Introduction to Quantum Mechanics
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Lecture – 06
Ehrenfest's theorems for the expectation values of x and p

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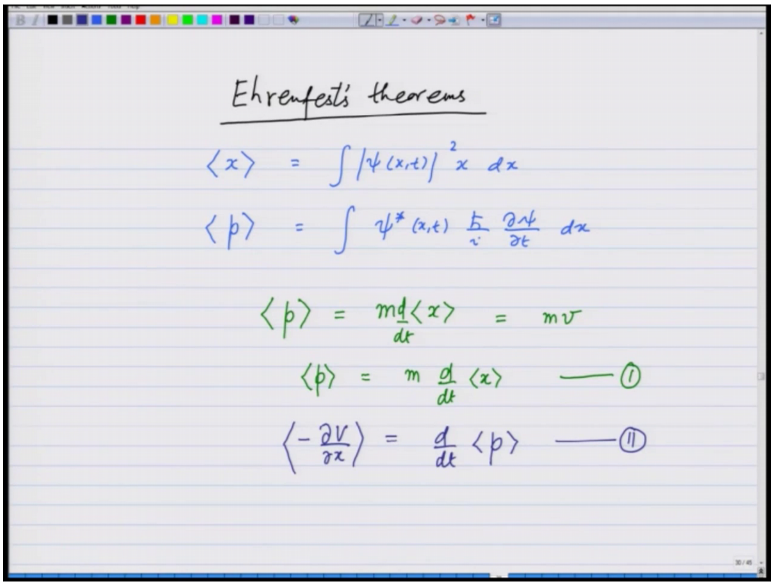

In this lecture, I am going to talk about Ehrenfest's theorems, and they kind of tell you how the expectation values of p and x change with time, and these theorems tell you how expectation value of p and x change with time and they gave it some meaning which we have been sort of anticipating. So, the recall that expectation value x is define as integration psi x t mod square x dx. And we said earlier that this is like the average value of x. Similarly, average value of p is defined as integration psi star x, t h cross over i d psi over dt dx and we said this is like the average value of momentum for a particle n wave function psi state. What Ehrenfest theorem tell you is the more meaning to this and show that p can be return as m x d by dt. So, this is like m v.

So, the expectation value of p that is average momentum that we have been talking about is m d by dt of the average value of x, this is first theorem. Second one tells you about the rate of change of p and that tells if I take the expectation value of minus dv dx that is the derivative of potential v to the minus sign in front which is the force this is equal to d by dt of average value of p. So, these again tell us at these expectation values as defined earlier when we talking about equivalence of Heisenberg and showed in this picture, do have a meaning of being the average position and average momentum of a particle in state sign.

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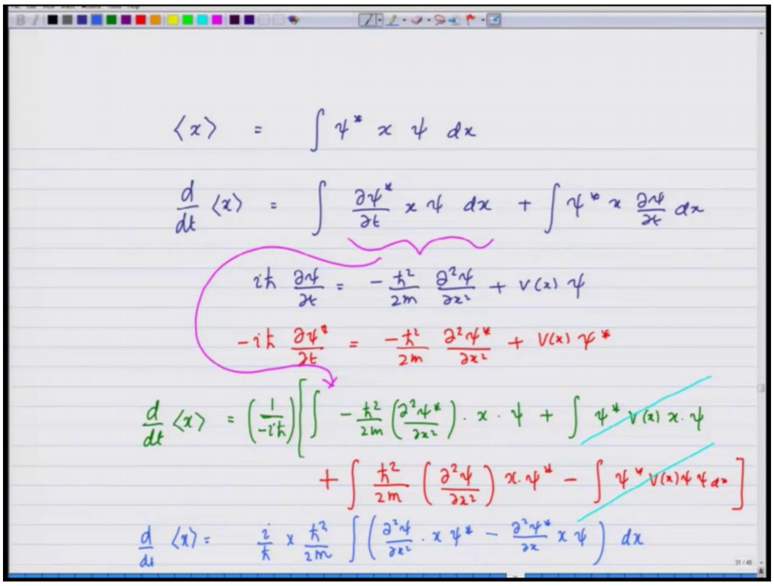

So, let us now proof these. x average is nothing but integration psi star x psi dx and d by dt of x is therefore, going to be integration d psi star by dt x psi dx plus integration psi star x d psi by dt dx. Notice that if you talking about the change of expectation value of x and p in time, the wave function necessarily has to be time dependent in a stationary state these things do not change in time as we showed earlier because the probability density is independent of time.

Now, I know that i h cross d psi over dt is nothing but minus h cross square over 2 m d two psi over dx square plus vx psi. And minus i h cross d psi star over dt is equal to minus h cross square over 2 m d 2 psi star over dx square plus vx psi star. I can substitute these value in the equation about to get d by dt of x is equal to 1 over i h cross with the minus sign integration of minus h cross square over 2 m d 2 psi star over dx square times x times psi. I put this in the bracket plus integration psi star v x x psi that is from the first term. So, this correspond to this term.

