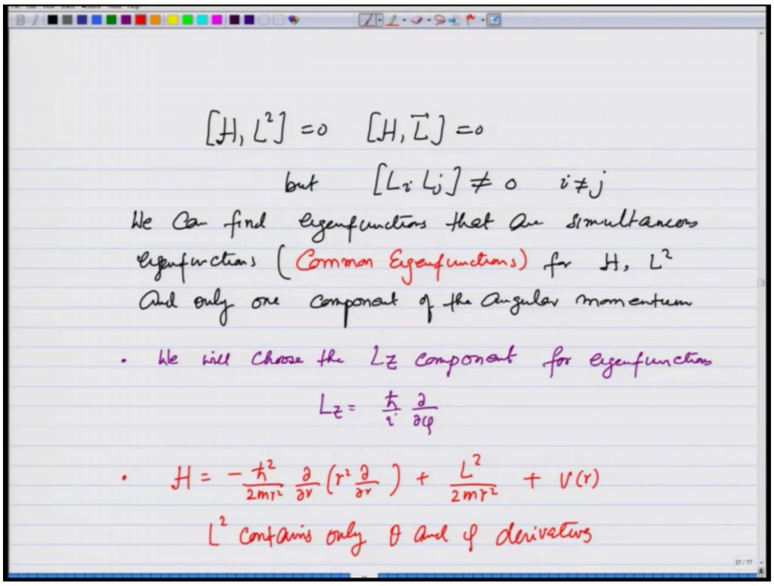
Introduction to Quantum Mechanics
Prof. Manoj Kumar Harbola
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 04
Angular momentum operator and its Eigenfunctions

(Refer Slide Time: 00:31)

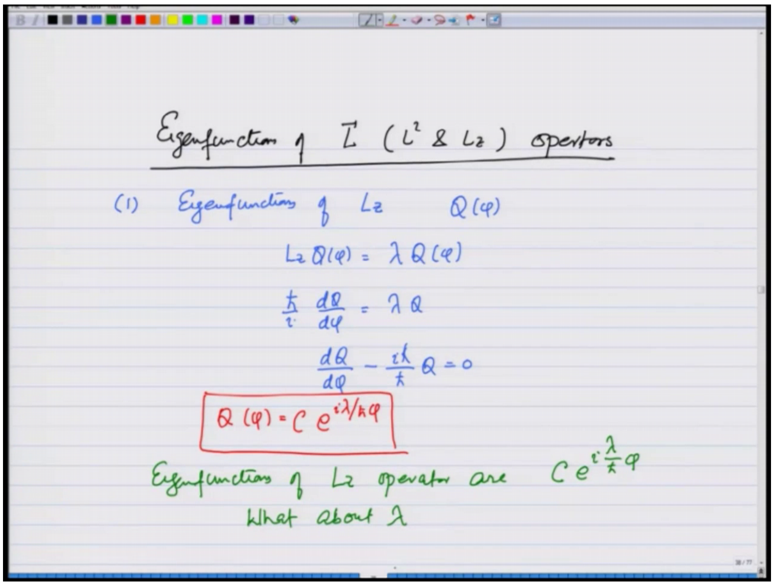


In the previous lecture, we set up the Schrodinger equation for this spherically symmetric systems and then argued and showed that the angular momentum is conserved for such systems and therefore, we can find them because of H L square being 0 H L being 0, but L i L j not being equal to 0 for i naught equal to j, we can find the Eigen functions that are simultaneous Eigen functions or what I will say is common Eigen functions for H L square and only one component of the angular momentum and what we will do is because of the simplicity we will choose the L z component for Eigen functions.

