Introduction to Quantum Mechanics
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Lecture – 05
Reflection and transmission of particles across a potential barrier

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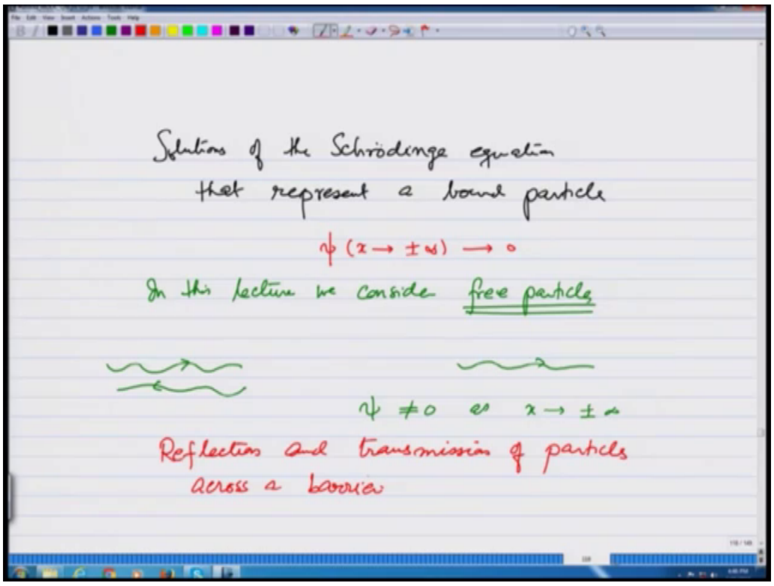

So far we have been looking for solutions of the Schrodinger equation that are that represent a bound particle and what does that mean; that means, the wave function psi x going to plus or minus infinity goes to 0. In this lecture, we want to do something different in this lecture, we consider free particles and by that I mean; they are coming from minus infinity may reflect back and go to plus infinity and so on. So, psi is not equal to 0 as exposed to plus or minus infinity and what you want to consider is the reflection and transmission of particles across barrier let me explain what means see earlier in one problem.

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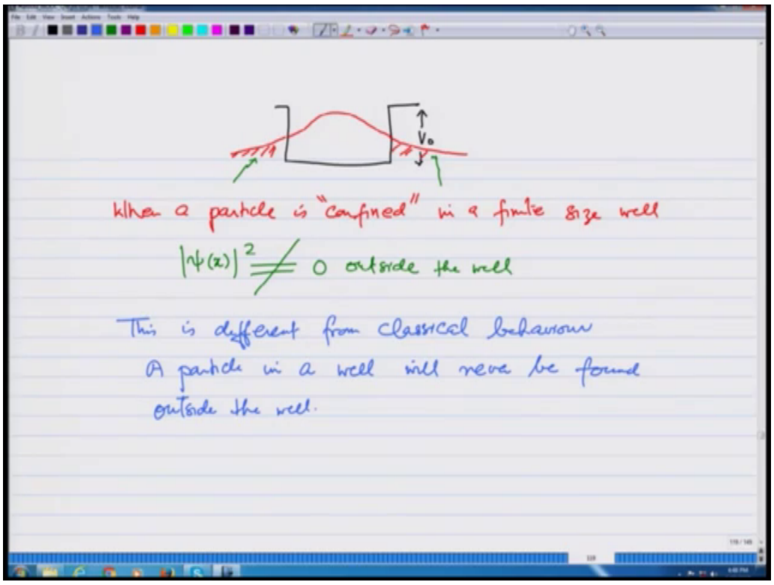

We had considered particle in a finite size box height being V 0 and what we found is that the wave function was non zero inside, but interestingly it also was non zero outside.

So, when a particle is confined and I confined; I will say in quotes in a finite size well, there is a probability that it is found outside the well in that case what we found is psi x square was not equal to 0 outside the well and; that means, there was a probability of finding the particle outside the well, if I probed here or probed here sometimes I would find that particle although the probability is small is largest in the well outside its small, but non zero this is different from classical behavior what happens if classical behavior is a particle in a well will never be found outside the well.

So, it does something similar happen in other cases and that is what we want to explore in.

