Introduction to Quantum Mechanics
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Lecture – 04
The uncertainty principle and it is simple applications

So, far we have discussed the Heisenberg and Schroedinger pictures of quantum mechanics. And what I have shown you is that both are equivalent both are equivalent. And in Heisenberg's picture things are defined through matrices, and in the Schroedinger picture we have the wave function and when you want to define a quantity xop it is done through operators for each quantity.

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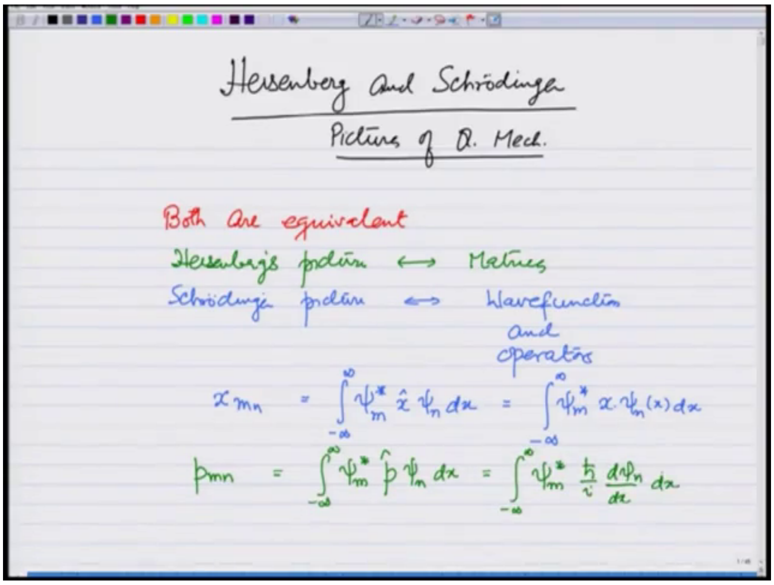

So, we had for example, x mn matrix element of Heisenberg picture as psi m star x operator on n dx. And when this operator x operator acts on psi n this means it just multiples by x psi m star x times psi n x dx. On the other hand we also defined p mn which was equal to minus infinity to infinity psi m star p operator acting on psi n dx and p operator when it operates on psi is a differential operator. So, it actually differentiates it. So, you get psi m star h cross over I d psi over dx, dx.

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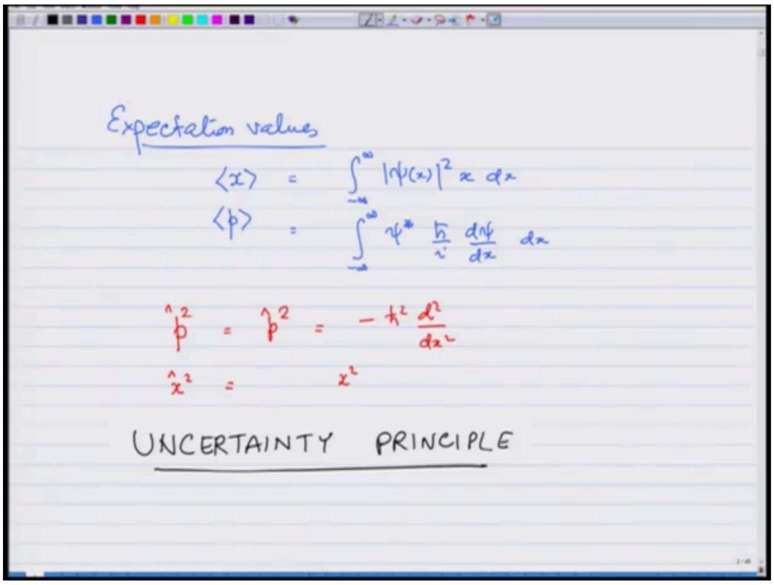

And then we also define the expectation values for these quantities. So, expectation value for x and expectation value of p were define as minus infinity to infinity psi x square x dx. And expectation value for p was defined as the integration over psi star h cross over I d psi over dx integrated over. And we also understood this as the average values of these quantities. Not only that we could also see that operators corresponding to p square was nothing but p operator square and this became minus h cross square d 2 by dx square. Operator corresponding to x square was nothing but x square multiplying the wave function. And similarly we can define higher powers of these operators and also their own multiplication.

What we are going to learn in this lecture through all this is a very important principle of quantum mechanics known as the uncertainty principle which states that quantity is in quantum mechanics cannot be measured in general with very high precision. In fact, 2 quantities which are called conjugate to each other have a relationship given by uncertainty relation if you measure one quantity to very high accuracy the other quantity becomes less accurate. And that is what we are going to see why it happens in quantum mechanics it actually is a result of the quantum condition and therefore, this is a principle which is very unique to quantum mechanics.

