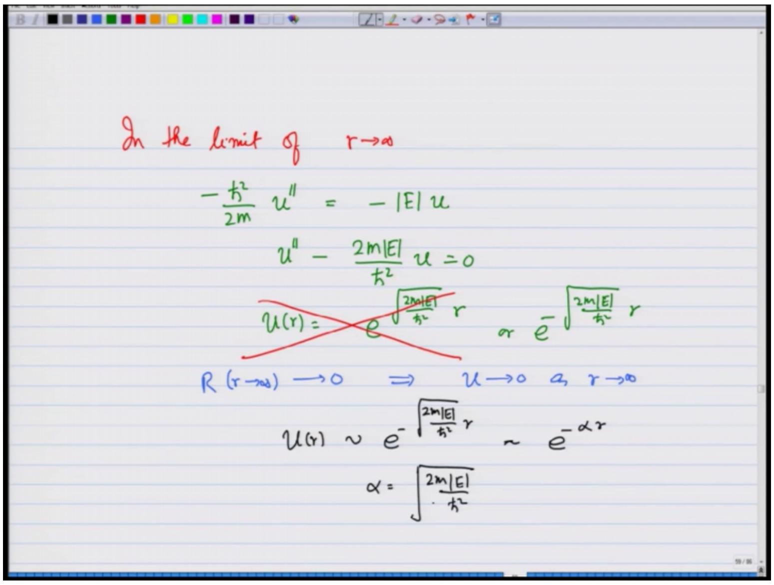

So, in the limit of r tending to infinity I can write my equation as minus h cross square over 2 m u double prime is equal to minus mod E u or u double prime minus 2 m mod E over h cross square u equals 0. This has immediate solutions u r equals either e raise to square root of 2 m mod E over h cross square r or e raise to minus square root of 2 m mod E over h cross square r.

Now, we would like the solution R r tending to infinity to go to 0, which implies u also should go to 0 as r tends to infinity. So, the first solution this one is out. So, what we have is that u r goes as e raise to minus square root of 2 m mod E over h cross square r which I will write as e raise to minus alpha r; alpha is square root of 2 m mod E over h cross square.

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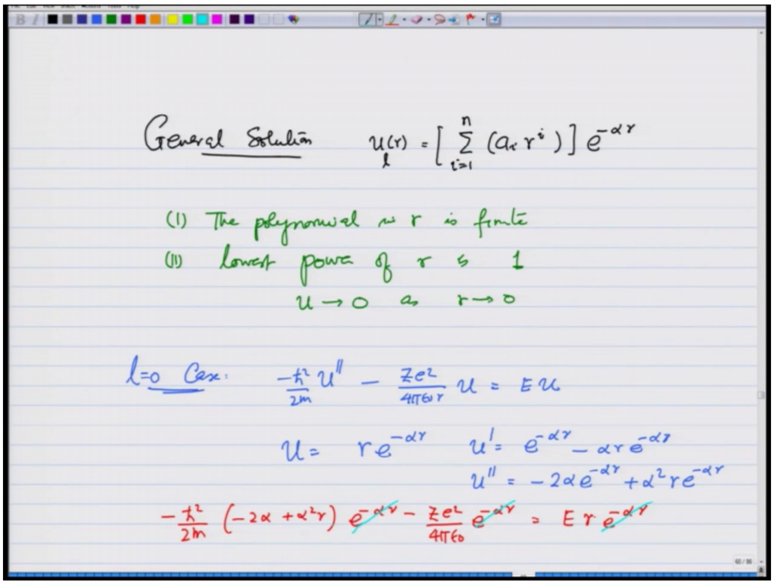

Now the general solution therefore: u r this will depend on l can be written as summation i equals 1 through whatever some polynomial n; a i r i times this is a polynomial e raise to minus alpha r. The polynomial is finite polynomial because you can show if n tends to infinity this blows up as e raise to plus alpha r and you do not want that to happen, because solution would not be going to 0 as r tends to infinity. So, you wanted to be a finite polynomial.

