Introduction to Quantum Mechanics

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Lecture - 05
Equation for the radial component of wavefunction for spherically symmetric potentials and general properties of its solution

We been looking at the Schrodinger equation for spherically symmetric systems.

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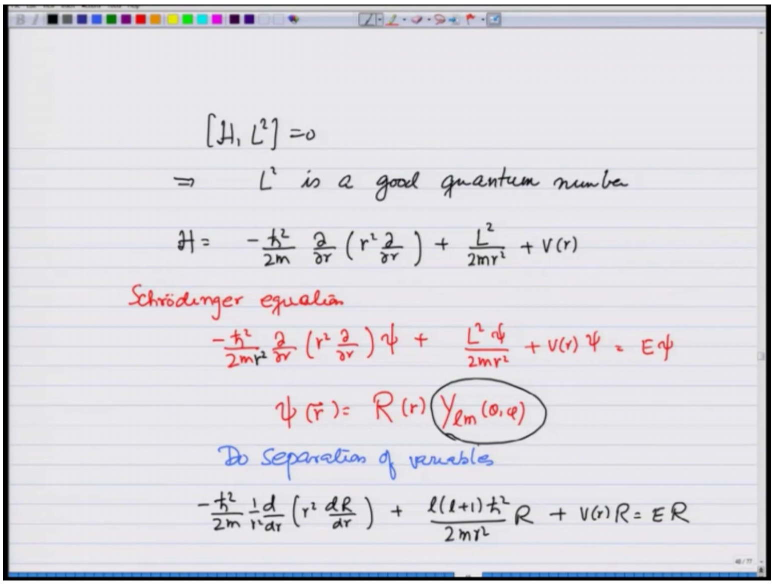

And we have shown that H and L square is equal to 0 which implies that L square is a good quantum number. And the Hamiltonian is shown is minus h cross square over 2 m d by dr r square d by dr plus L square operator over 2 m r square plus V r. And therefore, the Schrodinger equation is minus h cross square over 2 m d by dr r square d by dr times psi plus L square operating on psi 2 m r square plus v r psi equals E psi.

I could tell you that psi r vector can be written as R which depends only on r and some other function which I have given as Y lm theta and phi and then write this and do separation of variables. And what you would find is that equation becomes minus h cross square over 2 m there is r square here: 1 over r square d by dr r square d R by dr plus l, l plus 1 h cross square over 2 m r square R plus V r R equals E R and that is because Y lm's give you this l l plus 1 term. Let me do this explicitly and show it to you.

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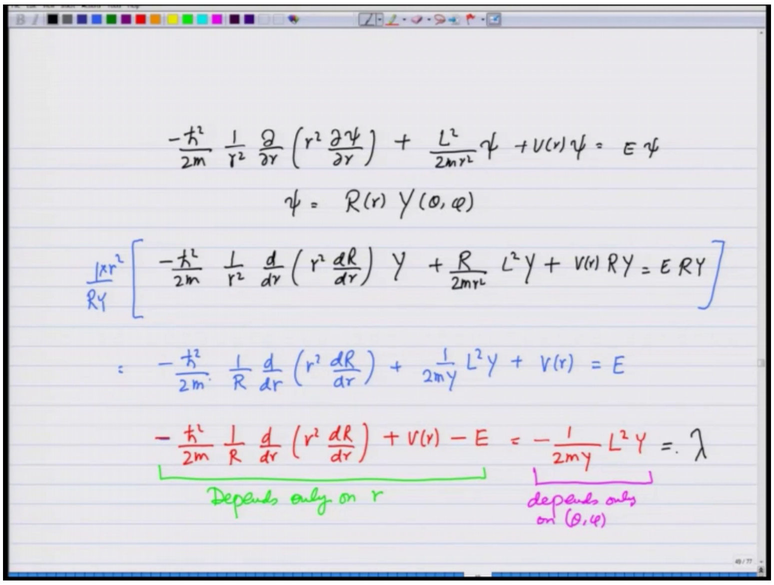

So, when I have minus h cross square over 2 m, 1 over r square d by dr of r square d psi over dr plus L square over 2 m r square psi plus v r psi equals e psi. And write psi equals some function R of r other function which I am writing y theta and phi; then what I get is minus h cross square over 2 m 1 over r square the full derivative of r square d R by dr times Y plus L acts only on those functions which is a functions of theta and phi because involves no derivative respect to R. So, I get R over 2 m r square L square on Y plus V r R Y equals E R Y.