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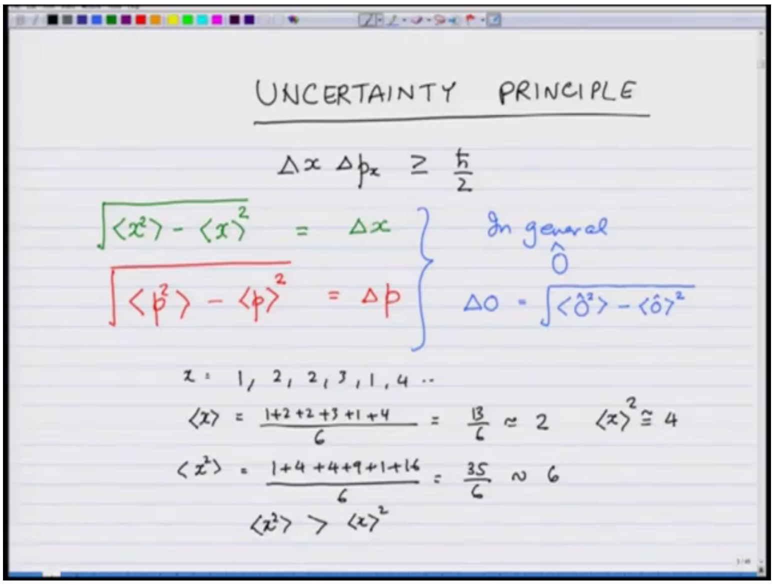

You may have learned about uncertainty principle in your 12 th (Refer Time: 04:56) this is the statement. So, let me just write it and then we will explore it further. You may have learned that for a given particle delta x delta px is greater than or equal to h cross by 2.

So, what does it mean? How do we define delta x how do we define delta px we are going to see it in a mathematical rigorous manner in this lecture. So, let us see if I am making a measurement of quantity x then I do get an average value x. And if I measure x square I get an average value of that if I subtract expectation value or average value of x from this and take square root, this is defined as delta x. Similarly if I take p square expectation value of that, subtract the square of the expectation value of p and take square root this is delta p. In general and let me write it on the right hand side, in general for any quantity right let us call it o, or operator O the uncertainty in O will be equal to square root of O square expectation value minus expectation value of O square. And that makes perfect sense let me explain how. So, suppose I have a quantity or say take some numbers, let us say I measure and I get 1 2 2 3 1 4 and so on. Let us restrict ourselves to 1 2 3 4 5 6 numbers and take the average. So, let us say this is x the average will be 1 plus 2 plus 2 plus 3 plus 1 plus 4 divided by 6. And that comes out to be 8 plus 5, 13 over 6 roughly 2.

And therefore, x square is roughly 4. Let us take x square average. And that comes out to be 1 plus 4 plus 4 plus 9 plus 1 plus 16 over 6, which is nothing but 5 plus 4 9 18 plus 16 34 plus 35 over 6 which is roughly 6. So, you see average of x square is greater than x square. And therefore, 2 are not the same because there is a spread in the values. On the other hand if all xs were the same then these 2 would have been equal. So, what this delta x or delta p or delta O in general gives you is the spread in any quantity. If I make several measurements and find that they are different this delta x defined in the manner given here gives me how much is the spread. And what uncertainty principle tells you is that the spread multiplied by the spread in the conjugate quantity is greater than h cross by 2, and let us now show that. Before that I just want to tell you what happens for an Eigen value in terms of it is spread.

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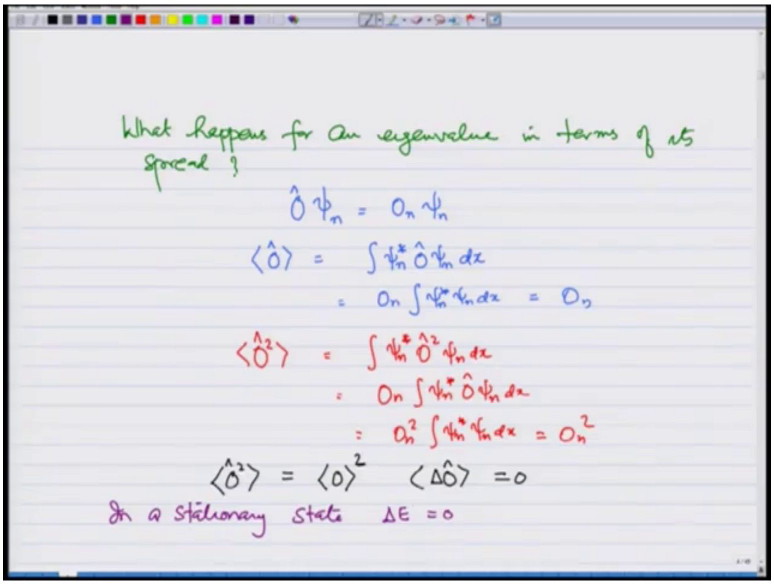

So, for example suppose I have an operator O which operates on a wave function psi and gives me the Eigen value O n psi n. Then the expectation value of O will be given by integration psi n star O psi n dx which is nothing but O n which comes out which is a number psi n star psi n dx which is O n.