So, number one notice that the polynomial n r is finite. And number two that the lowest power of r is 1. Why is that because you want u to be go into 0 as r tends to 0. So, these are two things that we keep in mind. And now what I will do is instead of solving for this polynomial in general I will build up solution for certain cases.

So, let me consider l equals 0 case first. In that case my equation is u double prime minus h cross square over 2 m, minus Z e square over 4 pi epsilon 0 r u equals E u. I am going to consider u equals r e raise to minus alpha r which means that u prime is e raise to minus alpha r minus alpha r e raise to minus alpha r, and therefore u double prime is minus 2 alpha e raise to minus alpha r plus alpha square r e raise to minus alpha r.

Let us substitute this in the equation I get minus h cross square over 2 m minus 2 alpha plus alpha square r e raise to minus alpha r minus Z e square over 4 pi epsilon 0 e raise to minus alpha r equals E r e raise to minus alpha r; e raise to minus alpha r term cancels throughout.

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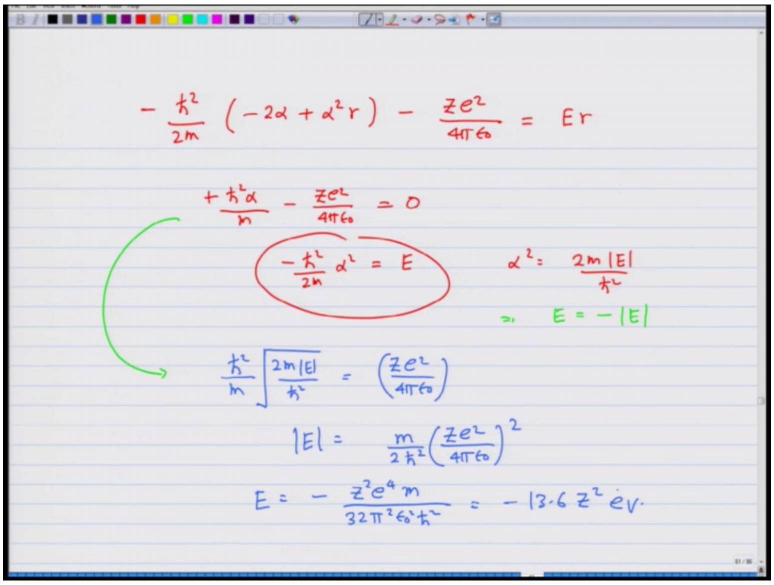

So, I am left with minus h cross square over 2 m minus 2 alpha plus alpha square r minus Z e square over 4 pi epsilon 0 is equal to E times r. Now terms or the same power should be the same and this immediately tells me that minus h cross square alpha over m that should be plus, minus Z e square over 4 pi epsilon 0 is 0. And minus h cross square over 2 m alpha square is equal to e.

This is automatically satisfied because alpha square is 2 m mod E over h cross square and this therefore gives me that E is equal to minus mod E. This is something we have been working with, but this equation gives me the energy because this tells me that h cross square over m square root of 2 m mod E over h cross square is equal to Z e square over 4 pi epsilon 0. And therefore, mod E comes out to be Z e square over 4 pi epsilon 0 square times m over 2 h cross square. And therefore, the energy for this state is minus Z square e raise to 4 m over 32 pi square epsilon 0 square h cross square which comes out to be minus 13.6 Z square electron volts.

So, we found that this is the ground state energy.