This case, all I want to say because of the wave nature of the particle in new quantum phenomena come although you are seeing it in the context of a particle going across the barrier it is nothing new in waves. So, let me just motivate you by saying.

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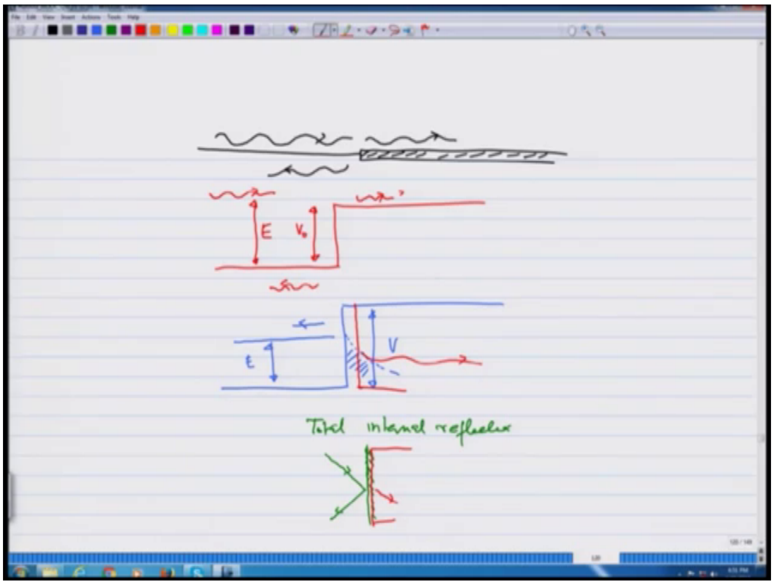

Suppose, I have a string a thin string and a thick string and what we find is; if there is a wave coming in always there will be a reflection and there will be some transmission no matter what you do. Similarly, if I have a step and suppose, I allow a particle to come in and symbolically, I am making this energy E to be greater than the step height classically the particle would just go straight quantum mechanically sound the particles have a probability of coming back and others go out in this case a majority would be going to the other side of the barrier as you would expect classically where all the particle go to the other side.

But quantum mechanically some of them would turned back and this is nothing, but because there is a wave associated and there is a boundary condition where waves have to match the other phenomena which is similar to this kind of wave mechanics is where suppose, I have a particle coming at energy lower than the barrier height, then you will find that even quantum mechanically all the particles go back none the less.

Sometimes you would have a probability of finding the particle on the other side, this is what happens in classical waves in total internal reflection in total internal reflection suppose there is a boundary of 2 media and total internal reflection is taking place even then there is some electromagnetic field on this side although all the light is going back and therefore, if I put another medium here or make this glass very thin some light goes through similarly here we will find if I make this step of finite size you will find the sum particles go through what is known as tunneling. So, these are all phenomena concerned with waves and when we consider particles as being driven by or their properties been driven by waves these phenomena come into particles behavior also.

So, let us now take these problems step by step. So, the problem I am going to tangle in this lecture is the reflection and transmission of particles across a barrier.

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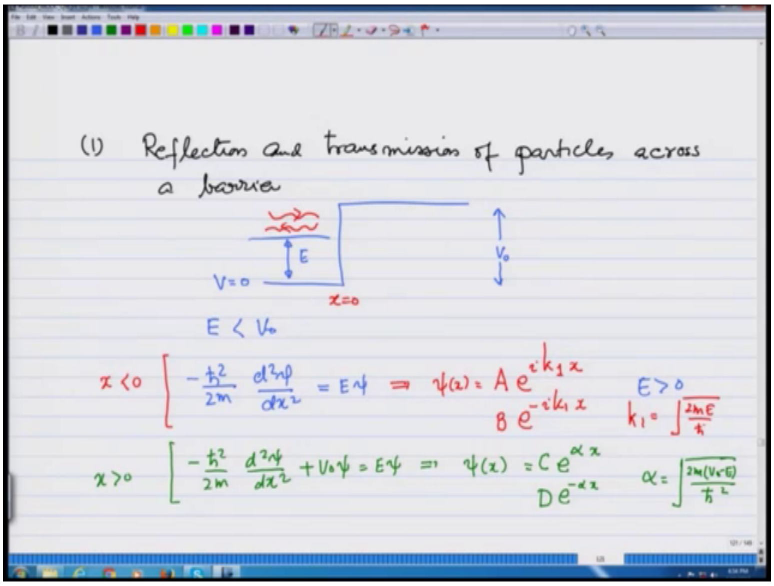