On the other hand if I calculate O square this will be integration psi n star O square psi n dx which is nothing but, O n comes out you left with psi n star O operating again on psi n dx and you get O n square integration psi n star psi n dx which is O n square. So, in this case what you see is that O square is same as expectation value of O square. And therefore, delta O is 0. So, if I take the expectation value of an operator, with respect to the wave function that are Eigen functions, and then calculate the spread in it or delta x or delta O in it comes out to be 0. Therefore, in stationary state we can write a wave in a stationary state delta E will come out to be 0. That is a fixed energy state all right. So, with this background I am now going to prove the uncertainty principle.

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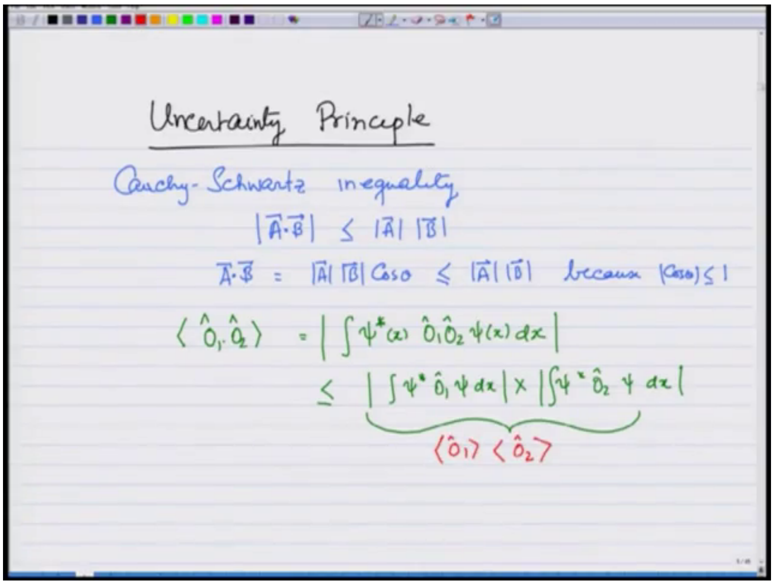

For that I need something called the Cauchy Schwartz, Schwartz will be swa inequality, which if I simply put in terms of vectors it says nothing but A dot B magnitude is less than or equal to magnitude of A and magnitude of B product which is very easy to understand because A dot B is nothing but magnitude of A magnitude of B times cosine of theta. And this is always less than or equal to mod A mod B product because cos theta mod is always less than or equal to 1.

In terms of functions what it tells you is if I have an operator O 1, and it is product with O 2 and take it is expectation value, what that means, is I am taking psi star x O 1 operator O 2 operator psi x dx. This and take it is modulus this would be less than or equal to the individual expectation values that is psi star O 1 psi d x, times psi star integration O 2 psi dx. Which is nothing but this whole quantity is nothing but expectation value of O 1 expectation value of O 2. This is known as the Cauchy Schwartz inequality and we are going to use this now to prove the uncertainty principle.

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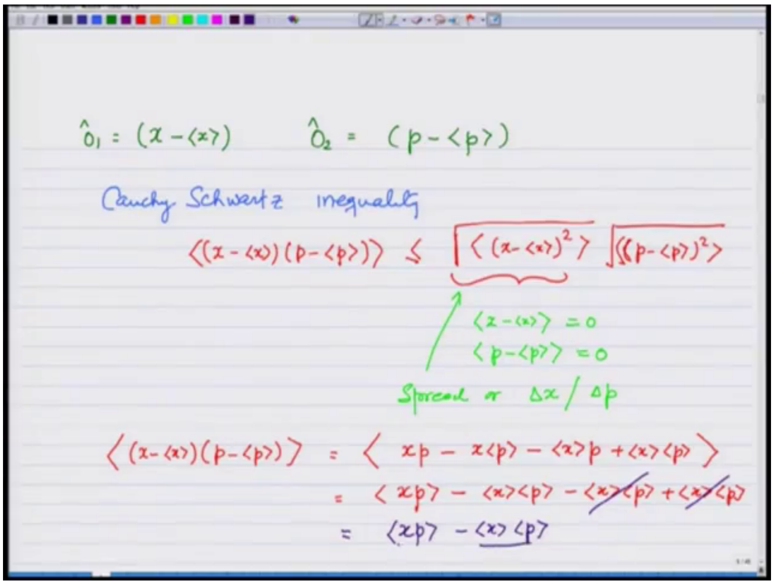

For that I choose the quantity x minus mod x as one operator. So, this is going to be O 1, and O 2 is going to be this operator p minus expectation value of p. So, I am going to have from Cauchy Schwartz inequality. So, I am going to use the Cauchy Schwartz inequality and take the, So what I am going to show is that take x minus mod x and p minus not expectation value of p expectation value. And this should be less than or equal to expectation value of x minus x square square root times square root of p minus mod p square.