Why is so, because L z has a very simple form h cross over i d by d phi and we also found that the Hamiltonian can be written as minus h cross square over 2 m r square partial r r square partial with respect to r plus L square over 2 m r square plus V r and L square contains only theta and phi L square contains only theta and phi derivatives and therefore, one can anticipate that the Eigenfunctions of H which are simultaneous

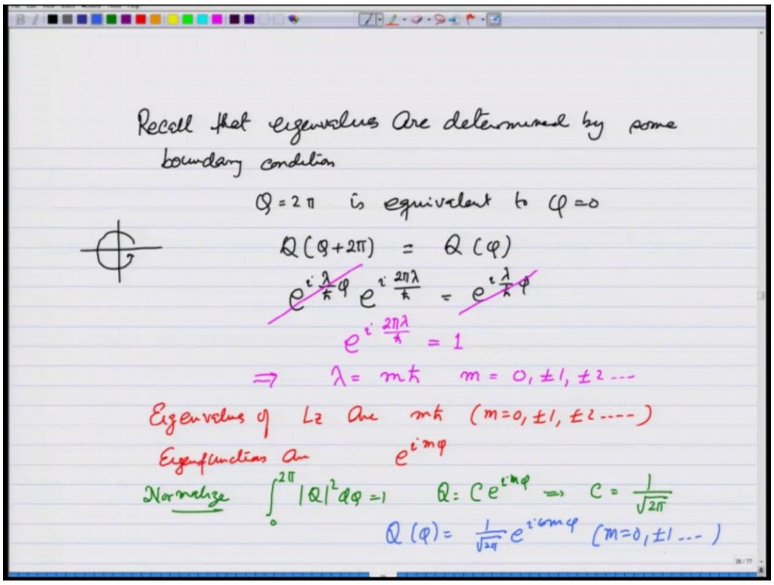
Eigenfunctions of L square are going to be such that the Eigen function L square are basically irrespective of what V r is because this V r out here in the in the Hamiltonian does not really contain any theta and phi terms.

(Refer Slide Time: 03:39)



So, therefore, L square is going to be independent of that L square Eigenfunction should be independent of that. So, what we are going to focus on in this lecture are the Eigenfunctions of L and therefore, L square and L z operators. So, let us do that first the simplest thing one Eigenfunctions of L z. So, these are going to be suppose these are let me write this as Q phi then I am going to have L z Q phi equals some Eigen value let us write it some Eigen value lambda Q phi. This is H cross over i d Q over d phi is going to be equal to lambda Q and therefore, I have d Q over d phi minus i lambda over H cross Q equals 0 the solutions for this is very simple. So, solution for Q is therefore, Q phi is equal to e raise to i lambda over H cross phi some constant in front the normalization constant, how do we find these are the Eigenfunctions. So, Eigenfunctions of L z operator are some constant C e raise to i lambda over h cross phi we are yet to find lambda, but these are the Eigen functions what about lambda?

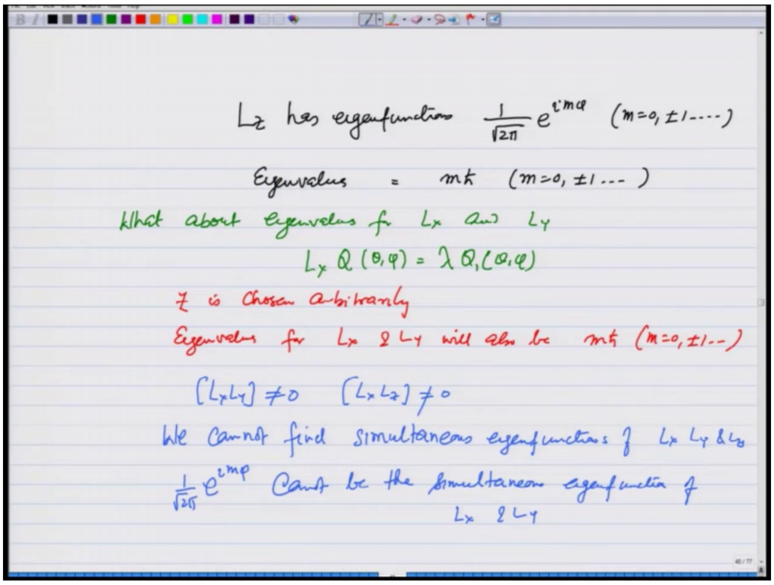
(Refer Slide Time: 05:49)