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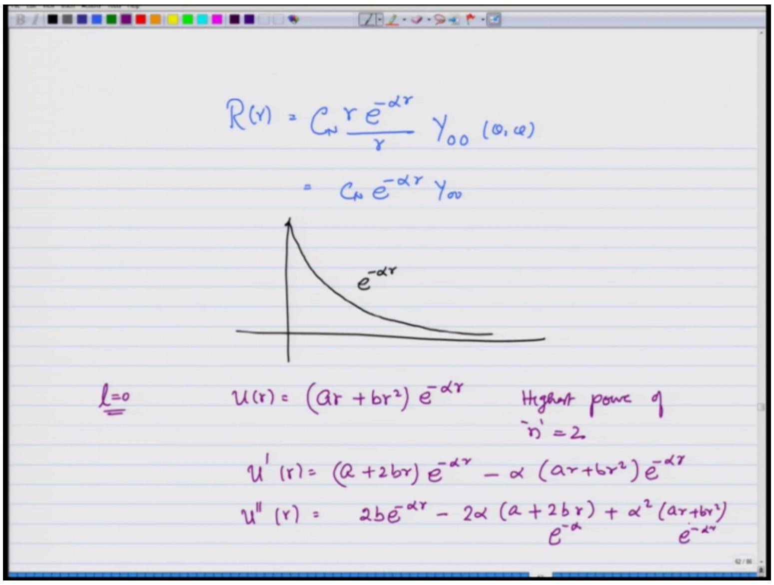

And the wave function R r for the ground state energy is going to be r e raise to minus alpha r divided by r times y 0 0 theta and phi plus some normalization constant which can be calculated. So, this state is e raise to minus alpha r Y 0 0 which is also a constant C n.

So, the ground state looks like finite at r equals 0 and this decays exponentially it is e raise to minus alpha r times a constant y 0 0. Still working on l equals 0; let us find out the solution which is a higher polynomial. So, I am going to take u r to be ar plus br square e raise to minus alpha r. So, this highest power of n in this is 2.

You again calculate u prime r which is a plus 2 br e raise to minus alpha r minus alpha ar plus br square e raise to minus alpha r, and u double prime r is equal to 2 b e raise to minus alpha r minus 2 alpha a plus 2 br plus alpha square ar plus br square; this is e raise to minus alpha r e raise to minus alpha r here. We substitute this n. So, next step is going to be.

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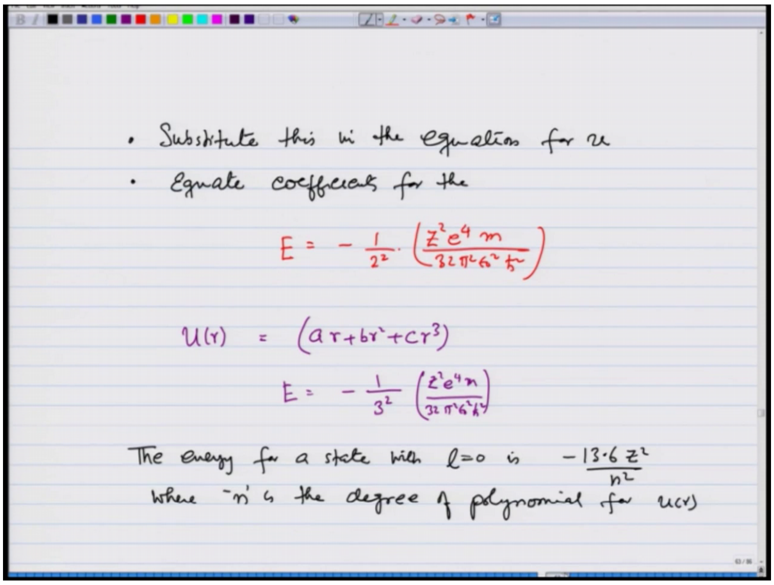

Substitute this in the equation for u, then equate coefficients for the equal powers of r. To do that and you are going to find that E comes out to be in this case as 1 over 2 square Z square e raise to 4 m over 32 pi square epsilon 0 square h cross square. You go further if I take u r to be as ar plus br square plus cr cubed and substitute in the equation you would find that E comes out to be minus 1 over 3 square Z square e raise to 4 n over 32 pi square epsilon 0 square h cross square.

So, if you do this exercise what you will conclude is that the energy for a state with l equal to 0 is minus 13.6 Z square over n square where, n is the degree of polynomial for u r.

So, I will leave that for you. You can go to higher and higher powers. As an illustration let us also take a case of l equals 1.