So, here is the barrier of height V 0 and first case I consider is a part of the coming in at energy e less than V 0 the left hand side is V equal 0. So, on the left hand side the wave function is calculated by the Schrodinger equation V 0. So, this is equal to E psi, E has to be greater than 0, otherwise the wave function decays and therefore, the solutions that I have are psi x equals e raised to i k let me call it k 1 x or e raised to minus i k 1 x some amplitude A, some amplitude B.

So, the wave could be coming in and because as the boundary could also be going back and. So, this is solution for x less than 0 where x 0 is where this step is for x greater than 0 I have minus h cross square over 2 m d 2 psi over d x square plus V 0 psi equals E psi and this gives me solution psi x which is equal to e raised to alpha x or e raised to minus alpha x where alpha is square root of 2 m V 0 minus E over h cross square. Let us also call this amplitude C and amplitude d k 1 was square root of 2 m E over h cross square.

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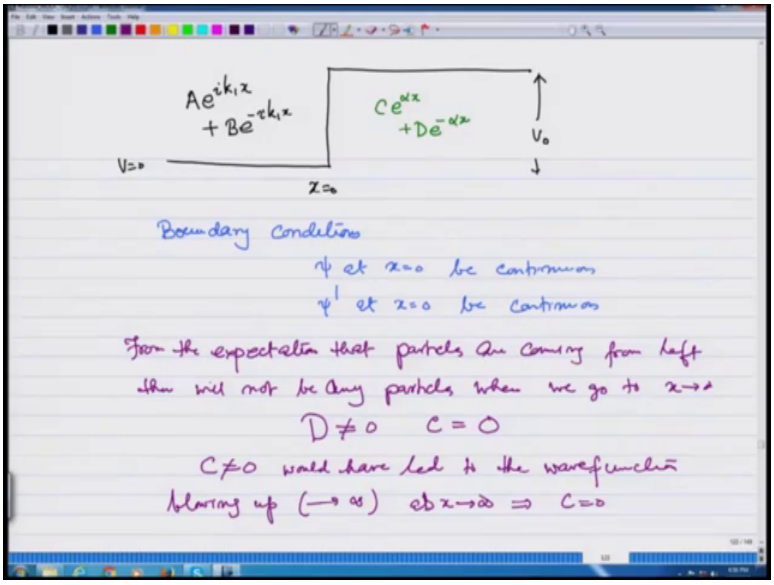

So, let us make this picture again this is x equals 0 in the middle and the height of the potential is V 0; V 0 here, here is the wave function which is a e raised to i k 1 x plus B e raised to minus i k 1 x. So, wave function is superposition and we will find A and B through boundary condition on the right hand side for x greater than 0 I have C e raised to alpha x plus D e raised to minus alpha x is the solution how do you find A, B, C and D through the boundary condition and boundary conditions are that is psi at x equals 0 be continuous and psi 1 at x equals 0 be continuous now our expectation is that when the particles are coming from left they cannot really go very far off on the right hand side noise.

So, from the expectation that particles are coming from left; so, initially they are no particles to the right. There will not be any particles when we go to x equals infinity means we can conclude that D is non 0, but C is 0 because C makes the wave function blow up as to infinity and therefore, will take it to be 0. So, C non zero would have led to the wave function blowing up and that is going to infinity as x tend into infinity and therefore, C is 0. So, physical expectation that we do not expect to see any particles far away from the boundary and also if we keep C non zero the wave function blows up both give you idea that C should be made 0.