For the second term, I am going to have minus integration h crosses square over 2 m d 2 psi over d x square I am write in d psi by dt times x times psi star minus integration psi star v x psi psi dx I have taken i h cross out. So, let us see now this is perfectly fine, this is except that the sign is going to be plus, yes. So, this is fine. Now, I can cancel the second term. And what I am left with his d by dt of x is equal to i over h cross 1 over minus I have written has i over h cross times h cross square over 2 m integration d 2 psi over d x square x psi star minus d 2 psi star over dx x psi dx that is what I am left with.

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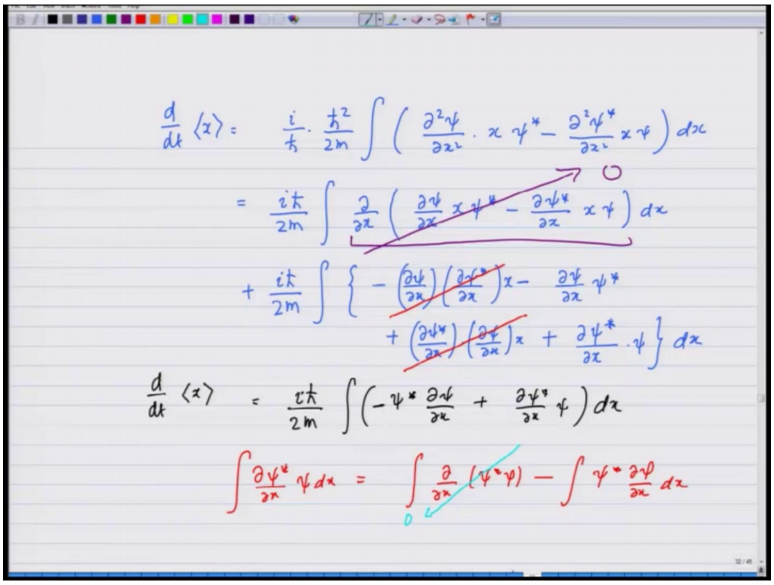

So, what I am getting is d by dt of x is equal to i over h cross h cross over 2 m integration of d 2 psi over dx square x psi star minus d 2 psi star over dx square x psi dx, which I can write as i h cross over 2 m integration of let me take d by dx out, and inside I get d psi over dx x psi star minus d psi star over dx x psi dx that the first term. Now, I have to make some correction because there are extra terms going to appears so i h cross over 2 m integration of I am going to subtract, minus d psi over dx d psi star over dx minus a times x minus d psi over dx times psi star minus minus plus d psi star over dx d psi over dx times x plus d psi star over dx times psi integration dx. Now, the first term can be integrated directly and if psi vanish at infinity boundary condition gives you that this is going to be 0.

In the second term, these two terms are going to be cancel. So, what I am left with is d by dt of x is equal to i h cross over 2 m integration of psi star d psi over dx with the minus sign plus d psi star over dx psi integrated over dx. The second term d psi star over dx psi dx can be return as integration of d by dx of psi star psi minus integration of psi star d psi over dx dx. Again the first integral vanishes because psi vanish that large distances.

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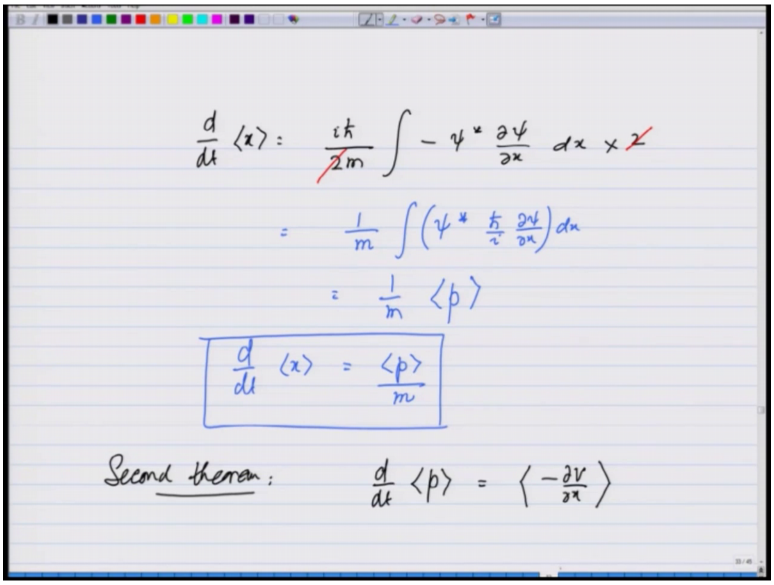

So, what I am left with is d by dt of x is equal to i h cross over 2 m integration of minus psi star d psi over dx dx times 2. This two cancels, and I can write this as 1 over m integration psi stars h cross over i d psi over dx dx, and this is nothing but 1 over m expectation value of p. So, what I have shown is the d by dt of x is expectation value of p divided by m. This is a first Ehrenfest theorem that again tells us that the expectation value of p or expectation value of x can be thought of as the average values of the momentum and position of the particle. The second theorem we have to show that the d by dt of expectation value of p is equal to nothing but minus dv by dx expectation value which is the expectation value of the force.