And then you divide this whole thing by 1 over R Y and multiply by r square to get minus h cross square over 2 m, 1 over R d over dr r square d R over dr plus 1 over 2 m Y L square Y plus V r equals E. And then I change sides of certain terms and write this as minus h cross square over 2 m, 1 over R d by dr of r square d R over dr plus V r minus E is equal to minus 1 over 2 m Y, L square Y.

And it is so happens that this term the term on the left hand side depends only on r; term on the right hand side here depends only on theta and phi. And for them to be equal therefore this must be equal to sum constant lambda.

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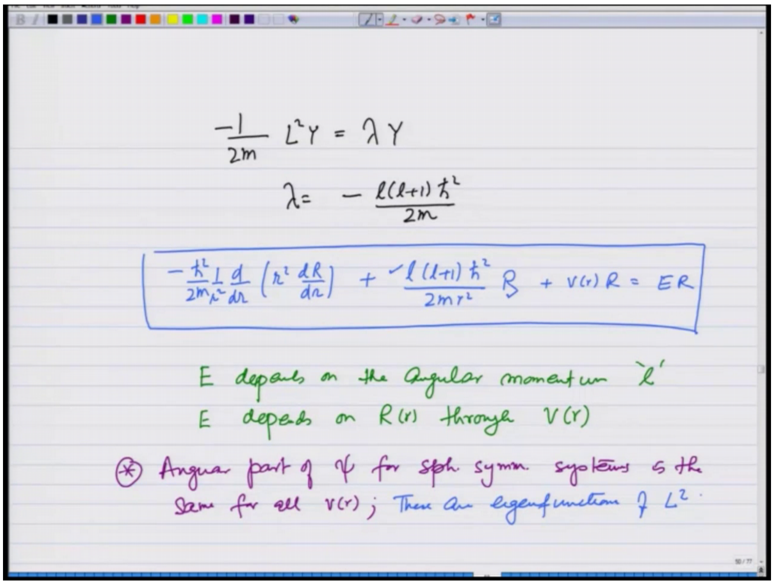

And therefore, what you find is that minus 1 over 2 m, L square Y is equal to lambda Y. Immediately you know that this is the Eigen function Y is the Eigen function for L square, and therefore lambda should be equal to nothing but minus l, l plus 1 h cross square over 2 m in this case because there is an extra 2 m sitting her. And you bring it back to the r equation and the equation you end up getting is minus h cross square over 2 m, d by dr of r square d R by dr and there will be 1 over r square in front plus l l plus 1 h cross square over 2 m r square R plus v r R equals E R.

So, the energy E depends on L and radial function r only. So, energy E depends on the angular momentum as is expected higher than low, but more the kinetic energy and E depends on R r through potential v r. So, you notice that the angular part of the wave function for a spherically symmetric potentials is the same. So, angular part of psi for spherically symmetric systems is the same for all v r. And what is it these are eigen functions of L square operator. It is only with r component that the person that depends on v r.

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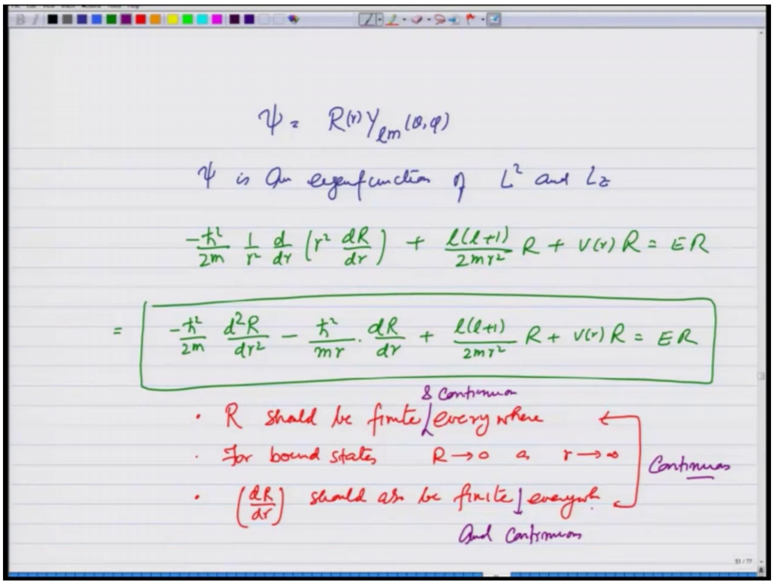