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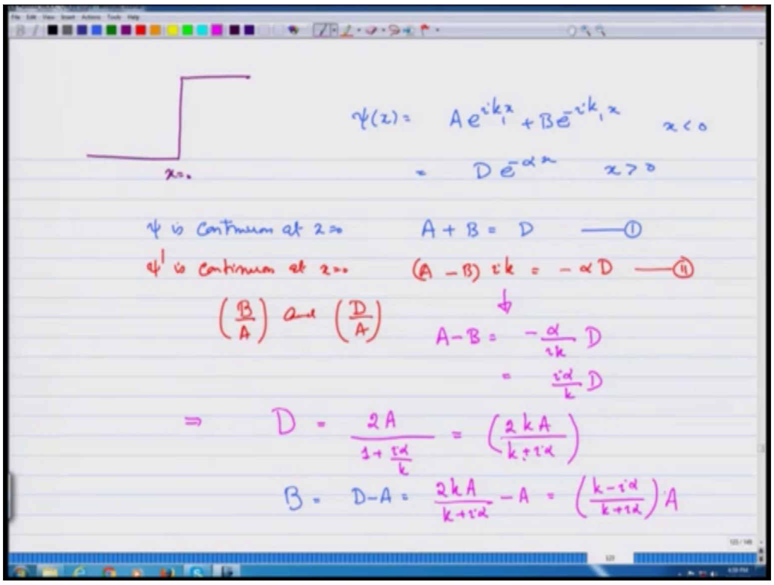

So, on the right hand side the only solution is D e raised to minus alpha x. So we can now write again this is the problem x equal 0 psi x equals a e raised to i k x plus B e raised to minus i k 1 x for x less than 0 is equal to D e raised to minus alpha x or x greater than 0. So, psi is continuous at x equals 0 gives you a plus B equals D that is my equation 1 and psi prime is continuous at x equals 0 give me a minus B times i k is equal to minus alpha D that is my equation 2.

There are 2 equations and 3 unknowns how do we solve this we are not interested in knowing all 3 unknowns all we are interested in knowing what is B over A and D over A, they will give you the ratio of the particles that go across. So, when we solve it. So, second equation I can write as A minus B equals minus alpha over i k D which is i alpha over k D and this immediately tells you that D equals 2 A divided by 1 plus i alpha over k which is equal to 2 k A over k plus i alpha and B which is equal to D minus A is going to be equal to 2 k A over k plus i alpha minus A and that gives you k minus i alpha over k plus i alpha A.

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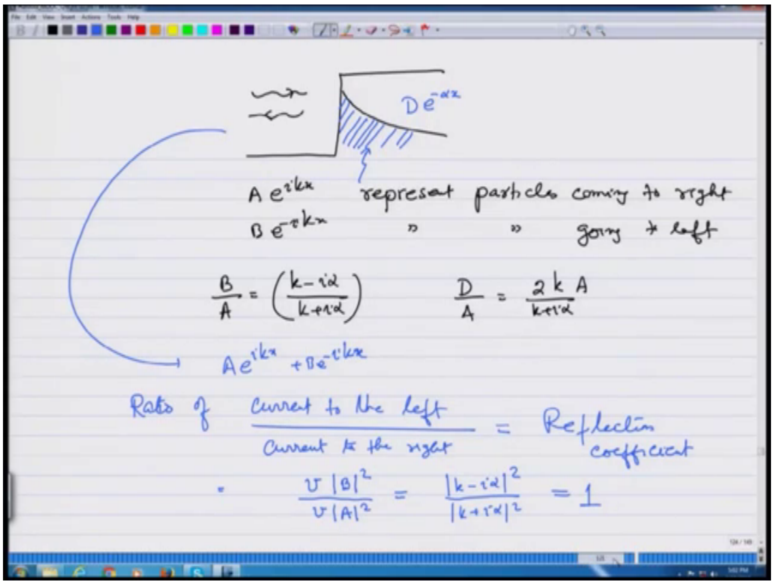

So, what we found is that in this case when particles are coming and going back and there is some decaying wave function here I have found A e raised to i k x which represent particles coming to right B e raised to minus i k x represent particles going to left I have found that B over A is equal to let see where it is k minus i alpha over k plus i alpha and D over A is 2 k A over k plus i alpha. Now the wave out here is a standing wave or maybe slightly travelling because A and B are not equal. So, this is basically when I write on the left hand side the solution has A e raised 2 i k x plus B e raised to minus i k x, I am writing a wave which is superposition of wave travelling to the right and wave travelling to the left and therefore, I could interpret A e raised to i k x as representing particles coming to the right and B e raised to minus k x as representing part of these going to the left.