So, recall That Eigen values are determined by some boundary condition. Now, what is the boundary condition in this case this is if I take the x y plane and let phi go around once. So, phi equals 2 pi, we will see is equivalent to phi equals 0 and therefore, the wave function Q phi plus 2 pi should be the same as wave function Q phi otherwise the wave function, we keep changing as you go around once twice thrice and therefore, I should have e raise to i lambda over H cross phi times e raise to i 2 pi lambda over H cross equals e raise to i lambda over H cross phi. This term cancels and I have e raise to i 2 pi lambda over H cross equals 1 and this implies immediately that lambda is going to be sum m h cross where m is an integer or 0 plus minus 1 plus minus 2 and so on. So, what we learn is that the Eigen values of L z are m h cross with m being 0 plus minus 1 plus minus 2 and so on and the Eigenfunctions are e raise to i. Now I can write this as m phi.

Eigenfunctions are e raise to i m phi and if I normalize the Eigenfunctions, they will be normalized over the range of phi which is 0 to 2 pi and I am going to have mod Q square d phi should be equal to 1. So, if I take Q to be C e raised to i m phi I will find that C comes out to be 1 over root 2 pi. So, my Eigen functions normalized Eigen functions Q phi for L z r 1 over square root of 2 pi e raise to i m phi m equals 0 plus minus 1 and so on.

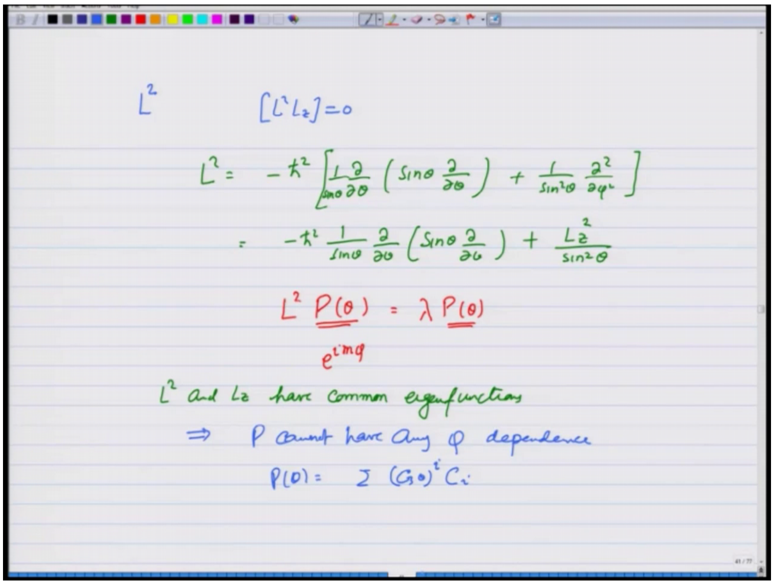
(Refer Slide Time: 08:49)



So, what we found is that L z has Eigen functions one over root 2 pi e raise to i m phi m equals 0 plus minus 1 and so on and Eigen values are m H cross again m equals 0 plus minus 1 and so on.

What about Eigen values for L x and L y? Suppose I were to solve L x times some Q, right which will be a function of theta and phi both equals lambda Q theta and phi. Now you see these direction z is chosen arbitrarily direction z is chosen arbitrarily. So, z is chosen arbitrarily. So, if it is chosen arbitrarily the next time this axis could be the z axis. So, Eigen value should not really depend on the direction. So, Eigen values for L x and L y will also be same m h cross m equals 0 plus minus 1 and so on except that the Eigen functions would be different because now the operator is different, but the Eigen values cannot depend what value L z takes or a component takes is the same the only thing that happens is now because L x L y is not equal to 0 or L x L z is not equal to 0 therefore, I cannot find simultaneous Eigen functions of L x L y and L z. So, I choose only 1 component L z and then I cannot that cannot be the Eigen function. So, e raised to i m phi 1 over root 2 pi cannot be the simultaneous Eigen function of L x and L y.