You may wonder why I am taking the square root, because you see expectation value of x minus mod x is 0. And expectation value of p minus expectation value p is 0. So, what I am really actually calculating is this spread or delta x delta p on the right hand side this is kind of rms deviation. Let us work on the left hand side. So, expectation value of x minus expectation value x p minus this quantity is nothing but expectation value of xp minus x p average minus x average p plus x average p average. And I am taking expectation value of this, which is nothing but expectation value of xp minus xp minus xp plus xp. And I am going to cancel a few terms. So, this fellow cancels with this and I get this equal to expectation value of x p minus x p. In all this keep in mind that whenever I am writing this angular bracket or the expectation value that is a number right. So, this quantity is a number and this quantity is a number.

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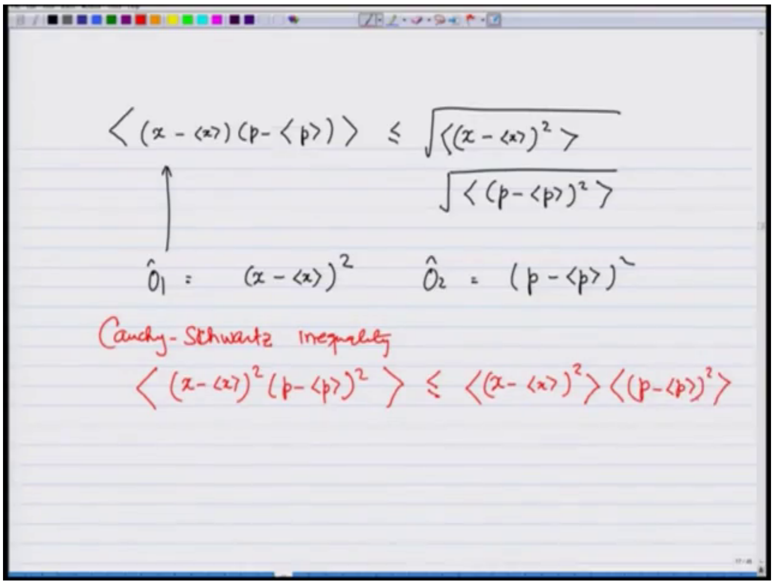

If you are uncomfortable in using x minus expectation value of x and p minus expectation value of p, expectation value less than equal to square root of x minus x expectation value square expectation value times square root of p minus expectation value of p square expectation value. Then you can instead use for O 1 this itself square and for O 2 the square of p minus p expectation value square. And write the Cauchy Schwartz inequality as x minus mod x square p minus mod p square expectation value, would be less than or equal to x minus expectation value of x square expectation value times the expectation value of p minus p expectation value square expectation value, leading to the same result as derived earlier.

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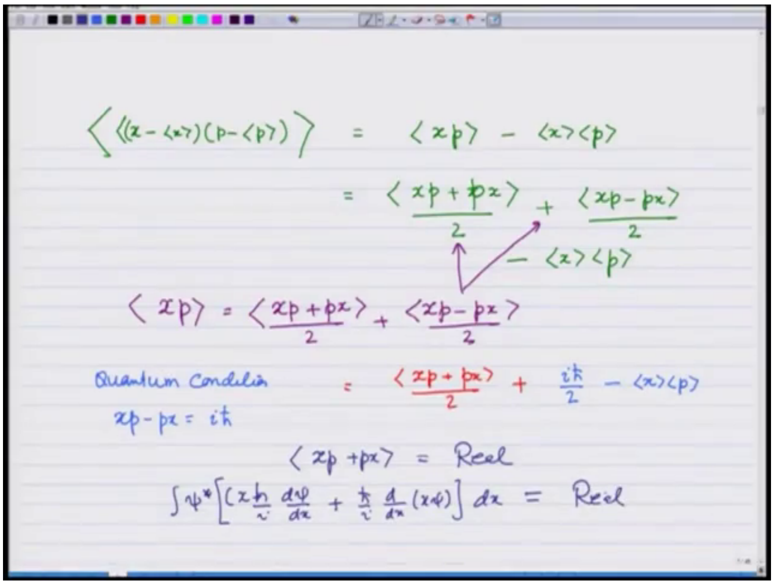

So, we have calculated x minus x p minus p expectation value to be, expectation value of x times p minus x p expectation value product. I am going to use this and do and check I will right this as x plus px expectation value divided by 2, plus x p minus p x expectation value divided by 2 minus x p. So, what have I done in this? What I have done is I have written x p product as xp plus px divided by 2 plus x p minus px divided by 2 even I take the expectation value on the 2 sides this is what I get and I will use this here and here. This then I know is nothing but expectation value of xp plus px divided by 2 plus, recall from the quantum condition that xp minus px this is nothing but I h cross. So, therefore, this becomes I h cross by 2 minus xp.