So, the ratio of current to the left divided by current to the right is going to be nothing, but the reflection coefficient unless you would the reflection coefficient is current to the left is going to be the speed this you please go back to the lecture where we took time dependent Schrodinger equation to calculate the current times mod B square current to the right is going to be speed divided by mod A square and this is going to be nothing, but mod of k minus i alpha square over mod of k plus i alpha square which is nothing, but 1.

So, all the particles are coming to the right are actually getting reflected at the same time there is a probability of finding the particles and I am showing by this blue in this figure because there is a D e raised to minus alpha x there is always a probability of finding particles in this region. So, although eventually all particles go back what happens is if I measure or try to locate a particle in this and I will find some particles sometimes because there is a finite probability, but the reflection coefficient is certainly one.

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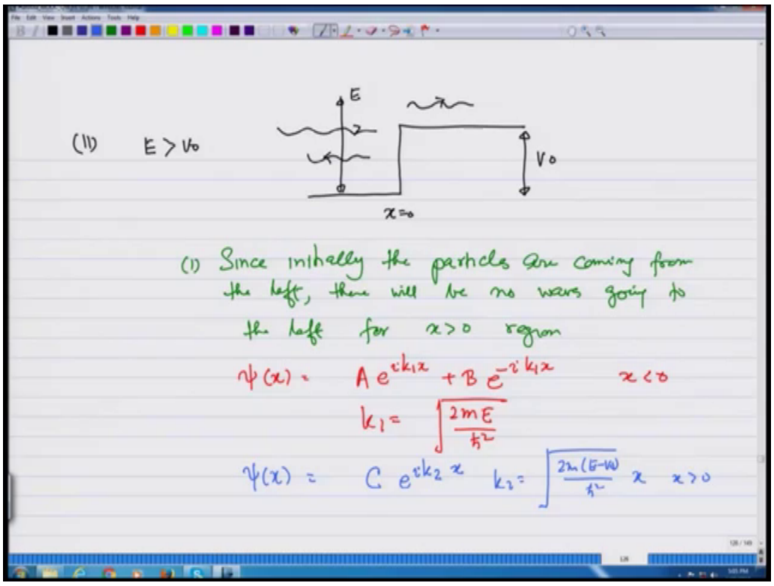

Let us see now what happens when we make energy through case 2 energy greater than V 0. So, what we are doing is we have this step at x equals 0 the height of this step is V 0 and the energy is large and initially the particles are coming from the left. So, again we expect that there will be some reflection and some transmission against is the particles are coming from the left initially there no particles coming from the right. So, the only thing I can see is that the particles after hitting the barrier can go only to the right, I will not take a solution coming to the left for x greater than 0. So, first observation since initially the particles are coming from the left there will be no waves going to the left for x greater than 0 region.

So, again I am going to have psi x equals A e raised to i k 1 x plus B e raised to minus i k 1 x for x less than 0 where k 1 is square root of 2 m E over h cross square and I am going to have psi x equals C e raised to i k 2 x where k 2 is square root of 2 m e minus V 0 over h cross square x for x is greater than 0 and how do I find C, A and B again through boundary condition.

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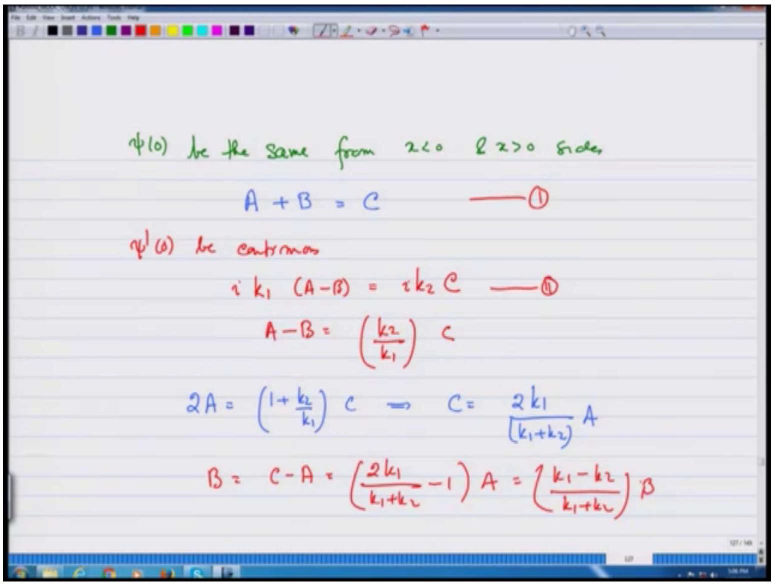