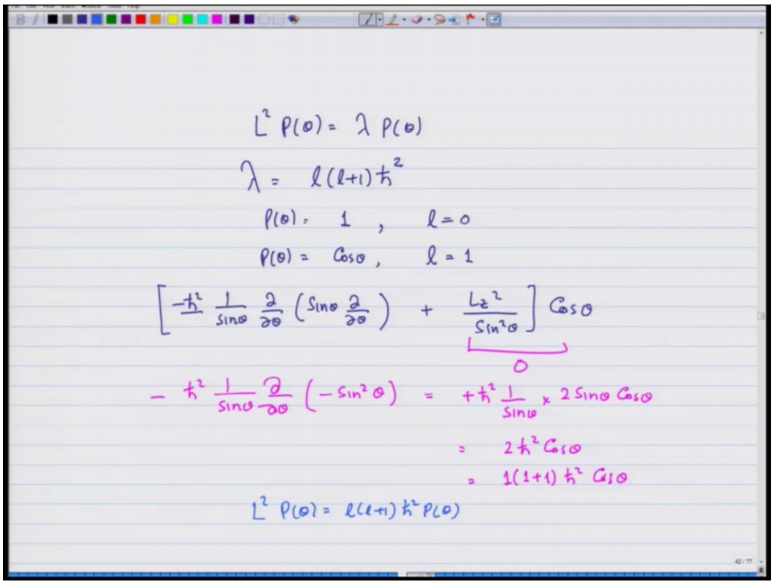
(Refer Slide Time: 11:43)



So, we can choose only one right and that Eigen value is this, now what about L square? L square L z turns out to be 0. So, I can find simultaneous Eigenfunctions of L square and L z. Now L square operator we have already seen L square is minus h cross square, d by d theta sin theta d by d theta is the 1 over sin theta sitting around here plus 1 over sin square theta d 2 by d phi square, which I can write as minus h cross square 1 over sin theta d by d theta of sin theta d by d theta plus L z square over sin square theta.

And if I want to find the simultaneous Eigen functions, I got to do L square; whatever those Eigen functions are let me call them P theta is equal to lambda P theta. You may wonder why I wrote P theta that is because I already know the L z Eigen functions and those are e raised to i m phi and since these are simultaneous Eigenfunctions of L square and L z they cannot be any phi term in P. If it was there then L z would operate on it. So, this p would depend only on theta, right. So, this is because L square and L z have common Eigenfunctions. So, this implies immediately that P cannot have any phi dependence.

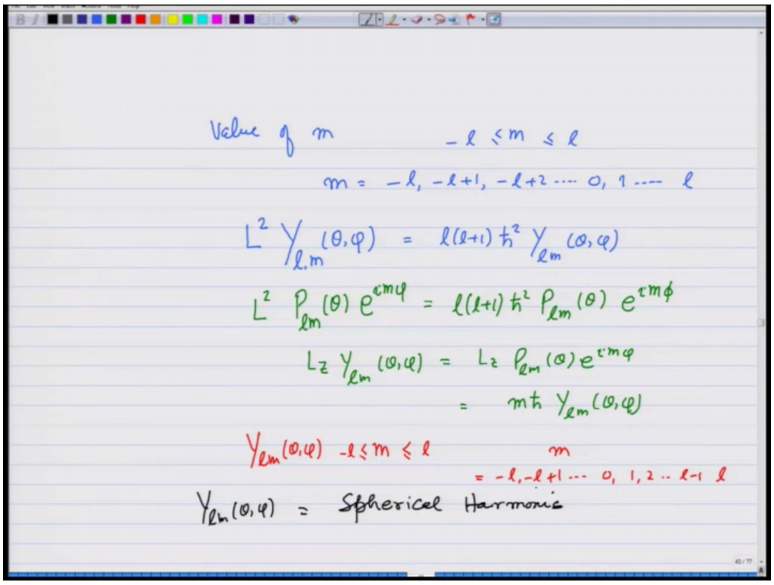
(Refer Slide Time: 14:22)



Now, the way one solves this equation is by writing P theta expands it in terms of cosine of theta and its powers i some coefficient C i, we are not going to do that I will just state the result and that is that these functions when we calculate P theta equals lambda P theta lambda value comes out to be l plus 1 h cross square. Now and p theta are different functions, one of the functions is P theta equals 1 corresponding l will be 0, P theta equals cosine of theta corresponding l is 1. Let us check the second case; first case is very easy to check this one.