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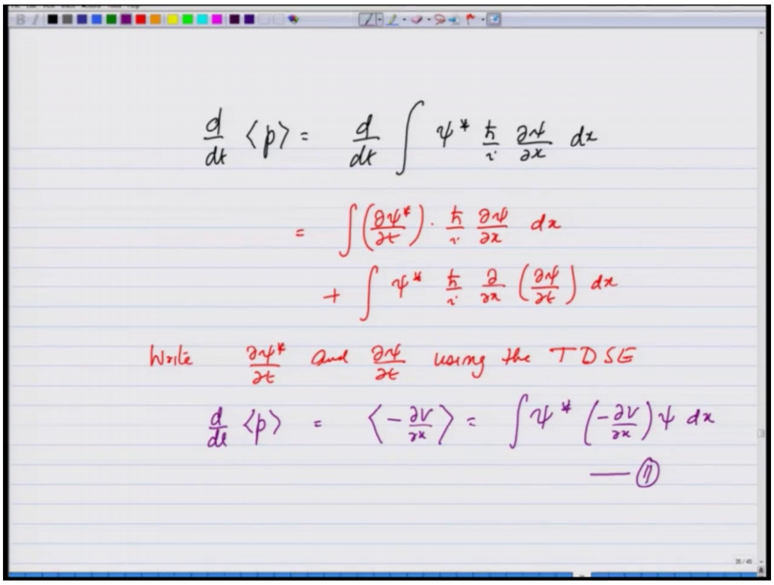

To proof Ehrenfest second theorem be again do what we did earlier for proving the first theorem that is write d by dt of p as d by dt of this expectation value which is psi star h cross over i d psi over dx dx. And this can be expanded has equal to integration delta psi star over dt h cross over i d psi over dx dx plus integration psi star h cross over i d by dx of d psi over dt d x. I interchange the partial derivative is with a respect to x and t that we can do.

The next steps going to be write d psi star dt and d psi over dt using the time dependent Schrodinger equation as we did earlier. And we do that and substitute this in the equation an d by dt p equals after the manipulations and partial integration and all that, you get this equal to minus dv by dx expectation value which is nothing but integration of psi star minus dv by dx psi dx which is the second theorem of Ehrenfest which tells you that average value of p or the expectation value of the p changes according to Newton’s laws. I have not done the derivation here I am leaving that has an exercise, because it is a very simple derivation although little longish. So, I leave it for you.

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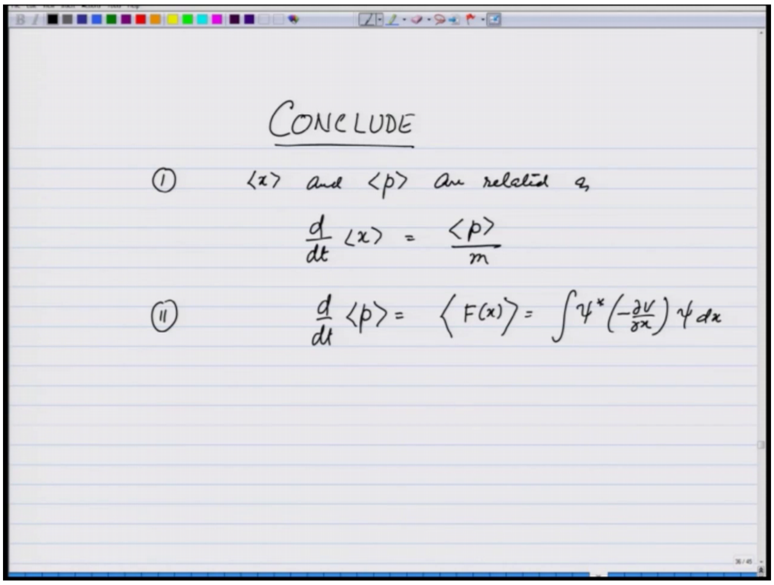

So, what we have shown in this lecture is conclude one at x and p are related as d by dt of x is expectation value p by m. And two - that d by dt of expectation value of p is nothing but the expectation value of the force which is same as psi star minus dv by dx psi dx. So, second tells you that these quantities are the average p and average x for particle n state of psi.