What is the boundary condition psi; 0 be the same from x less than 0 and x less greater than 0 sides that mean sides continuous.

So, I am going to have a plus B equals C number 2 psi prime 0 be continuous. So, I am going to have i k 1 a minus B equals i k 2 C, this is my equation 1, this is my equation 2 or A minus B is equal to k 2 over k 1 times C. Now we can solve the 2 equations and I get 2 A equals 1 plus k 2 over k 1 C or C equals 2 k 1 over k 1 plus k 2 A and I get B equals C minus A which is 2 k 1 over k 1 plus k 2 minus 1 A which is k 1 minus k 2 over k 1 plus k 2 B.

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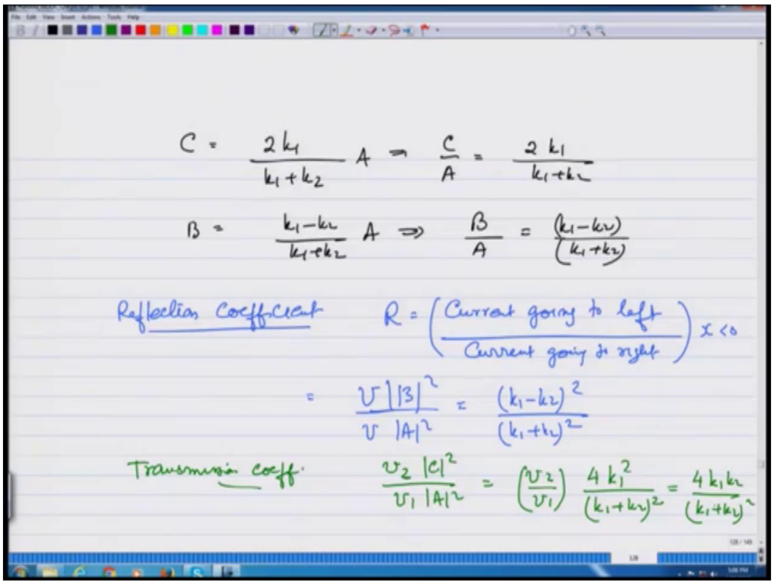

So, I get C which is 2 k 1 over k 1 plus k 2 A implies C over A as 2 k 1 over k 1 plus k 2 and B which is k 1 minus k 2 over k 1 plus k 2 A which implies that B over A is equal to k 1 minus k 2 over k 1 plus k 2.

Now, reflection coefficient R is the current to the left divided by current going to right in region x less than 0, this is going to be the speed which will be given by k times B square divided by speed A square. So, this comes out to be k 1 minus k 2 square divided by k 1 plus k 2 square and the transmission coefficient is going to be speed V 2 for x greater than 0 which is going to be smaller mod C square over V 1 mod A square.

So, V 2 minus V 1 now divided by V 1 times 4 k 1 square over k 1 plus k 2 square speed is proportional to k. So, this is going to be 4 k 1 k 2 over k 1 plus k 2 whole square.