Now, I leave this as an exercise for you to show that expectation value xp plus px is real all right. This is very easy to show if you write p in the operator form. So, this actually become integration psi star x times h cross over d psi over dx plus h cross over I d over dx of x psi dx. And you can do the manipulations you can do integration by part and show that this comes out to be real.

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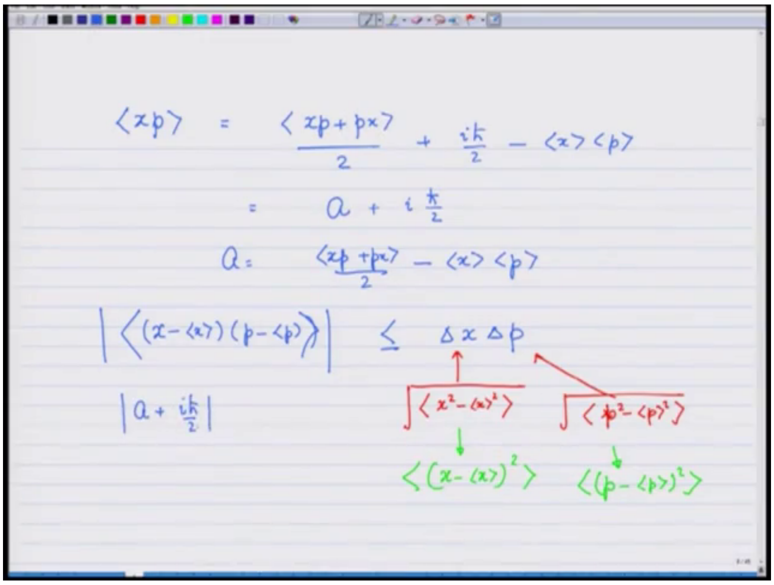

So, what you have is that xp expectation value is expectation value of x p plus px divided by 2 plus I h cross divided by 2 minus x p, which is real quantity a plus an imaginary quantity h cross by 2. Where a is defined to be xp plus px divided by 2 minus x p. So, to collect all this together what have we done? We have found that expectation value x minus mod x p minus mod p modulus is less than or equal to delta x delta p which were defined as let me remind you delta x was defined as square root of expectation value of x square minus mod x square and this was define as square root of x p square minus p square. Which is nothing but let me also write this as x minus expectation value x square expectation value, same thing for the other term this is nothing but p minus expectation value of p square expectation value they are all one and the same thing.

So, on the right hand side I have delta x delta p which is defined through all this operations. And on the left hand side now I have modulus of a plus I h cross over 2.

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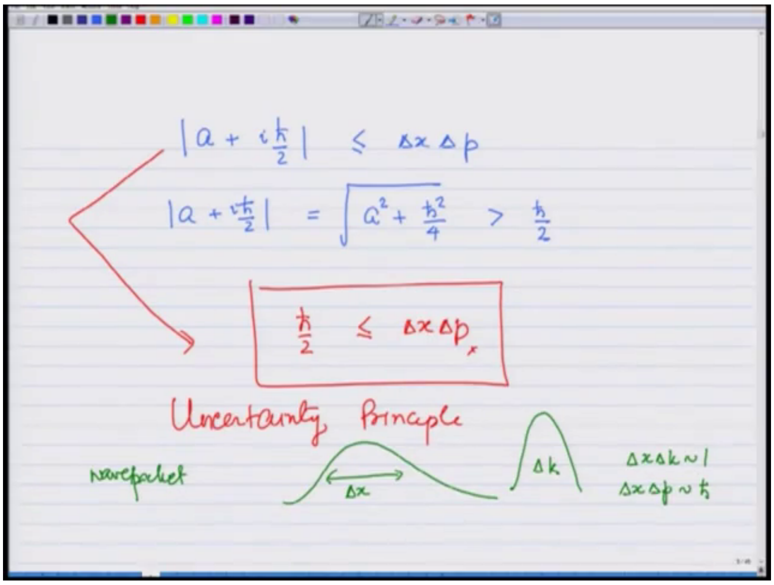

So, what do we have? We now have a modulus of a plus I h cross by 2 is less than or equal to delta x delta p. Modulus of a plus I h cross by 2 is nothing but a square plus h cross square by 4 square root, which is certainly greater than h cross by 2. Because a square is a positive quantity and therefore, from here I have my result that h cross by 2 is less than or equal to delta x delta p. And this is the uncertainty principle, which says that if you measure for a wave function x many many times and p many many times there will be a spread, but the spread is going to be such that it is product will be greater than or equal to h cross by 2, it is not a limitation of your measurement, but this is fundamentally inherent in quantum mechanics. I had alluded to it I hinted about it when we constructed a wave packet for a particles few lectures back. And what I had said there that if I have a wave packet which has a spread of delta x, the corresponding mixture of plane waves that gives you this has delta k spread in k space such that delta x delta k is of the order of one. You multiply by h cross on both sides you get delta x delta p of the order of h cross.