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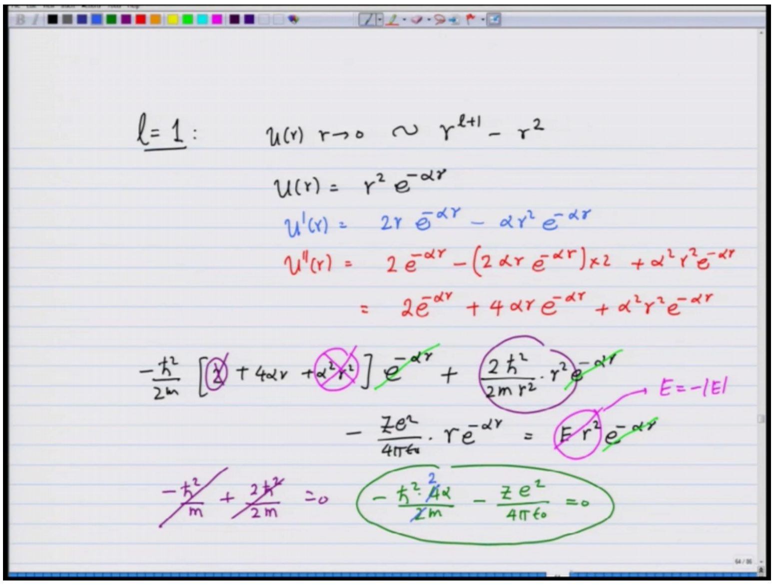

Now, remember for l equals 1 u r as r tends to 0 goes as r raise to l plus 1 which is r square. So, the minimum power of r that you take is r square. In this case I am going to take u r to be r square e raise to minus alpha r. Therefore, u prime r is going to be 2 r e raise to minus alpha r minus alpha r square e raise to minus alpha r and u double prime r is going to be equal to 2 e raise to minus alpha r minus 2 alpha r e raise to minus alpha r times 2 plus alpha square r square e raise to minus alpha r. So, this comes out to be 2 e raise to minus alpha r plus 4 alpha r e raise to minus alpha r plus alpha square r square e raise to minus alpha r.

Let us substitute this in the equation. So, I get minus h cross square over 2 m 2 plus 4 alpha r plus alpha square r square e raise to minus alpha r plus now l is not 0 so I am going to get 2 h cross square over 2 m r square and I have r square e raise to minus alpha r minus Z e square over 4 pi epsilon 0; one r will cancel I get r e raise to minus alpha r is equal to E r square e raise to minus alpha r. E raise to minus alpha r cancels throughout. And when I equate the terms of equal powers of r I get minus h cross square over m plus 2 h cross square over 2 m and this term actually cancels is 0 which is cancelling. So, let us do that, this term cancels with the first term this is 0.

Equate the terms with power r and I get minus h cross square times 4 alpha over 2 m minus Z e square over 4 pi epsilon 0 equals 0 and this gives me the energy. And the last term E r square term would actually cancel with this alpha square term that basically it give me E equals minus mod E which we already know.

So, the energy is determined by this term in green, so this cancels again this gives me 2.

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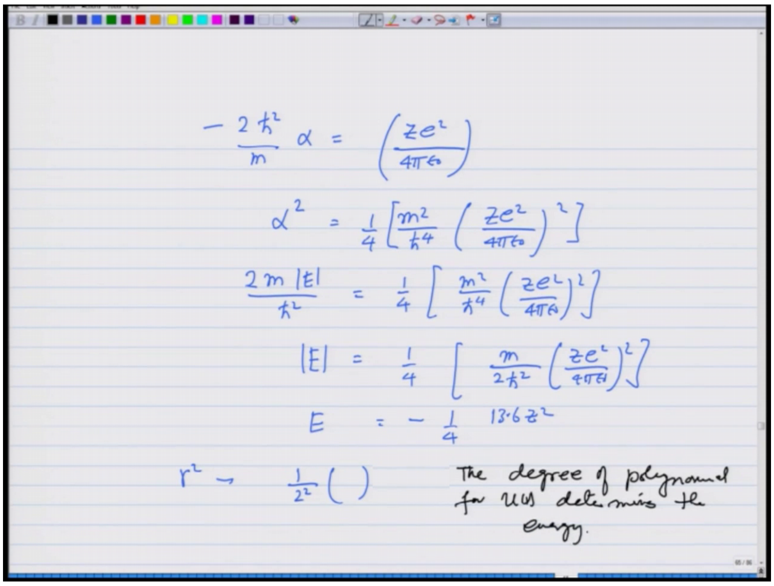