So, L square gives you 0 and L square operator is minus h cross square 1 over sin theta d over d theta sin theta d over d theta plus L z square over sin square theta and this operates on cosine of theta, what do I get the first one gives me 0, this term operating on this gives me 0 and the second term is going to give me minus h cross square 1 over sin theta d over d theta inside I get when I multiplied sin theta with the derivative of cosine theta, I am going to get minus sin square theta and this leads to immediately plus h cross square 1 over sin theta times 2 sin theta cosine of theta and that gives you 2 h cross square cosine of theta which is nothing, but 1 plus 1 h cross square cosine theta.

(Refer Slide Time: 16:32)

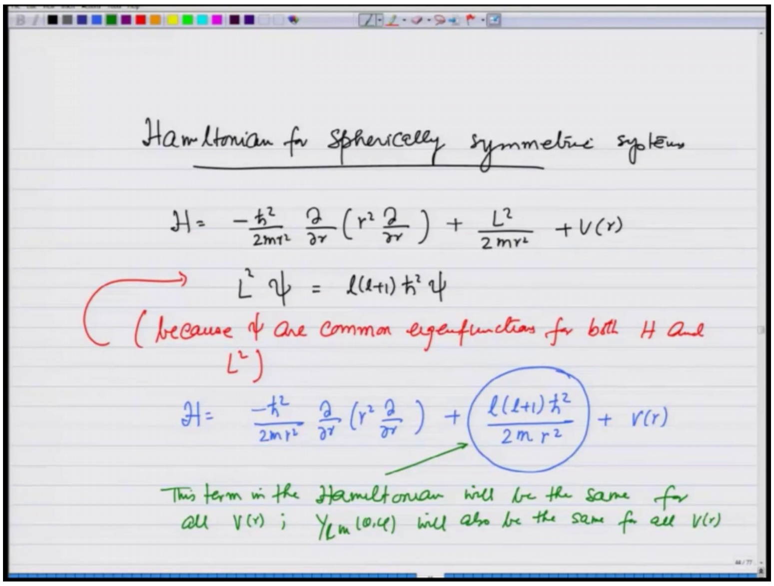


So, cosine theta is an Eigenfunction with l being 1. What is found is that when L square times this p theta is equal to l, l plus 1, h cross square P theta then the value of m; m is restricted to for a given L to being from minus L to plus L in steps of one. So, m comes out to be minus l, minus l plus 1, minus l plus 2 so on 0 1 all the way up to L and these Eigenfunctions are going to be labelled and they have a name L square, they are called spherical harmonics they depend both on L and m theta and phi is going to be equal to L, L plus 1, H cross square Y L m theta and phi and as I said earlier the phi dependence we know because of these functions beings simultaneous Eigenfunctions of L z also.

So, I have L square L m can be broken into P L m that depends only on theta, but does depend on L and m both, e raised to i m phi is equal to l, l plus 1 h cross square p l m theta only e raise to i m phi, and L z operating on Y l m theta and phi is nothing, but L z operating on P l m theta e raise to i m phi and that gives me m h cross Y l m theta and phi m for a Y l m theta and phi m is restricted between l and minus l in steps of 1. So, m can be equal to minus l, minus l plus 1 so on up to 0 1 to l minus 1 and l.

So, this is how the Eigenfunctions are given I have not given you details, I have just given you 2 Eigen functions right and that us, what it is this is where we stopped other things take us into mathematics of how to determine these Eigen functions, if you want to know the technical terms the Eigenfunctions comes out come out to be the associated Legendre polynomials P l ms multiplied by this these e raise to i m phi, but the Y l ms is have a name these are known as a spherical harmonics.