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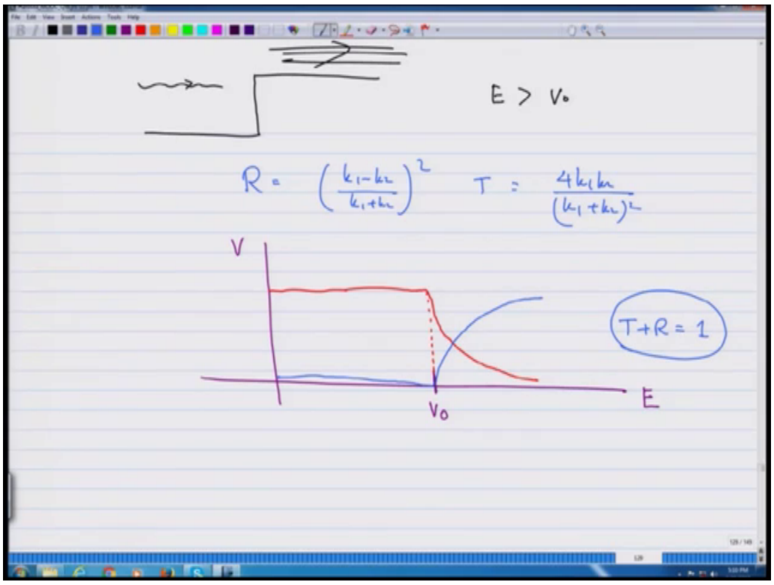

So, what we have calculated is in this case when a particle is going from the left hitting a barrier and the energy is such that this is greater than V 0 classical if we would expect that all the particles will go this below what we find now is that there is a reflection coefficient which is equal to k 1 minus k 2 over k 1 plus k 2 whole square and the transmission coefficient which is four k 1 k 2 over k 1 plus k 2 square if you plot these against the energy versus V then what we find is if e is less than V 0 the reflection coefficient.

Which I am going to show by red is one up to V 0 and the transmission coefficient which I am going to show by blue is 0 up to V 0 beyond this reflection coefficient does not become 0 all of a sudden, but slowly goes to 0 and beyond this, the transmission coefficient suddenly does not become one, but slowly approach is one such that the sum t plus r is always one. So, this is another result which is different from classical mechanics even for energy greater than V naught there is reflection and transmission is not one.

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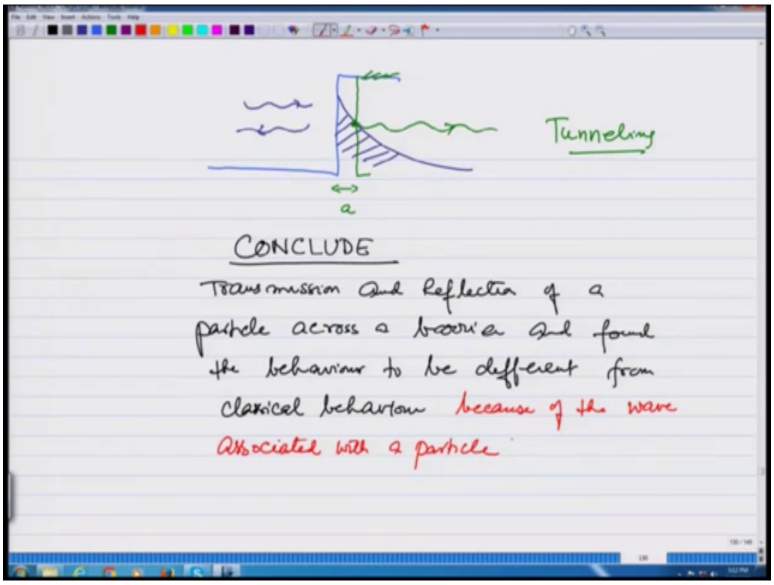

Now, what will happen in the case of energy lower than V 0 is what we found is particle comes in particle goes back, but there is a finite probability of the particle being found here. So, just like in total internal reflection if I make this barrier small; that means, I have V 0 only in this small region again there will be a probability of particle going through this is very countering intuitive very non classical.

Because is the probability of practicality here and then it can go through this is known as tunneling even if energy is lower than the barrier the particle can still go through and this is a quantum phenomena because of the wave nature of light to conclude this lecture what we have considered is transmission and reflection of a particle across a barrier and found the behavior to be different from classical behavior and this is because of the wave associated with a particle and I have motivated you to think that if I make this barrier of thin width they maybe particle transmitting through a barrier even if its energy is lower than the barrier height and that is known as the phenomena of tunneling which I will consider in the next lecture.