So, my psi is nothing but R Y and I can write this Y lm theta and phi. So, psi is also now psi is and Eigen function of L square and L z as is expected, because these are conserved quantities and in addition I get this r. And equation for r is minus h cross square over 2 m 1 over r square d by dr r square d R by dr plus L l plus 1 over 2 m r square R plus v r R equals E R. And this can be written further as minus h cross square over 2 m d 2 R over dr square minus h cross square over m r d R over dr plus L l plus 1 over 2 m r square R plus v r R equals E R when we expand this

Now, this equation involves both the second derivative and first derivative r; we would like to simplify it but before that let us say that R should be finite everywhere. Because psi should be finite, if I whatever solution I find r should be finite everywhere including r equals 0 and r going far away. For bound state R should go to 0 as R goes to infinity because for bound states part we cannot escape to infinity.

So, now we got we are going to do is simplify this equation by writing r in a certain term or before that I should also mention that d R over dr should also be finite everywhere and the finite. And with this also there is one more condition this should be continuous. So, finite and continuous; should also a finite and continuous everywhere.

Let us now simplify this equation.

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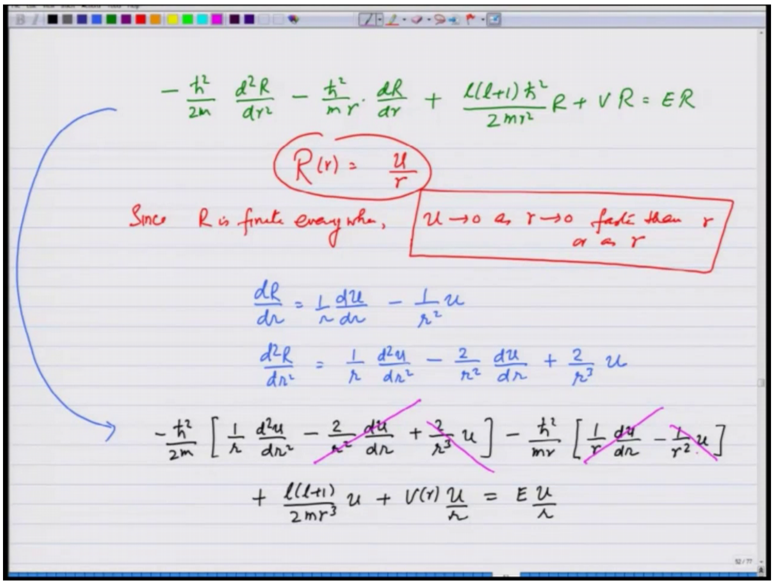

So, the equation I have is minus h cross square over 2 m d 2 R over dr square minus h cross square over m r d R over dr plus l l plus 1 h cross square over 2 m r square R plus v R equals E R. Let us write R r equals u over r. And sense R is finite everywhere u should go to 0 as r tends to 0 faster than and maybe equal to also. So, you should go to 0 as r tends to 0 faster than r or as r then only r would remain finite as r goes to 0. But what is does the; so in addition to this condition that I have on u which I am going to apply what does the substitution does is simplifies things for me. Because now d R over dr is going to be du over dr 1 over R plus minus 1 over r square u and d 2 R over dr square is going to be equal to 1 over r d 2 u over dr square minus 2 over r square du over dr plus 2 over r cubed u.

Let us now substitute this in the Schrodinger equation here and get minus h cross square over 2 m substitute for this 1 over r d 2 u over dr square minus 2 over r square du over dr plus 2 over r cubed u minus h cross square over m r in the bracket I get 1 over r du over dr minus 1 over r square u. Then I get plus l l plus 1 over 2 m r cubed u plus v r u is equal to E u over r.

Now let us cancel certain terms. The term with du by dr cancels with du by dr here. The term with u over r cubed cancels with this term here.