So, I get minus 2 h cross square over m alpha equals Z e square over 4 pi epsilon 0. So, alpha square is equal to m square over h cross raise to 4 Z e square over 4 pi epsilon 0 square and 1 over 4 here. So, alpha square is 1 over 4 m square over h cross raise to 4 Z e square over 4 pi epsilon 0 square. And alpha square is nothing but 2 m mod E over h cross square which is equal to one-fourth m square over h cross raise to 4 Z e square over 4 pi epsilon 0 square.

And you see this again gives me after simplifying modulus of E equals one-fourth of m over 2 h cross square Z e square over 4 pi epsilon 0 square, which is nothing but E equals minus one-fourth 13.6 Z square. So, again we see that the power r square leaves to energy 1 over 2 square times at minus 13.6 Z square.

So, what we conclude in this through this and you can keep trying it for other else that the degree of polynomial for u r determines the energy.

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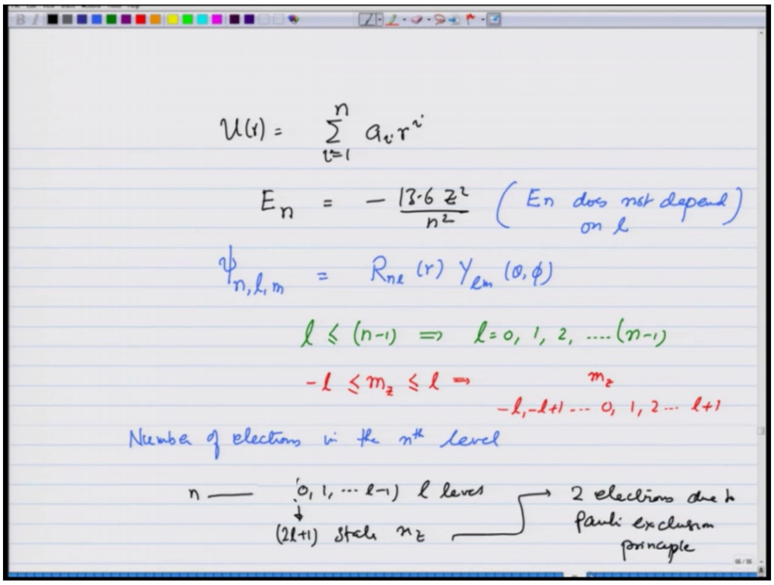

Then what does it give me? That if u r is summation i equals 1 through n a i r raise to i then the energy comes out to be it depends on n minus 13.6 Z square over n square. Keep in mind that E does not depend on l. So, this is quite a coincidence for hydrogen atom E n does not depend on l. But r does, so the wave function is written as n l and m and this is R nl r Y ln theta and phi. So, r depends on l, but the energy does not. And what is also found in this solution when you do it that l is restricted to below n minus 1. So, this means l can be 0, 1, 2, and so on up to n minus 1.

Other thing is that m; let me write it m Z is restricted to be below between minus l n plus l implies m Z as from minus l minus l plus 1 so on 0 1 2 up to l plus 1. So, the number of electrons in the nth level will be. So, for each n I have energy levels with the same energy because it depends only on n 0 1 up to l minus 1; l levels. And for each I have 2 l plus 1 states with m Z, and for each there can be two electrons due to Pauli exclusion principle.