So, the moment you try to represent a particle by a wave, you get this sort of uncertainty relationship. Because the moment you try to show it has a wave it cannot be localized to the accuracy which you wish to have there is a limitation. And then the momentum also gets a spread correspondingly and which is given by this uncertainty relationship. You can see if I make delta x smaller and smaller delta p as becomes larger, and larger if I make delta x larger and larger delta p x becomes smaller and smaller smaller. So, this is the kind of relationship that these 2 spreads have.

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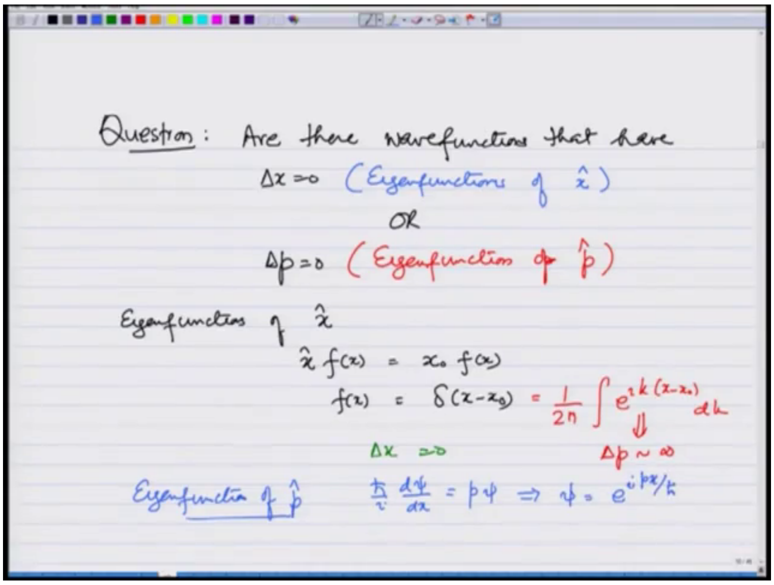

As I said earlier the wave functions that are Eigen functions do not have a spread. So, let us ask a question, are there wave functions that have either delta x is equal to 0? And these will be the Eigen functions of x operator, or delta p equal to 0. And they will correspond to Eigen functions of p operator. So, you can physically think that for Eigen functions of x operator x times it is fx should give you a definite x x 0. This is the only way it can happen is if fx is delta x minus x 0. There is also if you recall from previous lectures is nothing but 1 over 2 pi integration e raise to I k x minus x 0 dk.

Since the amplitudes of all plane waves are the same; that means, all momentum come with the same probability. So, you can see that delta p, this is the representation gives you the delta p is infinite. Where as delta x in this case is 0. So, the product I know remains finite. How about Eigen function of p operator? For this I am going to have h cross over I d psi over dx which is a p operator operating on psi is going to be a definite momentum p psi. And this immediately gives you that psi is e raise to I p x over h cross. So, what I have shown you is that Eigen function of p is e raise to I p x over h cross Which I can write as e raise to I k x.

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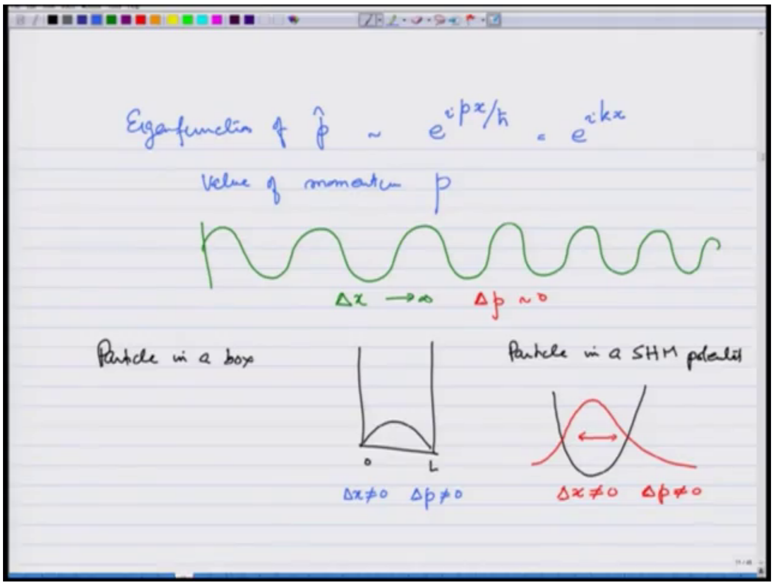

This has value of momentum for a particle in this status p fixed. On the other hand this is a plane wave. So, it goes all the way from plus infinity to minus infinity. So, delta x tends to infinity in this whereas, since it is momentum is fixed delta p is 0. These are 2 ideal cases, in one case the wave function is a delta function in the other case wave function is a plane wave spreading from minus infinity in a for a particle these are 2 ideal cases 2 extreme cases which are not same.