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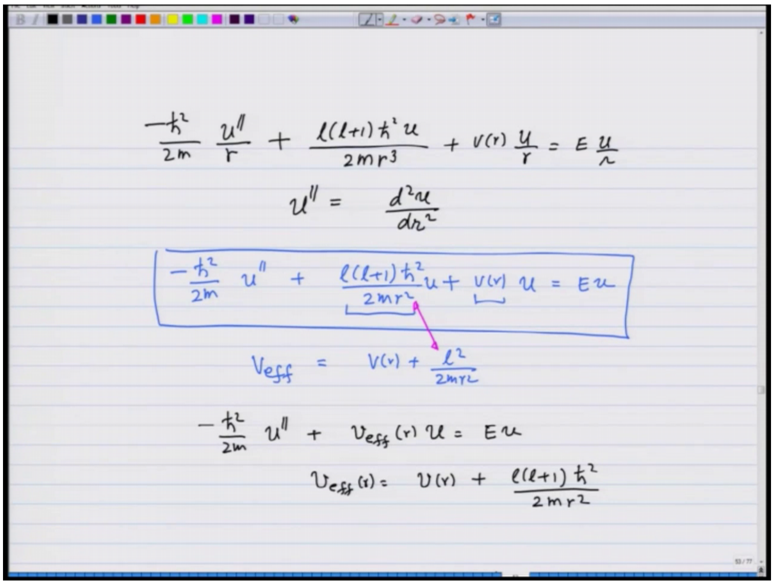

So, all I am left with is minus h cross square over 2 m u double prime over r plus l l plus 1 h cross square u over 2 m r cubed plus v r u over r equals E u r. Where you double prime is nothing but d 2 u over dr square. Multiplied by r throughout in the equation becomes minus h cross square over 2 m u double prime plus l l plus 1 h cross square u over 2 m r cubed plus v r u over r equals E u over r where u double prime is nothing but d 2 u over dr square multiply by r throughout, and the equation becomes minus h cross square over 2 m u double prime plus l l plus 1 h cross square over 2 m r square plus v r u equals E u

This is like the Schrodinger equation for one dimensional motion except that this v r also has an additional term here or this is a u. Here recall from classical mechanics that in motion in central forces there is v effective which was v R plus L square over 2 m r square. So, this term here is very similar to that you get a v effective which is L square over 2 m r square except in quantum mechanics I get l l plus 1. So, the equation is nothing but minus h cross square over 2 m u double prime plus v effective r u equals E u exactly like want equation where v effective r s v r plus l l plus 1 h cross square over 2 m r square.

Now for potentials that r regular near r equals 0 some general properties about r can be inform from this equation.

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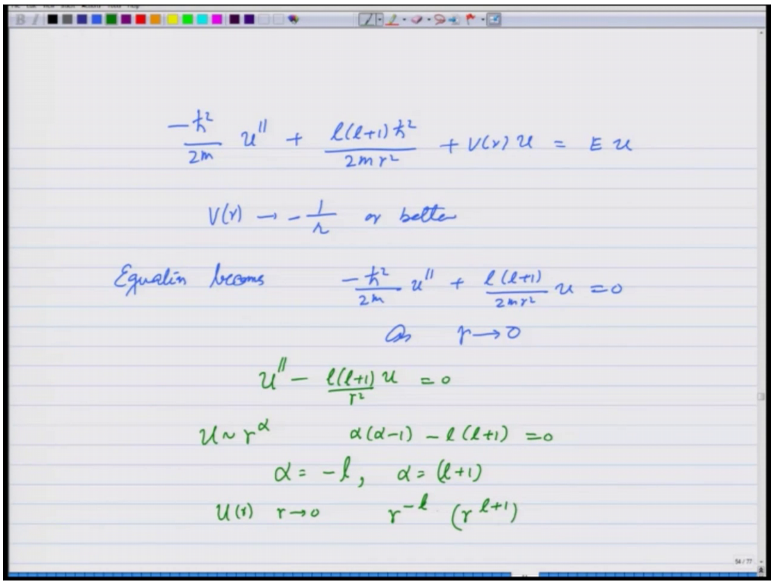

So, the equation that I have is minus h cross square over 2 m u double prime plus l l plus 1 h cross square over 2 m r square plus v r u equals E u. So, v r that goes as maybe minus 1 over r or better; better I mean it does not blow up faster than 1 over r. The equation becomes minus h cross square over 2 m u double prime plus l l plus 1 over 2 m r square u is equal to 0 as r tends to 0, because I can neglect E u I can neglect v r and therefore this is equation that becomes. And if you solve this u basically if you rearrange terms you get u double prime minus l l plus 1 u over r square is equal to 0.

If I take u to be of the order of r raise to alpha you get alpha alpha minus 1 minus l l plus 1 is equal to 0. So, alpha could be either equal to minus L or alpha could be equal to L plus 1. So, the solution u r as r tends to 0 goes as either r raised to minus L or r raise to L plus 1.