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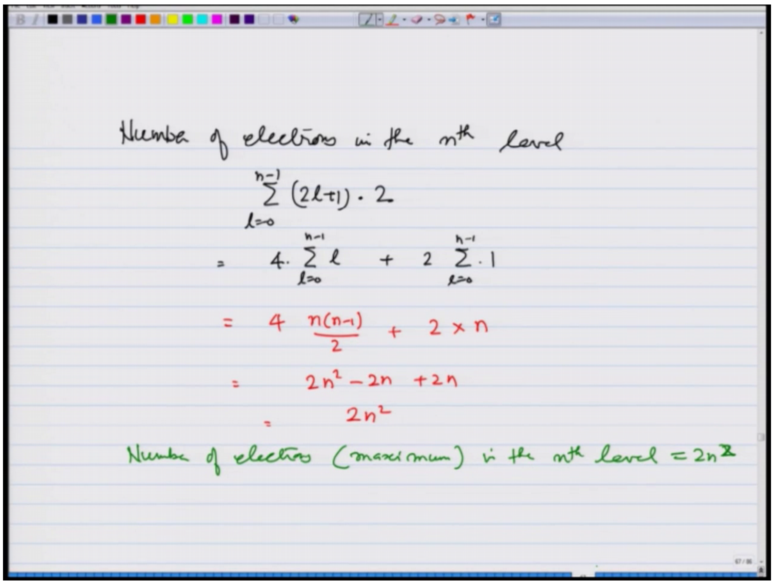

So, what we get is number of electrons in the nth level is going to be summation 2 l plus 1 times 2 l equal to 0 to n minus 1. So, this gives me 4 summation l equals 0 to n minus 1 l plus 2 summation l equals 0 to n minus 1 minus 1. So, this comes out to be 4 n n minus 1 divided by 2 plus 2 and there are n terms. So, this is going to be 2 n. So, this comes out to be 2 n square minus 2 n plus 2 n which is 2 n square. So, number of electrons maximum in the nth level is equal to 2 n square.

So, what I have done in this lecture is given to you an indication has to how hydrogen atom problem is solved.

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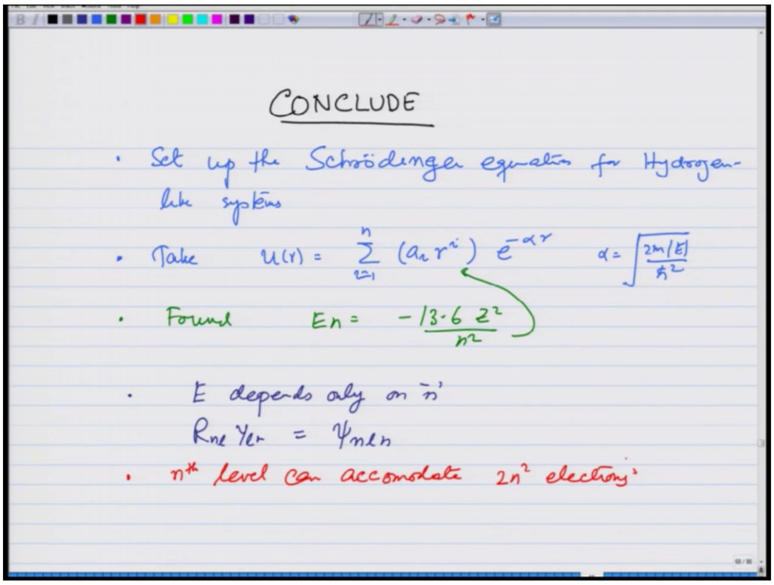

So, to conclude this lecture what we have done is: one, set up the Schrodinger equation for hydrogen like systems. Two, taken u r to be equal to summation i equals 1 through n a i r i e raise to minus alpha r where alpha is square root of 2 m mod E over h cross square. Then found E n to be equal to minus 13.6 Z square over n square where n reverse to this power. And finally, E depends only on n R nl Y ln is the wave function psi n l n. And n-th level can accommodate 2 n square electrons.

With this we stop the discussion on hydrogen atom, and next week we will begin with numerical solution of this radial equation.