What is seen is for example, particle in a box, where wave function is between 0 and l. Or you also see a particle in a simple harmonic motion potential, where if this is the potential wave function is like e raise to minus x square. So, it has some spread and this is where delta x is not going to be 0, and delta p is not going to be 0, but the product will always be greater than or equal to h cross by 2. Similarly an particle in a box delta x is not going to be 0; delta p is not going to be 0. In fact, I am going to give you the problems in the assignment where I would ask you to calculate delta x and delta p for these and see what their product is it should always come out to be greater than h cross by 2 as we have seen rigorously in our derivation.

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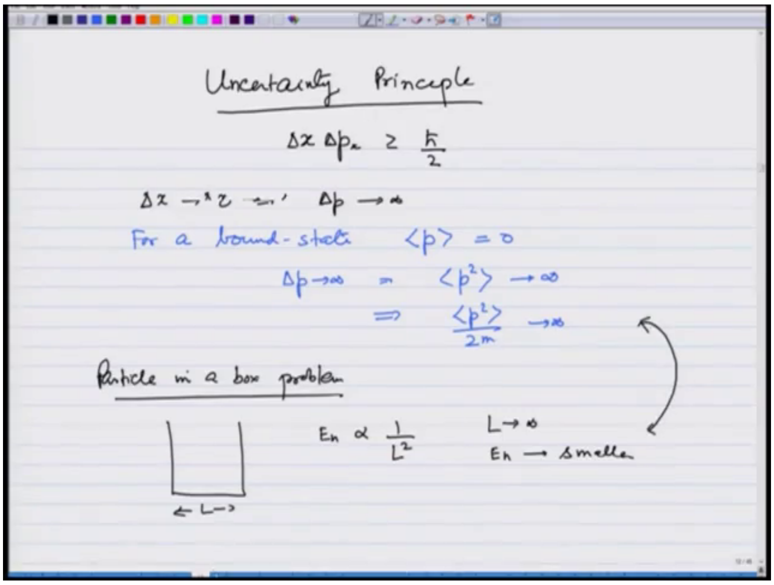

So, what you have learnt so far is the uncertainty principle and I have confined myself to one dimension, which says delta x delta p x is greater than or equal to h cross by 2. And it comes in very handy for quick back of the analog calculations. For example, now I know that delta x going to 0 means delta p goes to infinity. And recall from the previous lecturer for a bound state p expectation value itself is 0. So, delta p going to infinity means that p square expectation value is becoming very large, and this implies that p square over 2 m is becoming very large.

So, if you confine a particle more and more it is kinetic energy tends to increase because of the uncertainty principle. If let it spread it tends to become smaller and you seen this earlier. Again we go back to particle in a box problem, where a particle was confined in a box of length L and if you recall this energy en was proportional to 1 over L square. So, as L becomes larger the energy becomes smaller, this is precisely a demonstration of uncertainty principle.

We can use it also to estimate energies of different systems.

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For example again I will go to the example one, for using principle, for estimating ground state or lowest energy of a system. So, first example I take is this particle in a box of length L and you can see in this case delta x is of the order of L. And delta p which is defined as square root of p square minus expectation value of p square is nothing but the square root of p square, because for bound state expectation value of p is 0. So, delta p which is equal to square root of p square expectation value times L is going to be greater than or equal to h cross by 2. And therefore, p square itself is going to be greater than or equal to h cross square over 4 L square. P square over 2 m which is the energy is going to be greater than or equal to h cross square over 8 m L square, which you indeed find is the case.

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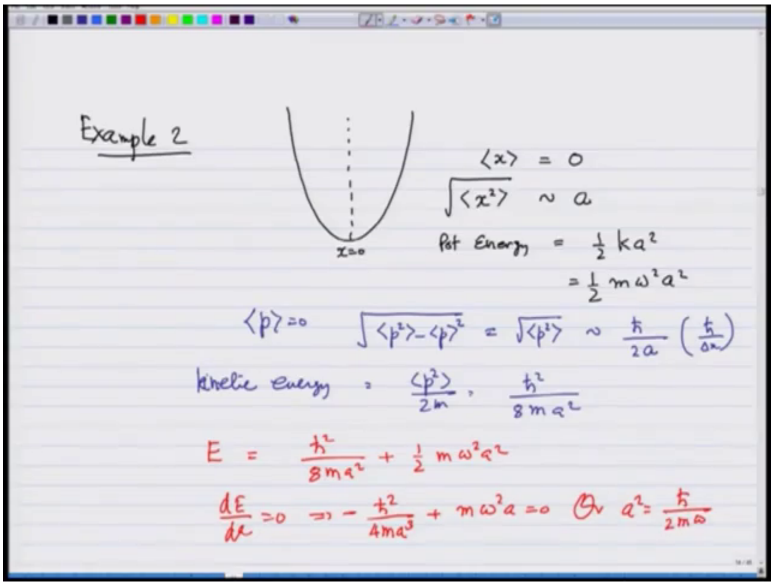

In the second example I will take the particle in a simple harmonic potential. And in this case if this is centered at x equal to 0, I know the expectation value from symmetry of x is going to be 0. And let me assume that x square square root is of the order of a, So that we can write the potential energy as one half k a square which is one half m omega square a square.