(Refer Slide Time: 19:29)

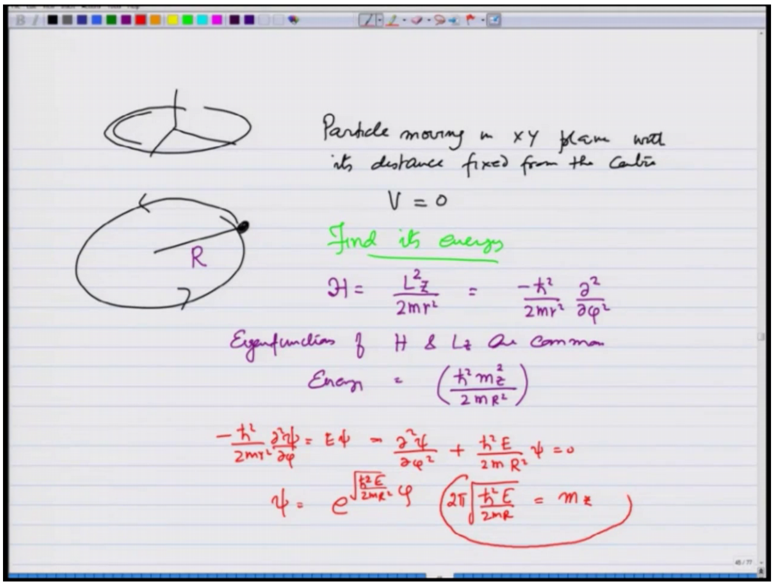


So, I given you 2 Eigen functions others I can keep giving and ask you to check what they are like and once you then go to the Hamiltonian for spherically symmetric systems this can be written as H equals minus H cross square over 2 m r square partial with respect to r; r square partial with respect to r plus L square over 2 m r square plus V r and since the Eigenfunctions psi are going to be simultaneous Eigenfunctions of the angular momentum operator. So, I can easily write that L square psi is going to be equal to L, L plus 1 H cross square psi y because psi are common Eigen functions for both H and L square. So, that automatically implies this and therefore, the Hamiltonian can be written as minus H cross square over 2 m r square partial with respect to r, r square partial with respect to r plus L, L plus 1 H cross square over to the mass r square plus V r for all Eigenfunctions for spherically symmetric system.

So, this is common V r could be anything this term would be the same. So, let me write this; this term in the Hamiltonian will be the same for all V r and those y L ms will also be the same for all V rs, this will see in more detail in the next lecture, but this is how the

Hamiltonian gets simplified and what you see in this is that now we are expected to solve only a differential equation which is for r the theta phi components have been taken care of.

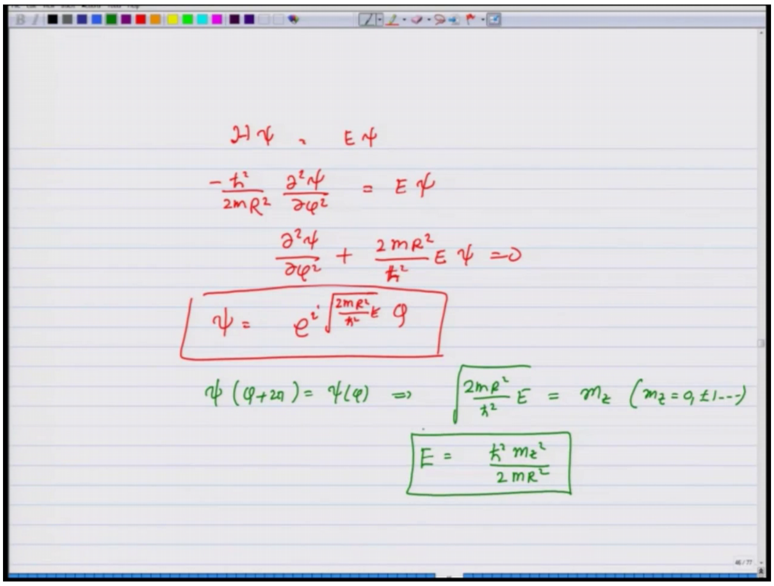
(Refer Slide Time: 22:24)



So, because of spherical symmetry the problem gets simplified now before I believe this we can consider because of L z suppose there is a particle moving in x y plane. So, particle is moving in x y plane with its distance fixed from the centre. So, for example, could be a rod the mass air and moving around with no potential. So, let us take V to be 0, find the energies. So, the Hamiltonian for the system is nothing, but the angular momentum square divided by 2 m r square where r is fixed r is let us say the radius is given to be r and angular momentum is only z angular momentum because V is 0 angular momentum is conserved. So, I can write it like this and this is going to be minus H cross square over 2 m r square d 2 by d phi square that the Hamiltonian and because V is fixed the Eigenfunctions of H and L z are common.