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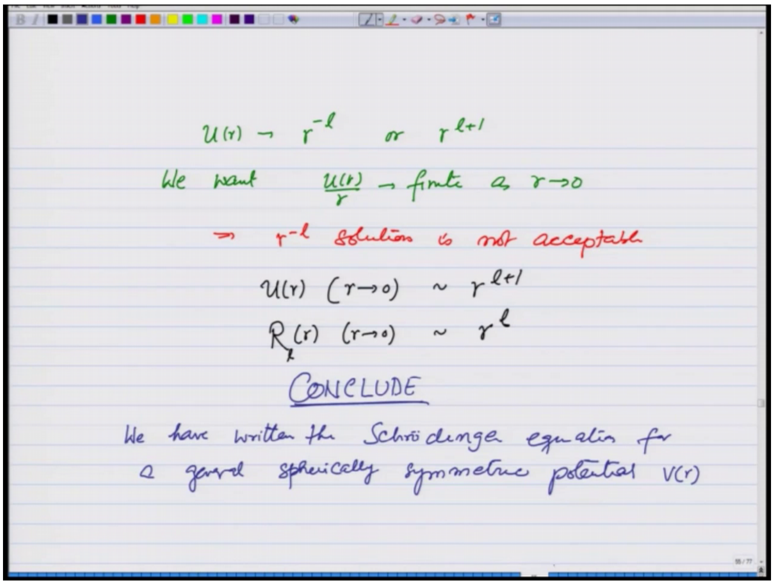

So, what we found is that u r causes either r raise to minus L or r raise to L plus 1. We want u r over r to be finite as r tends to 0 because r should remain finite, and therefore r raise to minus L solution is not acceptable. So, u r as r tends to 0 goes as r raise to L plus 1. The solution of the Schrodinger equation as r tends to 0 goes as r raise to l.

So, you see that r depends on L. Now through this we find out. So, that is one property we can see for u from here. Now other properties as you go far away r tend into infinity and all that those will depend on what v r is like. But this is general thing that you can find out from a u equation. And what will be solving now is the u equation because that is easier to solve

So, let me just now conclude this lecture by saying that we have return the Schrodinger equation for a general spherically symmetric potential v r.

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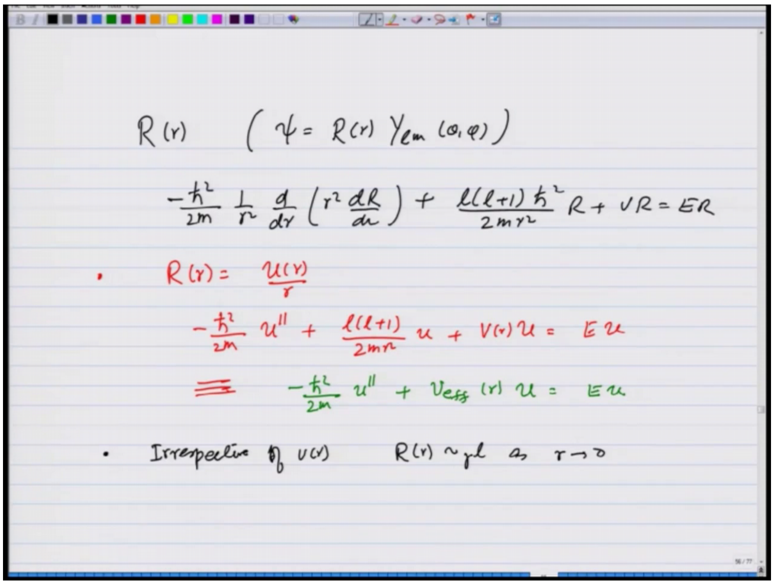

And then by writing the radial component where psi is given as the radial component times Y lm theta and phi we found that, equation for r s 2 m 1 over r square d by dr r square d R by dr plus l l plus 1 h cross square over 2 m r square R plus v R equals E R. Then by writing R r as u r over r we found the equation for u which is like the one dimensional equation u double prime plus L l plus 1 over to m r square u plus v r u equals E u; which is equivalent to minus h cross square over 2 m u double prime plus v effective r u equals E u. And, we finally found for irrespective of v r; as long as regular does not blow faster than 1 over r as r goes to 0 R r goes as r raised to L as r tends to 0.

In the next lecture we are going to focus on a particular system and that will be the hydrogen atom.