How about the kinetic energy? Now again p expectation value 0 therefore, square root of p square minus p expectation value square which is same as square root of p square is going to be of the order of h cross over 2 a, which is nothing but h cross over delta x and therefore, the kinetic energy is going to be p square expectation value over 2 m, which is h cross square over 8 m a square. And the total energy E is therefore, going to be h cross square over 8 m a square plus 1 half m omega square a square. And to find the lowest energy I minimize this energy with respect to the only parameter that is available to me and that is a. So, I do de over da is equal to 0, which gives me h cross square over 4 ma cube with the minus sign in front plus, m omega square a is equal to 0 or you solve this you get a square is equal to h cross over 2 m omega.

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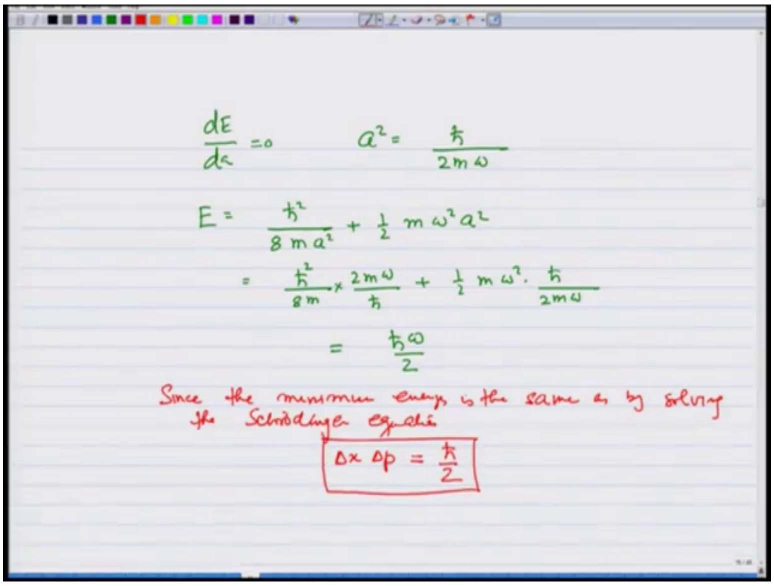

So, what we have found my minimizing energy with respect to a, that a square is h cross over 2 m omega. And the total energy E was given as h cross square over 8 m a square plus 1 half m omega square a square. This will come out to be h cross over a 8 m times 2 m omega over h cross square plus, 1 half m omega square times h cross over 2 m omega which comes out to be h cross omega by 2. Precisely the same answer as we got by solving Schroedinger equation. So, let me comment here since the minimum energy is the same as by solving the Schroedinger equation. The uncertainty delta x delta p for the ground state of simple harmonic motion should also be minimum. Or all other states is going to be greater than h cross by 2. So, this is an application to get an order of magnitude estimate of the energy using the uncertainty principle. I will also give you some problems related to this in your assignment.

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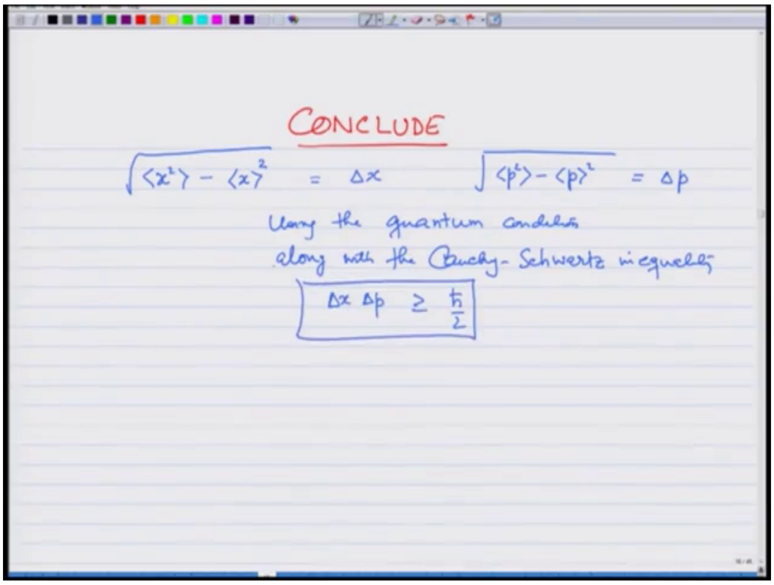

So, with this let me conclude this lecture, by saying that delta x is the expectation value of x x square expectation value when you subtract delta x from this and take square root this is how we define delta x. Similarly if I take p square expectation value and subtract the square of expectation value of p this is how we define delta p. And using the quantum condition along with the Cauchy Schwartz inequality, we got delta x delta p to be greater than or equal to h cross by 2 and that is the uncertainty principle. And then we applied it to estimate energies of 2 systems.