So, you can immediately write that the energy is going to be H cross square m z i am writing m z to differentiate from mass over to m r m z 2 square for to r square, but you can also do it if you want by solving the equation. So, I have minus H cross square over 2 m r square operating on psi equals e psi which gives me psi. This is d 2 by d phi square d 2 phi by d phi square plus H cross square e over 2 m r is fixed psi equals 0. So, immediately you get psi equals e raised to square root of H cross square e over to m r square phi and H cross square e over 2 m r square; square root should be equal to m z what times 2 pi should be equal to m z because the uniqueness of the wave function and therefore, you get your answer.

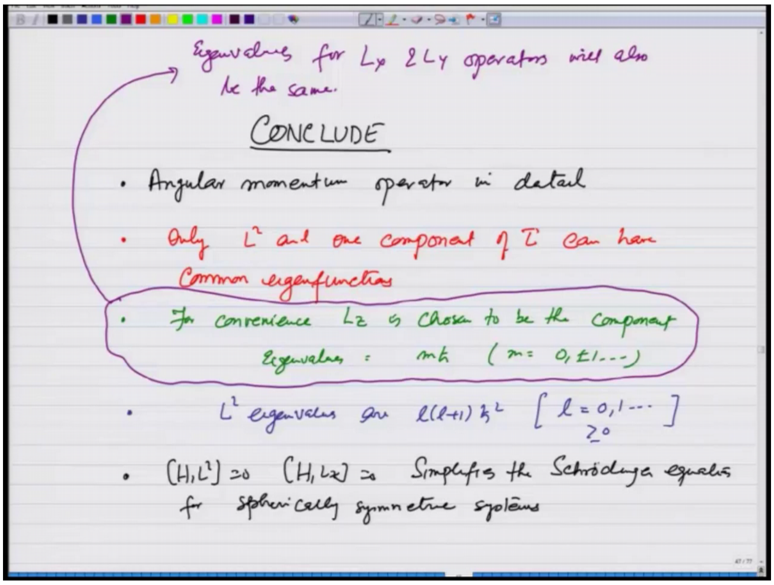
(Refer Slide Time: 25:21)



So, you can see this also directly by writing the Hamiltonian times psi equals E psi Hamiltonian, we have written is minus H cross square over 2 m radius is fixed r square d 2 psi over d phi square equals E psi. So, I am going to have d 2 psi over d phi square plus 2 m r square over H cross square E psi equals 0.

So, the solution is going to be psi equals e raise to i square root of 2 m r square over H cross square E times phi, this is a solution and if I demand again for the unique value of psi phi plus 2 pi should be equal to psi phi, it immediately gives me that 2 m r square over H cross square e square root should be equal to m z where m z is equal to 0 plus minus 1 and so on and therefore, E comes out to be H cross square m z squared over 2 m r square as obtained earlier.

(Refer Slide Time: 26:53)



So, you see this having a thought about simultaneity of the Eigen functions also gives me a tool to simplify my solutions what we have done in this lecture, I will just conclude now That will look that angular momentum operator in detail we found that only L square and one component of L can have common Eigenfunctions, we have also found that for convenience L z is chosen to be the component and Eigen values come out to be m H cross m equals 0 plus minus 1 and so on then we also found that L square Eigen values are L, L plus 1 H cross square and I; what I did not write there is L is 0 1 and so on. This is greater than or equal to 0 and L square and L z have common Eigenfunctions that we have already decided and this H L square being 0 and H L z being 0 simplifies the Schrodinger equation for spherically symmetric systems.

But I would also this comment with respect to the point for the Eigen values for L z and I am circling it here is that the Eigen values for L x and L y operators will also be the same if we decide to solve for them except that the Eigenfunctions are going to be slightly more complicated because the direction should not really matter L x L y and L z for a spherically symmetric systems are all the same we choose L z for convenience because the wave function becomes very convenient we could have chosen to work in terms of L x, L y and L square instead also the same results.